Mathematics Education Research Journal 2006, Vol. 18, No. 1, 47–77

Effects of Classroom Instruction on Students' Understanding of Quadratic Equations

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Two hundred and thirty-one students in six Grade 9 classes in two government secondary schools located near Chiang Mai, Thailand, attempted to solve the same 18 quadratic equations before and after participating in 11 lessons on quadratic equations. Data from the students' written responses to the equations, together with data in the form of transcripts of 36 interviews with 18 interviewees (a high performer, a medium performer, and a low performer from each of the six classes), were analysed. Using a rubric for assessing students' understanding, the analysis revealed that at the post-teaching stage students improved their performance on quadratic equations and had a better understanding of associated concepts than they had at the pre-teaching stage. However, many were still confused about the concepts of a variable and of a "solution" to a quadratic equation. After the lessons, most students had acquired neither an instrumental nor a relational understanding of the mathematics associated with solving elementary quadratic equations.

This report explores the impact of traditional teaching approaches on Grade 9 students' learning of mathematics associated with solving quadratic equations. Data were gathered in two government secondary schools in Thailand during the period from December 2002 to February 2003. The main focus in the report is not on what transpired in the lessons themselves, but rather on how, and to what extent, the lessons affected student performance on, and understanding of, quadratic equations.

At the outset it will be useful to comment on what we mean by "traditional teaching approaches". Analyses of lessons within the large collection of videotapes made for the 1995 TIMSS Video Study (Stigler & Hiebert, 1999) and the 1999 TIMSS Video Study (Hiebert et al., 2003) revealed that in all the participating countries except one – Japan was the exception – most secondary school mathematics lessons proceeded according to a review/introduction/model example/seatwork/ summary structure (Hiebert et al., 2003; Hollingsworth, Lokan & McCrae, 2003). When, in this article, the term "traditional" is used in relation to lessons it can be assumed that the lessons were structured in that way.

In traditional lessons the teacher talks more than the students and classroom discourse patterns feature many sequences of chorused answers, with teachers channeling students' thinking towards answers that they want the students to give (Vaiyavutjamai, 2004a, 2004b; Voigt, 1994). Vaiyavutjamai's (2004a) detailed analysis of algebra classrooms in Grade 9 classes in Thailand, and Lim's (2000) analyses of Grade 10 O-level algebra classrooms in Brunei Darussalam, revealed that most teaching, by local as well as by expatriate teachers, was of this elicitation variety, with exposition (“teacher telling”) also being common. Vaiyavutjamai (2204a) and Lim (2000) reported that there was always a strong emphasis on symbol manipulation, with less attention being given to the meanings of the symbols.

It would be wrong to think that such methods are confined to Asian nations, for TIMSS video studies suggest that forms of traditional instruction are still widely used in mathematics classes, including algebra classes, in many nations. Yet researchers have shied away from investigating the immediate effects of such teaching on student learning. This article responds to the need to become more aware of those effects.

Review of Related Literature

Research into the Teaching and Learning of Quadratic Equations

Considering the importance of quadratic equations in the history of mathematics and in secondary school mathematics curricula around the world, it is surprising that research into the teaching and learning of quadratic equations has been so sparse. Publications of the National Council of Teachers of Mathematics (NCTM) in the United States, for example, have rarely reported on research into the teaching and learning of quadratic equations. Although articles on algebra in various NCTM research publications (e.g., Kieran, 1992; Kieran & Chalouh, 1993; Wagner & Parker, 1993) have paid some attention to research into the teaching and learning of linear equations, they have been silent on quadratic equations.

The 13 chapters in Stacey, Chick and Kendal’s (2004) edited volume on *The Future of the Teaching and Learning of Algebra* provided commentary from scholars from numerous countries on the past, present, and future status of algebra in school curricula, and on possible renewal of pedagogy through creative use of modern technology. None of the contributors to the volume, however, provided a careful statement on the cognitive challenges faced by students trying to solve quadratic equations. As far as we are aware, such a statement is not to be found, anywhere.

It appears to be the case that difficulties that students experience in learning to solve quadratic equations are not part of the pedagogical content knowledge of secondary mathematics teachers or, for that matter, of authors of textbooks or articles on the teaching and learning of algebra. Vaiyavutjamai (2004a) reviewed sections on quadratic equations in mathematics textbooks and teachers’ guides widely used in Thailand, and found no reference to such difficulties. In his book on *Teaching and Learning Algebra*, French (2002) made no reference to difficulties students experience with quadratic equations. The same was true of Filloy and Sutherland (1996) in their chapter on algebra in the *International Handbook of Mathematics Education*.

