Just *perfect*

part 1

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Introduction

This article is about a very small subset of the positive integers. We say that the positive integer N is *perfect* if it is the sum of all its divisors, including 1, but less that N itself. (Notice that N is technically a divisor of itself.)

For example, $N = 6$ is perfect, because the (relevant) divisors are 1, 2 and 3, and $6 = 1 + 2 + 3$. On the other hand, $N = 12$ has divisors 1, 2, 3, 4 and 6, but since $1 + 2 + 3 + 4 + 6 = 16 \neq 12$, 12 is not a perfect number.

Table 1

N	Divisors	s(N)
2	1	1
3	1	1
4	1, 2	3
5	1	1
6	**1, 2, 3**	**6**
7	1	1
8	1, 2, 4	7
9	1, 3	4
…	…	…

Perfect numbers are not new: in fact the search for them began in ancient times. The first three perfect numbers were known to the ancient mathematicians at least from the time of Pythagoras (circa 500 BC).

Now if we are going to study the properties of the perfect numbers, we need more than one — and the ancient Greeks have laid down the challenge of finding at least three.

Finding perfect numbers

The most obvious first plan is to set up a table as shown in Table 1, with columns for the numbers N. lists of the divisors of N, and the divisor sums $s(N)$.

It is easy to fill in the entries, but you will need a little perseverance to find the next perfect number. What is it?

To find the third perfect number, my hint is: try a different method! If you have any computing experience, it is easy and satisfying to construct a little program for generating the first few perfect numbers. The program given in Figure 1 is in Pascal, but the notes alongside indicate the purpose of each line of code.

This program can be improved. For example, we do not need to test for $d = 1$ up to $N - 1$; up to $N/2$ will do, but of course $N/2$ may not be an integer.

```
program perfect
{to list the perfect numbers up to 10000}

  var  s, d, N: integer;              Set variables for the number N, an
                                      arbitrary divisor d, and divisor sum s.
  begin {program}
    for N := 2 to 10000 do            Set the range of testing for numbers N.
      begin{for}                      For each N we will build the sum s,
      s:=0;                           starting with s = 0. To do this, we divide
      for d:= 1 to (N-1) do           N successively by d = 1, 2, 3, ...  If d
        begin{for}                    divides N exactly, we add it to s.
        if N mod d = 0 then  S:= S + d;
      end{for};
      if s = N then writeln(N)        If the s we obtain equals N we write it
    end{for};                         down. Now go back to the beginning for
  end. {program}                      the next N.
```

Figure 1

The program in Figure 1 generates the short list:

6
28
496
8128

— the first four perfect numbers. You might like to check that

$$28 = 1 + 2 + 4 + 7 + 14.$$

In fact, the fifth and sixth perfect numbers are

33 550 336
8 589 869 056

This is still a fairly small sample, but it is enough for us to make some conjectures (intelligent guesses).

Conjectures and questions

1. Look at the six numbers. What can you say about the parity of the numbers? about the last digits? Are there any odd perfect numbers?

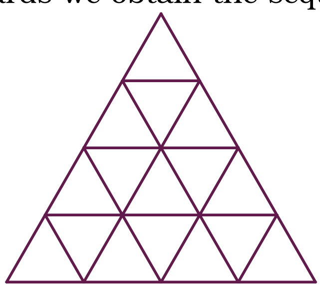
2. We have seen that $6 = 1 + 2 + 3$. Do the other numbers have a similar representation? In fact, there are many numbers of this form. Can we tell from the sum which of these are perfect?

3. Let's try factoring the early numbers.

$$\begin{aligned} 6 &= 2 \times 3 \\ 28 &= 4 \times 7 \\ 496 &= 16 \times 31 \\ 8128 &= \ldots \end{aligned}$$

Is there a pattern here? What is it?

4. Triangular numbers are numbers created by counting the vertices in a triangular grid. So in the diagram below, starting from the bottom left hand corner and working right and upwards we obtain the sequence:



1, 3, 6, 10, 15, 21, 28, ...

What do you notice? Do you expect other perfect numbers to appear? Why?

5. If we start with the second of the perfect numbers we have:

$$28 = 1^3 + 3^3$$
$$496 = 1^3 + 3^3 + 5^3 + 7^3$$

Is there an ongoing pattern here?

6. Perfect numbers greater than 6 also show other curious patterns. Let us try adding together the digits, then adding together the digits of this sum, and so on. For example:

28: $2 + 8 = 10$
$1 + 0 = 1$

496: $4 + 9 + 6 = 19$
$1 + 9 = 10$
$1 + 0 = 1$

8128: $8 + 1 + 2 + 8 = 19$
$1 + 9 = 10$
$1 + 0 = 1$

In these examples, we always finish up with a 1. Does this always happen?

7. Suppose for any perfect number N we take the reciprocals of the divisors of N which are less than N. We obtain:

6: $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$
28: $\frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28} = 1$
496: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{31} + \frac{1}{62} + \frac{1}{124} + \frac{1}{248} + \frac{1}{496} = 1$

Do the reciprocals of a perfect number always add to 1? In fact this is rather a trivial result; you should be able to show quite easily that the answer to this question is "Yes".

Clearly the perfect numbers are a remarkable resource for making conjectures. Some of the questions we have asked remain unresolved. Others we can answer, and we shall return to this in the next issue.

Bibliography

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From Helen Prochazka's

Scrapbook

Mathematics, rightly viewed, possesses a beauty cold and austere like that of a sculpture, without any appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.
Bertrand Russel, 20th century mathematician

May not music be described as the mathematics of sense, mathematics as music of the reason? The musician feels mathematics, the mathematician thinks music — music the dream, mathematics the working life.
Joseph Sylvester, 19th century mathematician

When mathematics is taught, it is presented mainly as a collection of slightly related techniques and manipulations. The profound, yet simple concepts get little attention. If art was taught in the same way, it would consist mostly of learning how to chip stone and mix paints.
George Boehm, 20th century educator