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Algebraic Generalisation Strategies: Factors Influencing Student Strategy Selection

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This study reports on the algebraic generalisation strategies used by two fifth grade students along with the factors that appeared to influence these strategies. These students were examined over 18 instructional sessions using a teaching experiment methodology. The results highlighted the complex factors that appeared to influence student strategy use, which included: (a) input value, (b) mathematical structure of the task, (c) prior strategies, (d) visual image of the situation, and (e) social interactions with the teacher and other student. Particular combinations of these factors appeared to increase the predictability of student strategy use. However, the complex nature of the factors influencing these strategies demonstrates the challenges that exist in encouraging students to move toward more sophisticated strategies.

The placement of algebra content within the mathematics curriculum has received considerable attention from educational policy-makers, professional organisations, and the research community. Rather than viewing algebra as a separate topic in secondary school courses, Kaput (1995) recommends that algebraic concepts be developed throughout the elementary and middle school grades. Documents from the Australian Education Council (1994), the National Council of Teachers of Mathematics [NCTM] (2000) in the United States, and the Department for Education and Skills (2001) of Great Britain concur with Kaput's view. These documents recommend the development of algebraic ideas at elementary and middle school levels through activities such as generalising numeric patterning situations.

However, student algebraic understanding has often lacked depth due to the traditional focus on symbol manipulation without a connection to meaning (e.g., Booth, 1984; Demby, 1997; Kieran, 1992; Lee & Wheeler, 1989; Mason, 1996). To circumvent this difficulty, generalising numeric patterns is viewed as a potential vehicle for transitioning students from numeric to algebraic thinking because it offers the potential to establish meaning for algebraic symbols by relating them to a quantitative referent. Kaput (1999) defines generalisation as:

> deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situation themselves but rather on the patterns, procedures, structures, and the relationship across and among them. (p. 136)

One facet of generalisation, as described by Kaput, involves examining varying quantities and describing relationships that exist among cases for a particular situation. Developing an understanding of the variant and invariant conditions can provide meaning for algebraic symbols.

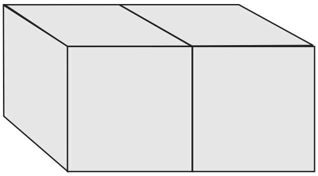
Recent research on early student algebraic reasoning has demonstrated that elementary students are capable of applying functional reasoning (e.g., Schliemann et al., 2003; Warren, 2005) and can develop generalisations (e.g., Carpenter & Franke, 2001; Fujii, 2003; Kaput & Blanton, 2001; Swafford & Langrall, 2000). These studies point to the potential for introducing algebra in conjunction with arithmetic in the elementary grades (Lins & Kaput, 2004).

In addition, a growing research base (Healy & Hoyles, 1999; Stacey, 1989; Swafford & Langrall, 2000) provides insight into the strategies students use to develop algebraic generalisations. However, little is known about what influences students to use particular generalisation strategies. Such knowledge is important for both teachers and curriculum designers. If certain factors could be identified that influence the use of particular strategies, tasks could be constructed to encourage students to use and reflect on various strategies, promoting the use of more sophisticated strategies throughout the elementary and middle grades. To address this issue our study pursued the following research questions: (a) How do various factors influence student use of generalisation strategies? and (b) How do these factors influence the use of a particular generalisation strategy? In examining these questions, we used a teaching experiment methodology to develop a theoretical model that can be used to analyse the factors that influence student strategy use in a variety of mathematical domains.

Prior Research on Generalisation

Students regularly make generalisations about the world in which they live (Mason, 1996). They reason that dogs and cats are household pets, but raccoons are not. When students are asked to generalise numeric situations in the mathematics classroom, they construct a variety of generalisations and use many different strategies. For example, Kenney, Zawajewski, and Silver (1998) found that students constructed a surprising number of valid generalisations for a particular eighth grade item on the U.S. National Assessment of Education Progress. These generalisations demonstrate that students are able to analyse problem situations in a variety of different ways. Other research studies (Carpenter, Franke, & Levi, 2003; Healy & Hoyles, 1999; Lannin, 2001; Stacey, 1989; Swafford & Langrall, 2000) have examined the strategies and reasoning that elementary and middle school students use to generalise numeric situations, such as the *Cube Sticker Problem* (see Figure 1). We discuss this research further in the following section.

A company makes coloured rods by joining cubes in a row and using a sticker machine to place "smiley" stickers on the rods. The machine places exactly one sticker on each exposed face of each cube. Every exposed face of each cube has to have a sticker, so this length two rod would need 10 stickers.



1. How many stickers would you need for rods of length 1-10? Explain how you determined these values.
2. How many stickers would you need for a rod of length 20? Of length 50? Of length 127? Explain how you determined these values.
3. Explain how you could find the number of stickers needed for a rod of any length. Write a rule that you could use to determine this.

Figure 1. Cube Sticker Problem.

Theoretical Perspectives

Generalisation Strategies

This study drew on previous research (Healy & Hoyles, 1999; Lannin, 2001; Stacey, 1989; Swafford & Langrall, 2000) regarding the strategies children use to generalise numeric situations. We recognised that these strategies (see Figure 2) often emerge through different reasoning. For example, children may select a recursive strategy (i.e., reasoning from term to term) for two different reasons: (a) the child could determine a recursive rule based on an understanding of a relationship that occurs in the situation (e.g., noticing that the number of stickers in the Cube Sticker Problem increases by four each time because a single cube can be inserted into the middle of the rod); or (b) the child could discover a numeric pattern in the consecutive values of the dependent variable, absent a strong connection to the context (e.g., noticing that the number of stickers increases by four each time: 6, 10, 14, 18, ...).

The framework in Figure 2 guided our analysis of children's reasoning and influenced our instructional goals. Similar to Healy and Hoyles (1999), our desire was for the children to be flexible in their reasoning so they could recognise the power and limitations of the various strategies. For example, we designed tasks that encouraged students to examine the use of whole-object reasoning (e.g., mistakenly thinking that if a rod of length five has 22 stickers, a rod of length 10 has 44 stickers) so the children would deepen their understanding of proportional reasoning and recognise why this method often over or undercounts the desired attribute.

Occasionally, after the participants developed a generalisation, strategies of fictitious students were introduced to encourage the participants to reflect on the strengths and limitations of alternative strategies in comparison to those that were originally developed.

We view the strategies in Figure 2 along a continuum from recursive to explicit. The use of the whole-object and chunking strategies are attempts by students to immediately calculate particular values. However, both strategies vary more conditions than do explicit strategies, leading to difficulties in general implementation. For example, in the *Cube Sticker Problem* (Figure 1) the whole-object strategy can be used to find the number of stickers for a length-20 rod when the number of stickers for a length-10 rod is known (by subtracting two stickers from the result of doubling the number of stickers from a length-10 rod). The use of the whole-object strategy to find the number of stickers for a rod of length 235 has the effect of creating challenges, as the student must consider the impact of combining multiple groups of 10 and an extra group of 5, for example. In a similar way, the chunking strategy resembles an explicit strategy. However, the starting value changes as the student attempts to calculate particular values. For example, the student may begin with knowledge of the 10th output value and determine the 25th output value. The 25th value is then used as the starting point to determine the next output value.