Database searches on the Internet led to the identification of many studies into the teaching and learning of linear equations, but hardly any on quadratic equations. A search of *Dissertation Abstract International* for the past 10 years identified just one dissertation concerned with quadratic equations. That dissertation was by Sproule (2000), who investigated the performance on different types of quadratic equations of high- and low-ability upper-secondary students (in Grades 11 through 14) attending a Northern Ireland grammar school. She concluded that only very high ability students operated at adequate conceptual levels, a finding which raises the question whether Grade 9 middle-secondary students in Thailand should be expected to cope with the conceptual challenges presented by even the simplest quadratic equation.

An ERIC search identified a study by Zaslavsky (1997) who investigated misconceptions with respect to quadratic functions of more than 800 Grade 10 and Grade 11 students in 25 different schools in Israel. Zaslavsky's research emphasis was quadratic *functions*, however, and her report touched only incidentally on students' responses to quadratic equations. In this article the emphasis is on how students responded to quadratic equations in one variable that were presented in standard format. The investigation sought to uncover how the students solved them, what meaning they gave to the variable, and how they interpreted solutions.

In the chapters on algebra in the last two four-yearly research summary publications of the Mathematics Education Research Group of Australasia (Warren, 2000; Warren & Pierce, 2004), the word "quadratic" was used just twice – when Warren and Pierce (2004) referred to a small study by Gray and Thomas (2001) into the use of a graphics calculator and multiple representations to explore quadratic equations. The Gray and Thomas study involved a sample of 25 students aged 14–15 years, and results indicated that the students did not improve their ability to solve quadratic equations.

The relative lack of research into the learning of quadratic equations has meant that peculiarities associated with variables in quadratic equations and, in particular, with the effects of these on student learning, have remained hidden. Thomas and Tall (2001) distinguished, among other things, between "algebra as generalised arithmetic" and "manipulation algebra", and commented that research indicated that students who completed secondary education were usually able to substitute values correctly for variables in expressions and equations, and were able to interpret variables in symbolic and graphical contexts. However, student thinking in such contexts appeared to be dominated by a need to achieve procedural mastery, and usually there was no guarantee that relational understanding was achieved.

Misconceptions Regarding Variables

There is considerable discussion in the international mathematics education literature on beginning algebra students' abilities to grasp the concept of a variable in the context of linear equations (see, e.g., Booth, 1984; Filloy & Rojano, 1984; Fujii, 2003; MacGregor, 1991; Stacey & MacGregor, 1997, 1999a). The findings of Fujii (2003), who investigated the understanding of the concept of variable of students in the United States and Japan, would appear to have direct relevance to likely students' misconceptions with respect to quadratic equations. Among the tasks that Fujii (2003) asked students to attempt were the two shown in Figure 1:

First Task, from Fujii (2003)

Mary has the following problem to solve:

"Find the value(s) of x for the expression $x + x + x = 12$."

She answered in the following manner (a) 2, 5, 5; (b) 10, 1, 1; (c) 4, 4, 4. Which of her answers is/are correct? Circle the letters that are correct. State the reason for your selection.

Second Task, from Fujii (2003)

Jon has the following problem to solve:

"Find the values of x and y in the expression $x + y = 16$."

He answered in the following manner (a) 6, 10; (b) 9, 7; (c) 8, 8. Which of his answers is/are correct? Circle the letters that are correct. State the reason for your selection.

Figure 1. Two tasks from Fujii (2003).

Fujii reported that within a sample of 6th, 8th, and 9th graders in Georgia, in the United States, only 11.5%, 11.5%, and 5.7% respectively gave correct answers to both of these tasks. With 6th, 7th, 8th, 9th, 10th and 11th graders in Japan, the results were similar, with 0.0%, 1.7%, 9.5%, 10.8%, 18.1%, and 24.8% respectively giving correct answers. According to Fujii (2003), for both countries the percentages of correct answers were "disturbingly low", especially insofar as the percentages did not dramatically increase according to the grades "as we may expect" (p. 53).

The main error made on the first task by students in Fujii's (2003) study, in both the United States and Japan, was that students felt that the x's on the left side of the equation need not necessarily represent the same value. For the second task, many students thought that because x and y were different letters they could never take the same value. If this kind of thinking were to be translated to a task like "Solve the equation $(x - 3)(x - 5) = 0$" then one might expect some students to think that the value of x in $(x - 3)$ would be 3, and, simultaneously, the value of x in $(x - 5)$ would be 5. In this article data relating to that conjecture are presented. It should also be observed that students giving *correct* solutions (i.e., $x = 3, 5$) to $(x - 3)(x - 5) = 0$ might think that the variable x had to take different values in the two sets of parentheses. Thus, it was important to build in an interview component into the research design.