Strategy	**Description**
Explicit	A rule is constructed that allows for immediate calculation of any output value given a particular input value [e.g., there are four sticker for each cube, so I took four times the length of the rod, then I added two stickers for the ends of the rod.]
Whole-Object (also referred to as Unitising)	The student uses a portion as a unit to construct a larger unit using multiples of the unit. The student may fail to adjust for any over or undercounting, when applicable [e.g., a rod of length 10 has 42 stickers, so a rod of length 20 would have 42 (2 or 84 stickers (incorrect)]. The student may adjust for over (or under) counting due to the overlap that occurs when units are connected. [A rod of length 10 has 42 stickers, so a rod of length 20 would have 42 (2 - 2 because the stickers between the two length 10 sections would need to be removed.]
Chunking	The student builds on a recursive pattern by building a unit onto known values of the desired attribute. [For a rod of length 10 there are 42 stickers, so for a rod of length 15, I would take 42 + 5(4) because the number of stickers increases by 4 each time.]
Recursive	The student describes a relationship that occurs in the situation between consecutive values of the independent variable. [Each additional cube adds 5 stickers, and one sticker must be removed when the new cube is added to the rod, making a total of 4 stickers added for each cube.]

Figure 2. Generalisation strategy framework.

Evolution of Research on Strategy Influences

The research base regarding the potential factors that impact student strategy use demonstrates the complex nature of these factors. For example, Piaget (1970) described the schema that he hypothesised to exist in the minds of the children he studied. These schema represented the cognitive structures that students evoked as they engaged in various tasks. Reflection on certain tasks by a child appeared to lead to reorganisation (accommodation) or an integration of ideas into the child's current schema (assimilation). Piaget's work alludes to the powerful influence that students' prior ways of thinking can have on strategy use and selection.

Whereas Piaget focused on the individual mental constructs of children, Vygotsky (1934/1978) examined learning as situated within the social culture of the child. He suggested that learning occurs through social negotiation as children interact with their environment. Through the process of social negotiation, children develop meaning for complex sign and symbol systems such as speech, language, and mathematics. The creation of these signs and symbols leads to new connections in the mind of the child. As stated by Vygotsky:

> Every function in the child's cultural development appears twice: first on the social level, and later, on the individual level; first between people (interpsychological, and then inside the child (intrapsychological). This applies to voluntary attention, to logical memory, and to the formation of concepts. (p. 57)

Vygotsky and Piaget identified the complex forces within and external to children that impact the strategies they select. In relation to algebraic generalisation, specific potential forces have been identified that could lead to the use of more sophisticated strategies. Healy and Hoyles (1999) identified the visual connection between the problem context and the corresponding symbolic representation as a factor that encouraged student sense making of explicit rules in a technology rich instructional environment. Similarly, Swafford and Langrall (2000) and Zazkis (2001) suggested that having students examine numerous and increasingly large input values can promote student use of explicit reasoning. Stacey and MacGregor (2001) noted that many of the tasks used to develop explicit reasoning tend to focus students on the recursive relationship that exist in the situation, rather than promoting the use of more sophisticated strategies. Instead, Stacey and MacGregor encouraged the use of tasks that diminish the emphasis on the recursive relationship, focusing students on the connection between the input and output values so that students begin to examine explicit relationships. As we began our research on strategy influences, we realised that a variety of factors could lead to changes in student strategy use. We initially sought to identify particular forces that drove the changes we observed in our study.

Method, Data Sources, and Analysis

Eight fifth-grade students were purposefully selected from the fifth grade population of 80 students in a U.S. elementary school using a pre-assessment consisting of three tasks similar to those used later in the study. The eight students represented a range of ability levels and strategy use; they were grouped into two high/medium and two low/medium pairs. The pairing of students allowed us to focus on the cognition of a particular student while maintaining an element of the social classroom environment. The students had previously experienced instruction in grades K-4 that focused on following procedures without much focus on developing student understanding. In grade 5, the school district adopted new curricular materials that were intended to develop a deeper understanding of mathematical concepts. One high/medium pair, Lloyd and Dallas, serves as the focus for this paper.

Our research served as a design experiment (Cobb, 2000) in which we attempted to understand the impact of various factors on student strategy use. A teaching experiment in the model of Steffe and Thompson (2000) was utilised throughout 18 instructional sessions occurring over a four-month period. In this design, a pair of students interacts with a lead teacher (first author) and a witness (second or third authors). The teacher's role during these sessions was to facilitate student thinking without overtly directing participants to any particular solution or strategy. As such, we followed the guidelines set forth in the NCTM Standards (2000) that students be allowed to solve problems and apply their reasoning to tasks. The teacher guided the students by asking them to explain their thinking and to consider why they thought their rules could be applied to various values. During each episode, students attempted to generalise algebraic situations. The tasks were purposefully chosen to facilitate the generation and testing of hypotheses about "students' unanticipated ways and means of operating as well as their unexpected mistakes" (p. 277). (See Figure 3 for a list of the tasks.) This allowed us to create models of student thinking that could be examined, tested, and modified.

Each episode was captured on video with a separate camera focused on each participant. Further evidence included the researchers' field notes, students' written work, and video screen-capture of students' computer spreadsheets.

Following the completion of the 18-week teaching experiment, the data were retrospectively analysed using a data-reduction approach (Miles & Huberman, 1994). Initially, a descriptive account of the events for each session was constructed. The research team used these rich descriptive accounts to identify and code each occurrence of the four strategies identified in our framework.

In addition to coding for each strategy, special attention was paid to the context surrounding a change in strategy. Using descriptive accounts, strategy schematics (see Figure 4 for a schematic for the *Cube Sticker Problem*) were constructed to visually represent individual strategy use, change, and influence during each session. Strategy types were placed vertically on the schematic with the chronological order of strategy-use depicted horizontally; this organisation resulted in a matrix where each position represented a different solution strategy. The identification of influencing factors was difficult; student comments were used in the identification when possible, but the identification of a single factor remained a daunting task.

A constant-comparative method (Glaser & Strauss, 1967) was applied to test and revise the coding of influencing factors resulting in the categories of

Session	Task	Mathematical Structure
1	Shoe Organisation	Introduction to Tasks
2	Magic Money Pot	Introduction to Tasks
3	Cube Sticker	Linear increasing
4	Cube Sticker Alternative Solutions	Linear increasing
5	Theatre Seats	Linear increasing
6	Cube Sticker Revisited	Linear increasing
7	Poster	Linear increasing
8	Beam	Linear increasing
9	Pyramid	Linear decreasing
10	Walking	Linear decreasing
11	Lollipop	Linear decreasing
12	Lollipop Revisite	Linear decreasing
13	Pizza Sharing	Non-linear decreasing
14	Border	Linear increasing
15	Floor Design	Linear increasing
16	Allowance	Non-linear increasing
17	Chocolate Box	Non-linear increasing
18	Straw Brick	Linear increasing Linear increasing

Figure 3. Tasks used during the study.

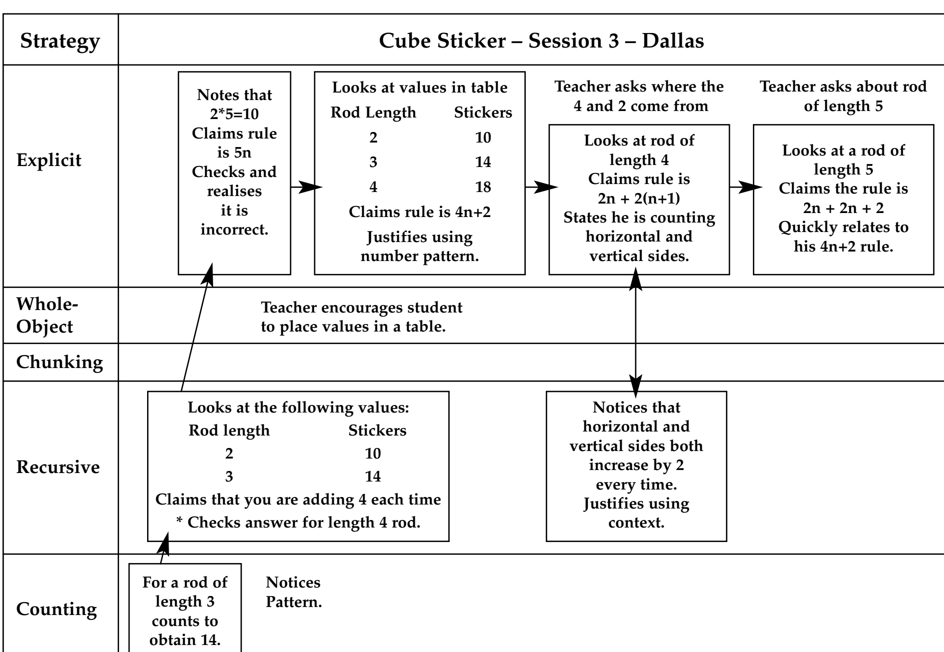


Figure 4. Sample strategy schematic for the *Cube Sticker Problem*.

mathematical structure, input values, social interactions, previous strategies, and visual image. Episodes from the other student pairings were then coded to check the scheme. Multiple members of the research team coded each episode and all discrepancies were discussed and resolved by the team.

As we coded the factors that influenced student strategy change, we discussed the challenge posed by attempting to identify a single influencing factor. We recognised that many factors appeared to simultaneously impact student strategy selection. Rather than attempt to identify a single influencing factor, we found that we could better characterise the complexity of the influences on student strategy use through examining multiple factors. Therefore, we created cross-case tables for each generalisation strategy to further detail the impact of the five influencing factors that emerged from the data: the mathematical structure of the task, the input values, social interactions, previous student generalisation strategies, and the perceived visual image that the student had of the situation. Each occurrence of a strategy in our data received a row in the table describing the nature and extent of each of the five influencing factors. The cross-case table allowed for a variety of influences to be documented for each strategy. These tables were analysed with an eye towards salient themes resulting in a picture of the factors influencing each strategy. A sample of cross-case table can be found in Figure 5.

For each student's strategy change we characterised the mathematical structure of the task (e.g., whether the problem was linear or non-linear, whether the task was increasing or decreasing, how the students viewed the task) the input values used (e.g. moving from a rod of length 10 to a rod of length 20), any apparent social influence by the teacher or students (e.g., if the teacher asked a particular question or another student mentioned using a particular strategy), prior strategies (i.e., the strategies used previously by the student in that particular session), and visual image (i.e., our assessment of the student's mental image of the situation). After coding was completed, we compiled the results by strategy to compare similarities in the impact of factors across sessions for both Lloyd and Dallas. As we examined these data, we assumed that all factors influenced student strategy use simultaneously. However, for each strategy, particular conditions for a subset of these factors appeared to consistently exist during student use of particular strategies; we referred to these as *determining factors*. For other factors, the conditions varied, leading to our designation of these factors as *contributing factors*.

Results

In the following sections we describe the factors that influence the use of the recursive, whole-object, chunking, and explicit strategies as well as the complex interactions among these factors. The strategies are presented in the general order they occurred during the sessions. However, the use of strategies varied by session and by student.

The impact of social interactions was always listed as a contributing factor due to the unpredictable nature of these interactions. In general, if an explanation

	Factors Influencing				
Session Number	**Mathematical Structure**	**Input Values for Task**	**Social Influence of Teacher & Students**	**Prior Strategies**	**Visual Image of Situation**
Dallas					
6	*Theatre Seats Revisited* (Linear Increasing) Dallas looks at the task recursively, observing the change from row to row.	Jump from the 5th to the 10th row in the theatre.	None	Recursive Strategy for N = 5, he says that he doesn't want to keep adding 3 all the way up to 10.	He does not appear to have a strong visual image of doubling in this situation.
8	*Beam Problem* (Linear Increasing) Dallas sees that four objects can be added to the end of each prior beam to create a new beam.	Jump from a beam of length 20 to a beam of length 40.	Lloyd is using the whole-object strategy and Dallas is questioned as to whether this is correct.	Dallas has an explicit rule when asked whether the whole-object strategy makes sense.	Strong visual image of what is occurring when doubling.
Lloyd					
6	*Theatre Seats Revisited* (Linear Increasing) Lloyd describes how the seats in the next row are added to the previous row (1.5 seats to each side)	Uses the same strategy for the 50th row, building from the fact that there are 34 seats in the 10th row.	Dallas did use the whole-object strategy earlier for N=10.	Used for recursive for row 10, whole-object for row 23.	He does not appear to have a strong visual image of doubling in this situation.
8	*Beam Problem* (Linear Increasing) Lloyd has difficulty seeing a relationship between the current and prior beam. Eventually, he is able to see where the 4 rods are added to the previous beam.	To find the number of rods on a beam of length 37, he recognizes the groups of ten (10 + 10 + 10 + 7)	Dallas draws and shows that Lloyd is missing a rod connecting the beams of length 10.	Recursive and whole-object.	He seems to understand that extra rods need to be added for the connectors between the length 10 beams, but does not see the need to an extra rod joining the length 10 piece and the length 7 piece.
18	*Straw Problem* (Linear Increasing) Lloyd sees that 3 additional straws can be added to the end of the straw to create the next term.	Recognizes that to find the 20th row he can use the 10th row as 10+10 = 20	The teacher draws him back to the context and to drawing out the cases to address his over count. He doesn't understand.	Recursive for 6 and 7 squares, Chunking for 10 squares.	No apparent use of a visual image. Lloyd quickly doubles his value for 10 and does not make the appropriate adjustments.

Figure 5. A portion of a cross-case strategy table for the whole-object strategy.

for a strategy was given or a question was asked, these interactions had the potential to impact student strategy use. However, the influence of the social interactions appeared to depend upon the nature of the comment and did not favour one strategy over another. For these reasons, the social influence will not be expanded upon for any of the following generalisation strategies.

Recursive

Input values, mathematical structure, and prior strategies were identified as determining factors for the use of a recursive strategy. Dallas and Lloyd were more likely to utilise a recursive strategy for a given task when the input values were relatively close, the task provided a clear connection to incremental change, and only recursion had been used previously. Student visualisation also impacted strategy use. At times, Dallas and Lloyd used recursion when they appeared to have a strong visual image of the situation and at other times when they focused on decontextualised numeric relationships.

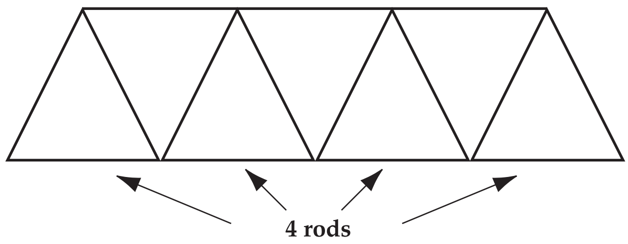
An example that illustrates many of the influencing factors of recursion is Lloyd's attempt to generalise the Beam Problem (see Figure 6) in Session 8. Lloyd began drawing and counting the number of rods required for beams of length 2, 4, 5, and 8, (see Figure 7 for the strategy schematic). When asked to determine the number of rods for a length-10 beam, he began using a recursive strategy, recognising that four rods could be added to the right side of a length-8 beam to generate a length-9 beam, and subsequently, a length-10 beam. Lloyd then used the whole-object and explicit strategies until he was asked by the teacher to find the number of rods for a length-103 beam given that a length-102 beam required 407 rods; at this point he returned to using a recursive strategy.