Another aspect of equations and inequalities that often causes difficulty with middle-secondary school students is that of checking solutions of equations. The report of the Mathematical Association (1962), in the United Kingdom, on the teaching of algebra in schools emphasised that part of the problem is that students who generate solutions to an equation often do not understand that the only number(s) which "make the equation true" are the solutions. Students who do not realise this may find it difficult to "check" "solution(s)." The problem is one of semantics, and gets disguised by students' participation in the ritual manipulations surrounding the processes of finding solutions to equations.

Thus, for example, the student who correctly gets the solution $x = 4$ to the equation $\frac{x-1}{3} + 4 = 9 - \frac{2}{5}(3x - 2)$ but then makes the following statement as a check, needs to be brought to realise that from a formal logical point of view the given check is illogical:

$$\frac{4-1}{3} + 4 = 9 - \frac{2(12-2)}{5}$$

$$\therefore 1 + 4 = 9 - 4$$

$$\therefore 5 = 5$$

In the first and second lines the "=" sign is being used as an expression of hope, not as a statement of fact. In any case, "by merely following the same procedure as was used to generate the solution, students are likely to repeat any mistake they made (such as errors in sign, or in learning fractions)" (Mathematical Association, 1962, pp. 23–24).

After being taught to solve linear equations presented in the form $ax + b = c$, Form 2 (i.e., Grade 8) students in Thailand are expected to learn to solve other equations that, although still linear, are not initially presented in standard form. For example, in order to solve equations in the form $ax + b = cx + d$, students are shown how to "get the x terms on the left-hand side, and the other terms on the other side." They are then expected to combine the x terms into one term (strictly speaking by using the distributive law, but often teachers prefer to use expressions like "you can add or subtract 'like' terms, but you cannot add or subtract 'unlike' terms"). Students are then told to "divide both sides by the coefficient of x" (which, mathematically speaking, would be justified by a combination of the multiplication property of equations and the inverse and associative laws for multiplication).

The TIMSS investigation of the late 1990s revealed that Grade 8 students do not solve equations in the form $ax + b = cx + d$ very well, with only 29% of Thai Grade 8 students in Thailand getting the correct solution for $12x - 10 = 6x + 32$, a result that was well below the international average (Mullis et al., 2000). Stacey and MacGregor (1997, 1999a, 1999b), after recognising that equations in the form $ax + b = c$ are more easily solved than equations in the form $ax + b = cx + d$, maintained that student performance on "$ax + b = cx + d$" equations and associated problems provides a litmus test for the extent of a student's algebraic development.

Part of the problem could be that with $12x - 10 = 6x + 32$, for example, some students believe that the x's on the two sides of the equation represent different values (e.g., 5 on the left side and 3 on the right side). Filloy and Rojano (1984) reported that many 12- to 13-year-olds thought that whereas the first x on the left side of the equation $x + x/4 = 6 + x/4$ had to be 6, the x's in the two $x/4$ terms could take any values. Similarly, students thought that with $x + 5 = x + x$, the first x on the left and right sides could take any value, but the second x on the right side had to be 5.

Hoch and Dreyfus (2004) argued that whereas, $30x^2 - 28x + 6$, for example, is equal to $(5x - 3)(6x - 2)$, students without "structure sense" may not realise that the quadratic trinomial and its factorised equivalent are "different interpretations of the same structure" (p. 51). Data related to that phenomenon have been reported by Vaiyavutjamai, Ellerton, and Clements (2005) and Lim (2000), who found that when faced with $(x - 3)(x - 5) = 0$, for example, many secondary-level mathematics students, some university mathematics students, and even some teachers choose to write the left side as $x^2 - 8x + 15$, then re-factorise before applying the null factor law.

There has been much research on the extent to which teachers, over the world, use potentially harmful expressions such as "take the term over to the other side (of the equation) and change its sign," "multiplication becomes division when it goes to the other side," and "cross-multiplication gives …" (Bodin & Capponi, 1996; Mathematical Association, 1962; Vaiyavutjamai, 2004a, 2004b). Although details are not given here of the first author's analyses of classroom observations of lessons on equations in the study (for details of the analyses of data on the language discourses in Grade 9 algebra classes in Thailand see Vaiyavutjamai, 2004a), interviews with selected students are reported. These revealed that many students could not explain what such expressions meant. They simply did what they thought their teachers told them to do (which, often, was not the same as what their teachers did tell them to do).