When the input values were relatively close, Lloyd recognised the recursive relationship and used the idea that the number of rods increased by four each time to calculate the number of rods for a length-10 beam from a beam of length 8. As the input values became larger and farther apart (i.e., jumped from 10 to 20), he abandoned the recursive strategy, attempting to find a more efficient way to determine the number of rods.

The ease in recognising the recursive relationship appeared to lead Lloyd and Dallas to use recursion as an initial strategy. For larger input values, the students rarely used recursion, as they appeared to eliminate this strategy in favour of more efficient methods. However, recursion often reappeared when students were questioned about consecutive input values, as evidenced by Lloyd in Session 8 for beams of length 102 and 103.

The ease in recognising the recursive relationship in the problem appeared to encourage Lloyd and Dallas to use recursion as an initial strategy. The structure of the task allowed Lloyd to visualize how rods could be joined to create a beam with the next integral length. For the Beam Problem, counting the number of "new" rods as the beam increased from length n to length $n + 1$ was similar to how they counted the total number of rods. For situations where the next instance was added onto one end of the previous length, Lloyd and Dallas were able to quickly recognise the recursive relationship.

Beams are designed as a support for various bridges. The beams are constructed using rods. The number of rods used to construct the bottom of the beam determines the length of the beam. Below is a beam of length 4.



How many rods are needed to make a beam of length 5? Of length 8? Of length 10? Of length 20? Of length 34? Of length 76?

How many rods are needed for a beam of length 223?

Write a rule or a formula for how you could find the number of rods needed to make a beam of any length. Explain your rule or formula.

Figure 6. Beam Design Problem.

Strategy	Beam Problem – Session 8 – Lloyd
Explicit	
Whole-Object	For a length-20 beam he doubles the amount from the length-10 beam (39) to obtain 78. For a length-37 beam takes 39 x 3+2 for the 3 groups of 10 and the 2 connecting rods. Then adds the number of rods for a length 7 beam. For a length 23 beam states that you would take 39 x 2 (for two length-10 beams) plus 11 (for a length-3 beam) plus 1 (to connect them).
Chunking	Input values increase from 10 to 20. Teacher asks student if there is an easier way.
Recursive	For a length-10 beam he adds two groups of 4 rods to the 31 he obtains for N = 8 to get a total of 39. When asked how he would calculate the 102nd term from the 103rd (and the 56th term from the 57th) uses a recursive rule and adds 4 (subtracts 4).
Counting	For a length -4, 2, 5, 8 beam he counts to obtain 15, 7, 19, 31.

Figure 7. Lloyd's strategy schematic for the *Beam Problem*.

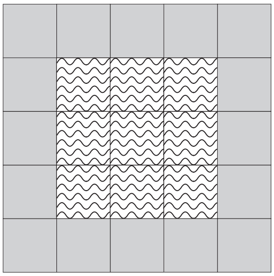
However, for situations such as the *Floor Design Problem* (see Figure 8), Lloyd and Dallas were unable to visualise the increase in the number of tiles when the length of the design increased from n to $n + 1$. For this task, tiles must be inserted into all sides of the design as the length of the design increases by one, a relationship that does not directly relate to how the students typically counted the tiles.

As Lloyd did in Session 8, the students occasionally established a visual referent for their recursive rules. At other times, the students appeared to notice a pattern in the output values after computing these values for consecutive input values. For example, Dallas noticed that the output values increased from 10 to 14 for rods of length 2 and 3 in the *Cube Sticker Problem* (see Figure 9). After testing this conjecture for a length-4 rod, he assumed that the increase in the number of stickers was four for any two consecutive values.

Whole-Object

The input values, mathematical structure of the task, visual image, and prior strategies were determining factors for the use of the whole-object strategy. When certain dimensions of these factors coincided, the whole-object strategy was likely to be utilised. For example, when a particular input value was a multiple, predominantly a double, of a previously used input value and the student did not appear to have a strong visual image of the problem situation, the student was more likely to use the whole-object strategy. Prior strategies also played a determining role, as students often moved from recursion to the whole-object strategy in search of a more efficient strategy. Finally, the mathematical structure of linear decreasing situations, such as the *Lollipop Problem*, appeared to eliminate student consideration of the whole-object strategy. The inverse variation of the input and output values appeared to invalidate the use of the whole-object strategy as a means for determining the output values. Students used the whole-object strategy only for increasing linear situations.

The AKME floor design company creates square floor patterns made of shaded square tiles surrounded by a grey border. The size of the floor is determined by the side length of the shaded square in the centre of the pattern. Below is the example of a floor of length 3 (shaded tiles).



How many grey tiles would you need to make a floor of length 6 (shaded tiles)? 10 (shaded tiles)? 20 (shaded tiles)? 47 (shaded tiles)? 139 (shaded tiles)?

Write a rule or a formula for how you could find the number of grey tiles needed to make a floor of any length (in shaded tiles). Explain your rule or formula.

Figure 8. Floor Design Problem.

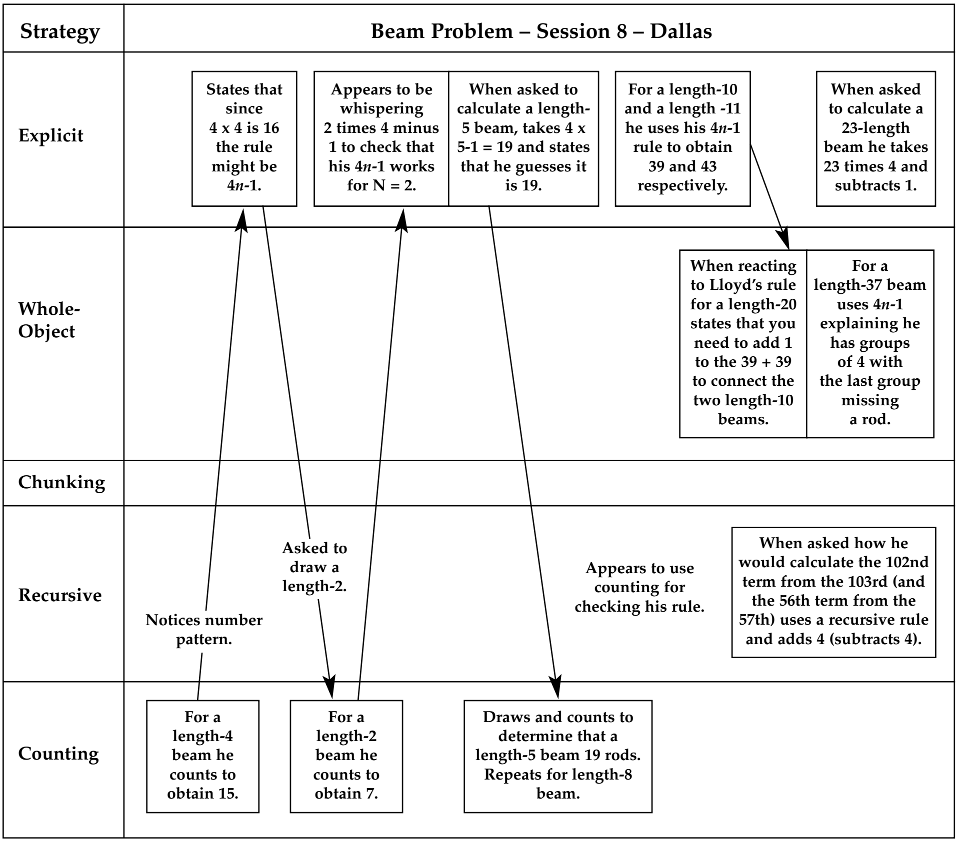


Figure 9. Dallas's strategy schematic for the *Beam Problem*.

The *Beam Problem* (see Figure 6) illustrates how the aforementioned factors influenced the use of the whole-object strategy. As seen in Lloyd's *Beam Problem* schematic (see Figure 7), after finding the number of rods for a length-10 beam, Lloyd was asked to determine the number of rods for a length-20 beam. He doubled the number of rods for the length-10 beam (39), obtaining 78 rods. To justify his strategy, Lloyd explained that since 10 + 10 was 20, he added 39 and 39 to determine the number of rods required for a length-20 beam. Doubling the prior input value often provoked the use of the whole-object strategy. However, the whole-object strategy was also used in non-doubling situations. For example, following Lloyd's initial use of the whole-object strategy in session 8, he again employed the whole-object strategy to ascertain the number of rods needed for a length-37 beam. Building upon the number of rods (39) for a beam of length 10, Lloyd noted that 10*3 + 7 =37. He then reasoned (incorrectly) that 39*3 added to the number of rods needed for a length-7 beam would calculate the number of rods for a length-37 beam.

In both of these instances, Lloyd appeared to lack a strong visual image of the problem situation, leading to incorrect results when he used the whole-object strategy. A stronger visual image of how doubling or tripling the number of rods for a length-10 beam related to the problem situation could have led Lloyd to realise that he undercounted the number of rods. Dallas, though he did not use the whole-object strategy initially, was able to recognise the errors that Lloyd made. Dallas appeared to have a strong mental image of how doubling and tripling the number of rods for a length-10 beam related to the problem context. He stated that after adding 39 and 39 to obtain the number of rods for a length-20 beam, Lloyd must also add an additional rod where these two length-10 sections join to correctly count the number of rods (see Figure 10). Dallas's strong mental image of the problem situation allowed him to use the whole-object strategy correctly, whereas Lloyd's poor mental image led him to an incorrect solution.

As previously noted, Lloyd's use of the whole-object strategy coincided with his prior use of recursion. He appeared to recognise recursion was inefficient for determining output values as input values increased. During the study, Lloyd was more likely to employ a whole-object strategy than Dallas. While this was certainly due to the afore-mentioned factors, Lloyd's apparent focus on particular values in lieu of obtaining a general rule also seemed to influence his use of the whole-object strategy.

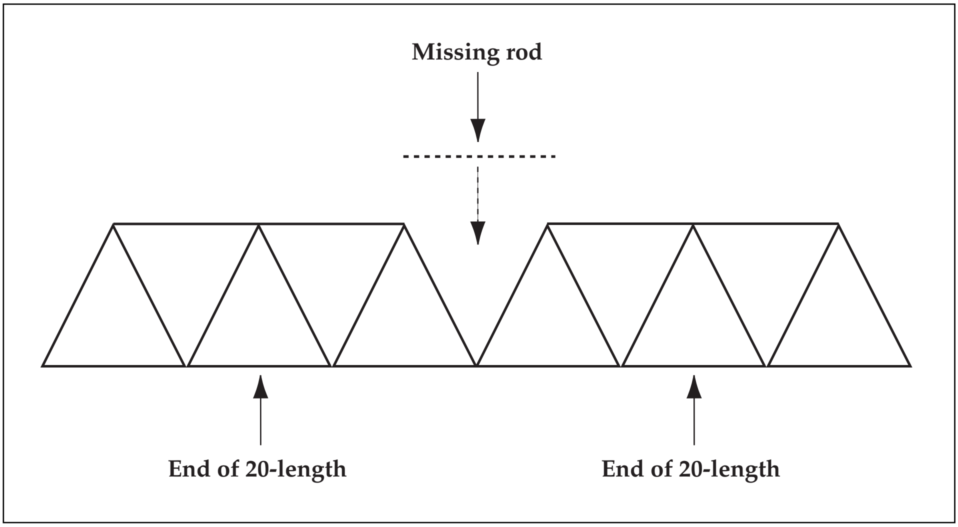


Figure 10. Lloyd's incorrect use of Whole-Object Strategy for *Beam Problem*.

Chunking

Determining factors for the chunking strategy include input values, mathematical structure, and prior strategies. Students were more likely to use a chunking strategy when input values were relatively close together, the task involved a linear decreasing situation, and when they had previously used the recursive strategy. Visualisation, while having an effect on the use of chunking, was not considered a determining factor.

Lloyd's use of the chunking strategy during the *Lollipop Problem* (see Figure 11) provides an illustration of these factors. For this situation, the students were required to determine the number of lollipops left in the box after four days and after six days, among others; hence, the situation is linear decreasing and, for these values, involves input values that are relatively close. To determine the number of lollipops left in the box after six days, Lloyd used the number of lollipops left after four days, 772, and subtracted 14 (7 (2) to account for the two days between day four and day six. This strategy provided the correct result of 758 lollipops after six days. Lloyd continued to use the chunking strategy for days 10, 20, 34 and 35.

Ms. Principal has decided to have a "Best Reader" contest for all of the students at Eastview School. The student who reads the most books in their grade level each day will receive a lollipop. Ms. Principal has purchased a box with 800 lollipops. Each day 7 lollipops are taken from the box and given to a "Best Reader" (one for each grade, K-6).

How many lollipops will there be in the box after the contest has lasted for 4 days? 6 days? 10 days? 20 days? 34 days? 45 days?

Write a rule that will calculate the number of lollipops after any number of days.

How long will it take until Ms. Principal runs out of lollipops?

Figure 11. The Lollipop Problem.

As previously noted, the relatively small difference between input values served as a determining factor for the chunking strategy. For instance, Lloyd and Dallas used chunking for differences in the input values ranging from 2 to 15, as illustrated by Lloyd's use of the chunking strategy in the example above. These differences in input values were typically larger than those values that evoked recursive reasoning and may explain why recursion is often used prior to chunking, as the differences between input values generally increased in the problems we provided the students.

The mathematical structure, in particular the use of linear decreasing situations, was a determining factor for the chunking strategy. Use of the chunking strategy in these situations was partly explained by the difficulty students exhibited in using the whole-object strategy in linear decreasing situations. For example, Lloyd used both whole-object and chunking in linear increasing situations, but used only chunking with linear decreasing situations. As the input values increased and Dallas and Lloyd searched for more efficient strategies, the chunking strategy was a viable option for these students.

Explicit

Determining factors for the use of the explicit strategy were the input values, prior strategies, and students' visual images of the problem situation. Lloyd and Dallas were more likely to generate an explicit rule when the input values were large and relatively distant from the previous input values. The explicit strategy was rarely the first strategy used; students often used explicit rules when they sought more efficient methods for calculating output values. The students' visual images appeared to contribute to their success in generating correct explicit rules. When Lloyd and Dallas had poor visual images and focused on particular values, they were more likely to use incorrect guess-and-check (explicit) strategies. They were more successful generating correct explicit rules when they were able to connect their rules to the problem situation.