Effects of Traditional Teaching on Student Understanding

There is evidence that traditional, elicitation/exposition-type teaching in mathematics classrooms isolates skills and fails to draw attention to connections (see, e.g., Farrell & Farmer, 1988; Good, Grouws, & Ebemeier, 1983; Hiebert, 2003; Hiebert & Carpenter, 1992; Phomjwi et al., 1999; Skemp, 1976). As Hatano (1988) stated, over-emphasising skills is likely to result "in a sacrifice of understanding and of the construction of conceptual knowledge", to the extent that it is difficult for understanding to be achieved at some later time because "it is hard to unpack a merged specific rule to find the meaning of any given step" (p. 64). Resnick and Ford (1981) concluded that research indicated that presenting mathematical concepts through stand-alone examples and repetitious practice does not foster understanding.

Quadratic Equations in the Grade 9 Mathematics Curriculum for Thailand

Chaysuwan (1996) reported that immediately after 661 Grade 9 students in secondary schools in Bangkok had participated in lessons on quadratic equations, 70% of their responses to standard quadratic equations tasks were incorrect. Many students did not seem to have the prerequisite algebraic skills needed to cope with quadratic equations. That raised the question whether most Grade 9 students should be expected to study quadratic equations. Until recently, it was a compulsory topic within M 012, a mathematics unit taken by most secondary students in Thailand (Institute for the Promotion of Teaching Science and Technology, 1998a, 1998b).

Relational and Instrumental Understanding

The authors decided to adopt, and adapt for the purposes of the study, Skemp's (1976) distinction between "instrumental" (or procedural) understanding, and "relational" understanding. According to Skemp (1976), instrumental understanding involves "rules without reason" (p. 20), and relational understanding "knowing both what to do and why" (p. 20). Skemp maintained that with relational understanding:

1. The means become independent of particular ends to be reached thereby;
2. Building up a schema within a given area of knowledge becomes an intrinsically satisfying goal in itself.
3. The more complete a pupil's schema, the greater his feeling of confidence in his own ability to find new ways of "getting there" without outside help.
4. A schema is never complete. As our schemas change, so our awareness of possibilities is thereby enlarged. Thus the process often becomes self-continuing, and (by virtue of 3) self-rewarding. (p. 26)

The authors were aware of more recent approaches to defining and investigating mathematical understanding in school children (e.g., Borgen & Manu, 2002; Pirie & Kieren, 1994; Schoenfeld, 1992), but decided that Skemp's (1976) seminal instrumental/relational distinction would be sufficient for the purposes of the study.

The Newman Interview Technique, and Additional Questions Used by Booth

Given that the research study would aim at assessing student understanding of quadratic equations it was essential that a valid and reliable procedure be devised for assessing understanding in that context. The authors decided that the extent of student understanding could best be evaluated by comparing what students wrote, when attempting to solve quadratic equations on pencil-and-paper tests, with what they said about the same equations in one-to-one interviews. A valid and reliable pencil-and-paper instrument for assessing students' knowledge and skills with respect to quadratic equations was therefore needed, as was an associated pre- and post-teaching interview strategy. For the latter, the authors decided to develop an interview protocol that combined interview approaches developed by Newman (1983) and Booth (1984).

Newman's (1983) diagnostic interview technique. With the Newman approach the interviewer makes five key requests of interviewees:

1. Please read the question to me.
2. Tell me, what does the question mean?
3. What will you need to do to answer this answer?
4. Now answer it, and tell me what you are thinking as you do it.
5. Now, write down your actual answer.

These requests are associated with error classifications that Newman called Reading, Comprehension, Transformation, Process Skills, and Encoding.

Booth's extended set of categories. In her study into the teaching and learning of algebra in lower secondary classes in the United Kingdom in the early 1980s, Booth (1984) used a semi-structured interview schedule that extended the Newman approach to interviewing. Booth's schedule included requests equivalent to all five Newman requests, and in addition she made requests aimed at finding out whether interviewees:

- knew what their answers meant in relation to the original question.
- could check their answers.
- would stick to their answers if challenged with other possibilities.
- could identify other questions similar to a question they had just answered.
- could generalise solutions to solve more complex, but nonetheless similar, tasks.

The authors thought that Newman requests could be used to investigate interviewees' instrumental understanding of quadratic equations, and Booth's additional requests could be used to explore the extent of their relational understanding.

Design of the Study

Aim

The aim of the study was to investigate how traditional lessons on quadratic equations, in which the emphasis was on teaching Grade 9 students to solve quadratic equations by factorisation (and application of the null factor law), by "completing the square", and by the quadratic formula, influenced student development of knowledge, skills, concepts, and understanding with respect to quadratic equations.

The Samples of Students and Teachers

Altogether, 231 students in six Grade 9 classes in two government secondary schools in the Chiang Mai Province of Thailand, and four teachers, participated in the study. During the 2002/2003 school year most, but not all, Grade 9 students at one of the participating schools ("School *X*") took M 012, and were therefore expected to study quadratic equations. At the other participating school ("School *Y*") all Grade 9 students took M 012.

The six participating classes in the study were taking M 012. There were 109 students in three participating classes at School X and 122 in three classes at School Y. Analyses of national data reported by the Office of the Basic Education Commission (2003) suggested that if all Grade 9 students in Thailand in 2002 were regarded as a population, the sample in this study would be of "average mathematical ability".

At the beginning of their Grade 8 year, students at School *X* and School *Y* who participated in the study were "streamed" into high-, middle-, or low-stream classes on the basis of overall academic performance in Grade 7. These students remained in the same classes when they progressed to Grade 9. At School *X*, "Teacher *A*", taught three classes (the high-stream Grade 9/1, the medium-stream Grade 9/2, and the low-stream Grade 9/3), and at School Y, "Teacher *B*", "Teacher *C*" and "Teacher *D*" taught the high-stream Grade 9/4, the medium-stream Grade 9/5, and the low-stream Grade 9/6, respectively. Each teacher was experienced and well-qualified.

The Lessons

At the time this study was conducted (during the 2002/2003 school year) the Grade 9 national mathematics curriculum for Thailand stipulated that M 012 students should receive 13 lessons on linear equations and inequalities and 11 lessons on quadratic equations (Institute for the Promotion of Teaching Science and Technology, 1998b). Teachers who participated in the study permitted the first author to observe and audiotape a total of 18 50-minute lessons – one lesson on linear equations, one on linear inequalities and one on quadratic equations, for each of the six classes.

The teachers agreed to teach "as they normally would" during lessons observed, and to follow, as they usually did, approaches recommended in the "official" Grade 9 student textbook and the associated teachers' guide (Institute for the Promotion of Teaching Science and Technology, 1998a, 1998b). The teachers gave permission for transcripts to be produced and reported. During the 11 lessons on quadratic equations the teachers taught a standard secondary school mathematics sequence of lessons on quadratic equations encompassing factorisation and the null factor law, completing the square, and the use of the quadratic formula. Analysis by the first author of transcripts of the six lessons observed on quadratic equations revealed that these lessons were all taught in a traditional way (Vaiyavutjamai, 2004a).

Sources of Data

Student interview data. The main data source for this report was a set of 36 transcripts of interviews that the first author conducted with 18 student interviewees (two interviews with each interviewee). The selection of interviewees was based on relative within-class student performance on two pencil-and-paper tests that had been administered to all six classes in August 2002 – a *Language of Equations and Inequalities Test* and a *Linear Equations and Inequalities Test* (see Vaiyavutjamai, 2004a, for further details of these tests). Each test was marked out of a possible 27, and so any overall mark within the range 0 to 54 was possible. From each of the six classes the highest performer, a medium performer, and the lowest performer (on the basis of total score on the two tests) were selected to be part of the interview sample.

Eighteen of the interviews took place at the pre-teaching stage in December 2002, immediately before the lessons on quadratic equations. The other 18 interviews took place in January 2003, soon after the six classes had participated in 11 lessons on quadratic equations. About 360 pages of interview transcripts, in Thai, were generated from the audiotapes, and these were translated into English by the first author.

Student performance data. In addition to interview data, pre- and post-teaching performance data generated by student responses to the *Quadratic Equations Test* were also analysed. This test, which comprised 18 standard questions on quadratic equations, had a Cronbach alpha reliability of 0.90, and the four teachers agreed that after the lessons on quadratic equations their students should have been able to answer each question correctly (Vaiyavutjamai, 2004a). The 18 questions are shown in Figure 2 (which appears later in this article). For the purposes of performance analysis, student responses to a question were scored 1 for a response deemed to be correct, or 0 for a response deemed to be incorrect, and 0, also, if no response was given. At both the pre- and post-teaching stages, all 231 participating students were given ample time to complete all 18 questions.