Session 14 demonstrated the various ways that students constructed explicit rules. To find the number of squares for a border of length four in the *Border Problem* (see Figure 12), Dallas counted the number of squares in the border, arriving at 12. He noted that the rule "times 2 plus 4" ($S = 2n + 4$ where S is the number of squares and n is the border length) calculated the correct number of squares for this particular instance. When asked whether this rule would always work, Dallas stated that he was unsure. He looked at the diagram of a length-seven border and determined that his "times 2 plus 4" rule only applied to a border of length five and abandoned this rule. Instead, he said that "times 3 minus 1" ($S = 3n - 1$) could possibly serve as the explicit rule for this situation. He returned to the length-four border to verify his new rule, but quickly realised that this rule was incorrect as well.

The teacher then directed Dallas to consider a diagram where the length of the border was 25, encouraging him to focus on the diagram to develop his rule. At this point Dallas stated that he saw two groups of 25 squares and two groups of 23 squares (see Figure 13), appearing to connect the rule [$S = 2n + 2(n-2)$] to his mental image of the situation. When asked whether his rule would always work,

The student council is creating designs with a dotted pattern on the border. The council would like to know how many squares are needed with the dotted pattern. They have asked the 5th grade class for help.

1. How many squares are in the border of a 4 by 4 grid? A 7 by 7 grid? A 10 by 10 grid? A 16 by 16 grid? A 25 by 25 grid? A 100 by 100 grid?
2. Write a rule to find the number of squares in the border of any size grid

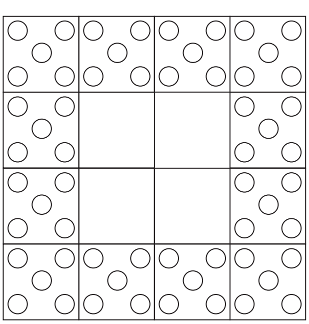


Figure 12. The *Border Problem.*

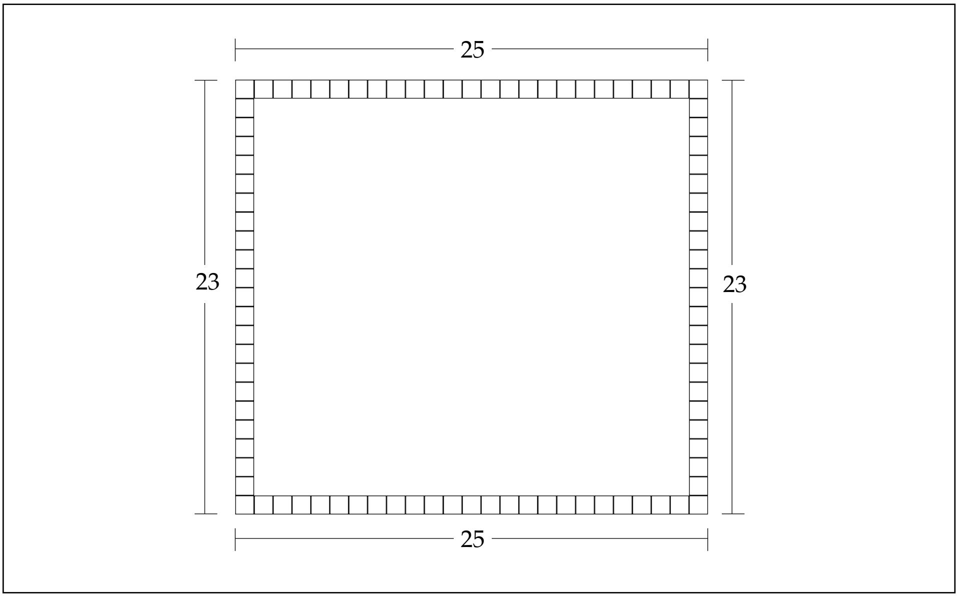


Figure 13. Dallas's view of a length-25 border.

Dallas said that it would because a similar method of counting could be used for a border of any length. He then applied the rule to a border of length 57.

This session was typical in that large input values often encouraged the use of an explicit rule. Dallas often desired to develop explicit rules when he attempted to generalise problems in an effort to find an efficient strategy that would allow for the quick calculation of output values. However, Lloyd and Dallas often employed other strategies prior to developing explicit rules, though the explicit strategy was typically a final strategy.

When these students had poor visual images of the problem situation, a guess-and-check variation of the explicit strategy was often used-a strategy that led to difficulties in determining how generalisable their rules were. The guess-and-check strategy frequently led to the development of incorrect rules, as occurred when Dallas initially tried to generalise the *Border Problem* (see Figure 12). When Lloyd and Dallas had a strong visual image and made connections between the context and the explicit rule, a correct rule was almost always found. This visual connection was sometimes a result of teacher questioning about how students' rules related to the context or diagram for a situation.

The mathematical structure appeared to have an effect on the use of the explicit strategy. Students used explicit rules for linear increasing and linear decreasing situations. In addition, some tasks, such as the *Border Problem*, appeared to encourage the use of explicit reasoning, as the recursive relationship was not as evident as it was in the *Beam Problem* (see Figure 6) and the *Cube Sticker Problem* (see Figure 1).

Discussion and Implications

Based on the results of our study, we created a conceptual framework to examine the potential factors influencing student strategy use. In the following sections, we describe this framework and provide implications for designing curricular tasks and guiding instructional decision-making when introducing similar algebraic tasks.

Factors Influencing Strategy Selection

Three broad categories emerged for grouping the factors that influenced student strategy selection (see Figure 14): social factors, cognitive factors, and task factors. We recognised that as a student engages in a task, she simultaneously interacts and is influenced by the other students and the teacher (social factors), her existing mental structures (cognitive factors), and the problem situation (task factors). This theoretical lens allowed us to examine the complex combination of factors that influenced student strategy selection in our study. Below we describe these three factors in detail. We contribute to theory about student strategy selection through our view of the simultaneous impact of factors influencing strategy selection. Previous studies (Healy & Hoyles, 1999; Stacey & MacGregor, 2001; Swafford & Langrall, 2000; Warren, 2004; Zazkis, 2001) have focused on one or two factors. The examination of social, cognitive, and task factors appears to allow for more reliable predictions of student strategy selection, though further research is necessary to examine how these factors influence student reasoning.

Student cognitive structures represent one factor influencing student strategy use. Piaget (1970) utilised such a perspective to develop conjectures

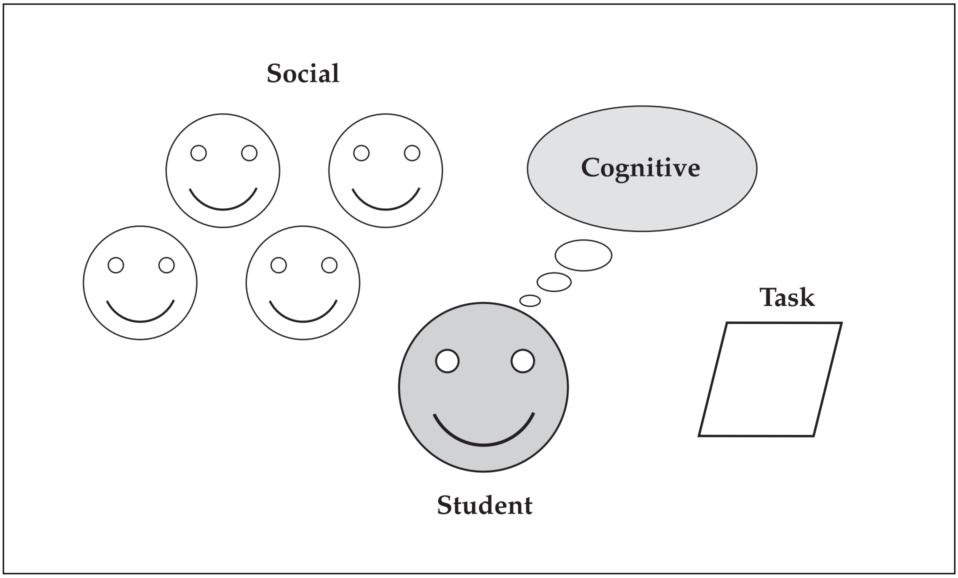


Figure 14. Factors influencing student strategy selection.

about the mental structures that exist in the mind of the individual. Based on a child's current mental structures, she either assimilates new knowledge into his current structure or accommodates his mental structure to better fit his new understanding. As students apply various strategies when generalising numeric tasks they often encounter situations that challenge their current cognitive structures, causing a change in strategy. Related to algebraic reasoning, student cognitive structures include the student's prior knowledge of mathematical operations, the strategies that the student used on previous tasks, the mathematical disposition of the student (National Research Council, 2001), and the ability of the student to visualise the physical structure of the situation in relation to the mathematical model that he or she creates (Healy & Hoyles, 1999).

Another influencing factor is the design of the task itself, including the mathematical structure of the task and the input values selected to encourage generalisation. In our study, the mathematical structure of the task includes whether the task involves a linear increasing (i.e., as the input values increase the output values increase) or a linear decreasing (i.e., as the input values increase the output values decrease) situation. The format of the task, as to whether a recursive or explicit relationship is more accessible, was also an important factor. Warren (2000, 2004) and Healy and Hoyles (1999) have noted the importance of encouraging visual connections when students construct generalisations, emphasising the use of geometric tasks. Another significant feature of the task includes the selection of input values for the situation. Choosing consecutive or non-consecutive input values or values that are multiples of each other (e.g., 5, 10, and 20) also serve as potential factors influencing student strategy selection.

Through our work we recognised that student desire for efficiency plays a role in student strategy use. In fact, this desire for increased efficiency can encourage a change in strategy use. For example, in the Theatre Seats problem, Dallas began the session using the recursive relationship that exists between consecutive rows to find the number of seats for the fifth row. However, his desire for efficiency appeared to contribute to his use of the whole-object strategy when finding the number of seats for the 10th row. Dallas quickly realised that this strategy was incorrect for this particular value, so he returned to the recursive strategy. After another unsuccessful attempt using the whole-object strategy and a subsequent return to recursion, Dallas once again changed strategies, in part due to his desire for an efficient strategy.

We view efficiency as an influence embedded within our theoretical model. Student desire for efficiency, along with their perceived inefficiency of certain strategies often served as the catalyst for their choice of strategy. For example, Dallas began to use explicit rules early in the teaching experiment. As the teaching experiment progressed, he recognised the efficiency of explicit rules and often sought to find such rules for each situation that was provided.

The Effects ofTask Structure on Strategy Selection

A student's visual image of an algebraic situation appeared to influence strategy choice in our study. Tasks for which the previous instance could clearly be observed within the following instance allowed students to more easily observe the change between terms, and hence use a recursive strategy. Figure 15 illustrates this point. Students may exhibit difficulty recognising the recursive pattern in the second example as the change occurs on four sides of the figure instead of just one. However, just because the recursive relationship can be easily seen in the situation does not guarantee that a student will observe this relationship.

The *Floor Design Problem* (see Figure 8) provides a situation for which the previous case is not easily observed in the next case. In this example the increase between terms is four; however, where the four squares are added to the figure is not easily observable. The use of a recursive strategy is possible, but the observation of the recursive pattern is more likely to come from the values themselves or a table, rather than the visual representation.

A visual representation may also help students develop explicit rules for a particular situation. The visual representations that lead to explicit rules are often connected to the counting strategy that a student employs. For instance, a student may look at the *Floor Design Problem* and observe that the bottom and top pieces each has *n tiles* and that *n* - 2 tiles exist on the sides between the top and bottom bar to develop a 2*n* + 2(*n* - 2) rule (see Figure 16).

In order for a student to use a visual cue when developing a strategy, the student must recognise the visual relationship in a general manner. Our contention is that some problems facilitate student recognition of certain relationships that lead to the use of particular generalisation strategies. Hence, we found that some tasks may promote the use of recursive reasoning whereas other tasks may encourage the use of explicit reasoning.

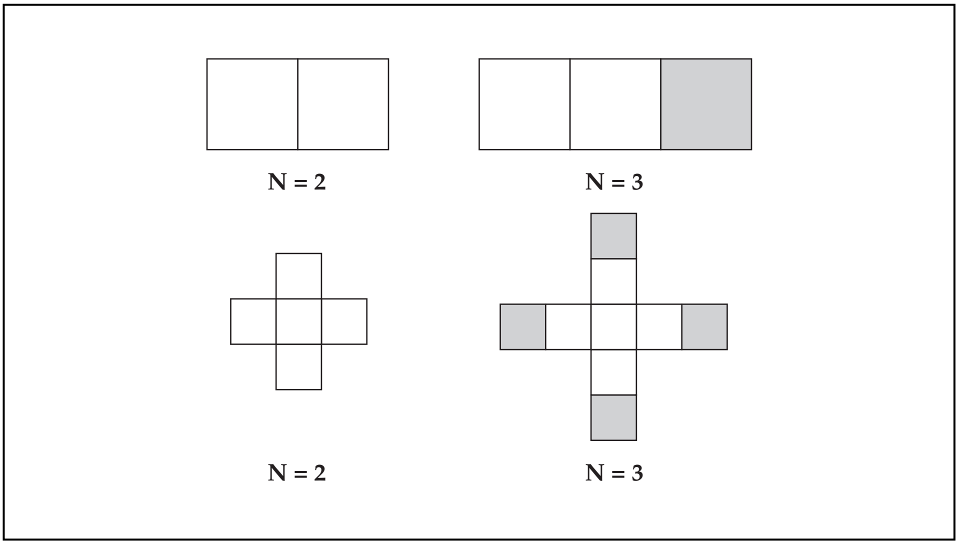


Figure 15. Recursively oriented patterns (grey shaded components denote changes between steps).

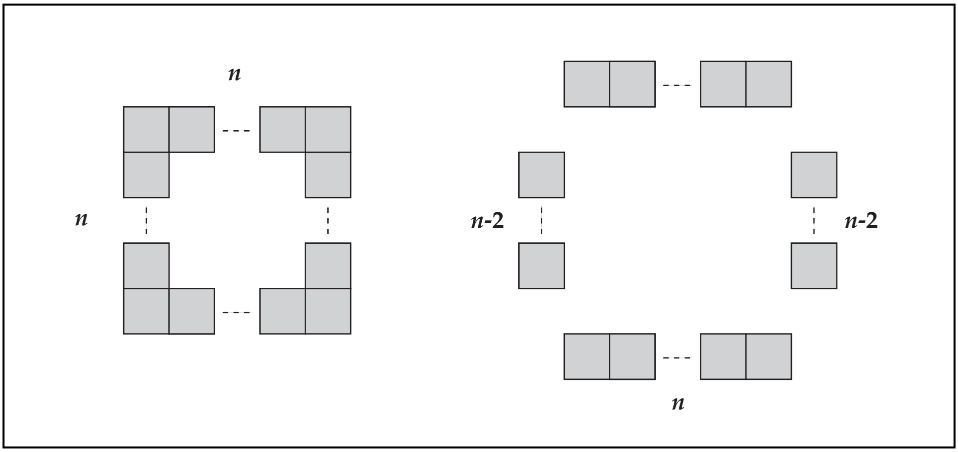


Figure 16. A visual representation of the $2n + 2(n - 2)$ rule.

Implications

We begin the discussion of this section by describing a general pattern of strategy use based on particular influencing factors that we identified. Next, we describe the curricular and instructional implications relevant to this pattern of strategy use.

Figure 17 details our predicted strategies based on three influencing factors: (a) visualisation (poor or strong), (b) mathematical structure(linear increasing or linear decreasing), and (c) input values (near, multiples of previous known values, and distant values). The figure was constructed with the assumption that students are approaching a task that they have not already seen and/or for which they have not developed a sophisticated generalisation strategy.

Students tend to use recursive rules when input values are near, regardless of the type of task or visual image of the situation. However, a student's visual image of situation often leads to markedly different views of their recursive rules. Students with a strong visual image of the situation attempt to connect their recursive rules to the problem situation (e.g., as Lloyd did for the *Beam Problem*), whereas those students focused on numeric values may have little sense of how their recursive rules relate to the problem context.

When provided with input values that are multiples of prior known values, students often apply whole-object strategies for linear increasing situations and chunking strategies for linear decreasing situations. Again, the use of the whole-object strategy differs for students who have a poor rather than a strong visual image of the situation. Students with a poor visual image often apply the whole-object strategy incorrectly whereas those with a strong visual image recognise the need to adjust the whole-strategy due to over or undercounting the desired attribute, just as Dallas pointed out the missing rod for the Beam problem (see Figure 10).

The use of "distant" input values can encourage the use of explicit strategies. Students with a poor visual image of the situation often resort to "guess-and-

Visualisation	P	P	P	P	P	P	S	S	S	S	S	S
Structure	I	I	I	D	D	D	I	I	I	D	D	D
Input Values	N	M	D	N	M	D	N	M	D	N	M	D
Projected Strategy	Recursive	Whole-Object	Explicit (Guess & Check)	Recursive	Chunking	Explicit (Guess & Check)	Whole-Object with Adjustment	Recursive	Explicit	Recursive	Chunking or Explicit	Explicit

Visualisation: P for Poor Visualisation, S for Strong Visualisation

Structure: I for Linear Increasing, D for Linear Decreasing

Input Values: N for Near Values, M for values that are multiples of previous values, D for distant values.

Figure 17. Predicted strategies based on influencing factors.

check" strategies to develop generalisations, focusing on numeric relationships rather than relationships that connect to the context of the situation. Such differences in strategy use occurred when Dallas dealt with the Border Problem discussed earlier.

Figure 17 represents a tentative initial model for the various strategies that students may use when provided with generalisation tasks. Based on the input values, mathematical structure of the task, and the visual image of the situation, the figure provides a predictor of the strategies that students may apply. However, we also realise that other factors, such as the geometric nature of the task, may also impact the strategies used by students. For example, the structure of the Border Problem (see Figure 12) appears to encourage explicit strategies over the use of other strategies due to the connection between student counting strategies and the explicit rules that they construct.

Curricular implications. When constructing numeric generalisation tasks for elementary and middle school students, curriculum designers need to consider the variety of factors that influence student strategy selection. Student progress toward more sophisticated strategies can be provoked through using a variety of linear problem situations, encouraging students to connect their generalisations to geometric representations of problem contexts, and through providing questions that encourage examination of the advantages and limitations of various strategies.

Instructional tasks should be arranged in a manner that promotes a wide range of strategies. Tasks should include both linear increasing and decreasing situations as well as situations that encourage both recursive and explicit reasoning. Mixing situations that encourage students to use a multitude of strategies and to construct a variety of rules allow students to reflect on the advantages and limitations of these ways of reasoning. The tasks that are used in the classroom should provoke students to use correct and incorrect generalisation strategies, encouraging them to examine when the various strategies should be applied. Teachers must promote reflection on student errors so they better understand why these errors occur (Hiebert et al., 1997). Errors such as the misapplication of the whole-object strategy can be rather resilient, just as Lloyd consistently returned to the whole-object strategy over many sessions.

Situations should be provided that develop connections between the visual image and the calculations in the generalisation. Tasks that involve geometric relationships, such as those used in this study, allow students to connect the meaning of operations to the context of the situation. Similarly, Healy and Hoyles (1999) found that the use of particular computer software encouraged students to connect their symbolic representations to iconic representations. Geometric situations allow students to conjecture whether rules will provide valid results and to test their conjectures by examining a model of the situation. As students develop a stronger visual image of the situation, they can better understand their errors and relate their calculations to the context, potentially leading to a decrease in the use of unsuccessful numeric generalisation strategies that are disconnected from the context of the situation.

Instructional Implications. Classroom teachers must also understand the intent of generalising numeric tasks at the elementary and middle school levels. Students at these grade levels should experience and construct generalisations (NCTM, 2000) that encourage them to use sophisticated strategies and to make connections among these strategies. As such, it is important to bring out the general nature of their rules (Mason, 1996) by asking students to examine whether their generalisations apply to all values in the domain and then to justify why their rules can be applied to all cases.

Classroom interactions should also encourage the sharing of strategies and discussing the advantages and limitations. Connections to geometric representations should be encouraged as a means of explaining, justifying, and refuting student generalisations. Similar to Bednarz, Kieran, and Lee (1996) we believe that "the process of generalisation as an approach to algebra appears ultimately related to that of justification" (p. 8). Therefore, we should consistently ask students to justify their general rules, requiring them to explain why their rule applies to all values in the domain.

Future Research

This study resulted in the construction of a theoretical lens to examine the various influences on student strategy use. We focused on two students to develop this theoretical model, but encourage further study of a larger group of students using this perspective. Further examination of the complex interactions among these influencing factors could yield a deeper understanding of the potential factors that encourage students to consider and use efficient and appropriate generalisation strategies. In addition, further teaching experiments conducted over a number of years would provide further insight into student use of these strategies. Further study of strategy use from diverse student groups and with a variety of age levels would also prove useful for researchers and teachers.

A primary implication of this study is the recognition that altering a single influencing factor, such as the mathematical structure of a task, does not necessarily lead to changes in student strategy use; the task, student cognitive structures, and social influences all contribute to student strategy selection. However, varying aspects of the task can provoke change in student strategy use. Prior student strategies and/or limitations of student perception of the problem situation can lead students to continue using inefficient or error-laden strategies.

Another important issue involves using student errors as opportunities to deepen student understanding of generalisation. Stacey (1989) raised questions about students' understanding of the generalisation process, asking, "Did students know that these cases did not follow the rules they were proposing to use, but ignored the fact? Did they think it to be of no importance or were they unaware that the rule could be applied to that data?" (p. 161). We also question how students attempt to reconcile the errors they make while generalising. For example, Lloyd often used the numeric whole-object strategy even after previously recognising that it did not apply to previous situations. Further study of this topic would inform teachers' and curriculum designers' decisions regarding how to encourage students to reflect more effectively on their understanding.

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