The Four Interview Questions

In the interviews, the interviewer (the first author) explored the knowledge and understanding of interviewees with respect to four so-called "interview questions".

- $(x - 3)(x - 5) = 0$
- $x^2 - x = 12$
- $x^2 = 9$
- $2x^2 = 10x$

These "interview questions" differed in subtle but mathematically important ways. During interviews, interviewees were invited to solve the equations, showing all their working, and to respond to questions asked by the interviewer. Transcripts of excerpts from interviews in relation to the equations $(x - 3)(x - 5) = 0$ and $x^2 - x = 12$ are reproduced later in the article.

Most of the participating Grade 9 students had their first-ever class lessons on quadratic equations *after* they had attempted the (pre-teaching) *Quadratic Equations Test*, and after the pre-teaching interviews for quadratic equations had been conducted. It was expected that at the pre-teaching stage only a small proportion of responses given by the 231 students to the equations would be correct.

It was a matter of interest whether students confronted with a quadratic equation at the post-teaching stage would know, before they attempted to solve it, that the equation might have two solutions, or one solution, or perhaps no solution. Would students tend to have a better appreciation of the mathematical principles involved, for example, of how the null factor law, "if $ab = 0$, then $a = 0$, or $b = 0$, or both a and b are zero," might be applied? Would they realise that in equations like $(x - 3)(x - 5) = 0$, $x^2 - x = 12$, and $2x^2 = 10x$, the two x's represent the same variable? Would they know that a solution to a quadratic equation was a number which, when substituted in the equation, generated a true statement? These were among the questions probed in the interviews associated with the *Quadratic Equations Test*.

During interviews the first author concentrated on aspects of the problems that were not necessarily revealed by an examination of students' pencil-and-paper responses. For example, with $(x - 3)(x - 5) = 0$, did students think that the x in $(x - 3)$ stood for a different variable from the x in $(x - 5)$? Lim (2000) reported that many students attempted a similar question by expanding the two brackets, refactorising, and then equating each factor to zero. It was expected that the interviews would reveal whether a similar tendency existed within the six classes involved in the present study. With each of the equations "$x^2 - x = 12$" and "$2x^2 = 10x$" it was expected that the interviews would reveal that, even at the post-teaching stage, some students would think that the x in the x^2 term and in the other x term stood for different variables. It was also expected that some students would divide both sides of $2x^2 = 10x$ by x. And, with the "$x^2 = 9$" question it was expected that the most common error would be the single solution ($x = 3$) answer.

Developing a Rubric for Measuring Understanding

The authors believed that analysis of interview data generated by student responses to the interview questions had the potential to make clear the extent to which student understanding, as distinct from unconnected knowledge, was present in a student's schema. The authors set out, therefore, to develop a rubric that would enable reliable distinctions to be made between aspects of students' understanding.

A Rubric for Assessing Understanding

The rubric that was developed and used in this study for allocating "understanding scores" to students is now elaborated. An interviewee's set of responses to questions asked in relation to an "interview" question was assessed on a 0, 1, 2, 3, or 4 basis, where the score was allocated according to the following criteria:

Score of 0: The interviewee does not *comprehend* the meaning of the question. This could be due to an inability to *decode* one or more symbols used in the question, or to grasp the overall meaning of the question as it is presented.

Score of 1: The interviewee has some idea of the meaning of the question, and is able to *transform* the task by choosing appropriate mathematical procedures for answering it. However, he/she is not able to apply those procedures correctly.

Score of 2: The interviewee not only identifies an appropriate sequence of procedures for the question, but also applies the procedures accurately, or reasonably accurately, when carrying out associated *process skills*. He/she is able to encode his/her answer in an appropriate way, but cannot explain the meaning of the answer obtained in relation to the original question. Also, he/she is not aware of mathematical principles that underlie the algorithms used when attempting to solve the equation.

Score of 3: The interviewee can apply an appropriate procedure, or set of procedures, accurately, has some awareness of the mathematical principles that underlie the algorithms, and has some idea of how the answer relates to the original question. However, he/she is either not able to check whether his/her answer is correct or, if he/she can check the answer, cannot interpret the check.

Score of 4: The interviewee can apply an appropriate procedure accurately, has some awareness of the mathematical principles that underlie the algorithms, and knows how the answer obtained relates to the original question. He/she can check the answer and can link the various representations of the answer(s) – written, verbal, and symbols – to each other and to the original question.

Specific criteria for the four interview questions from the Quadratic Equations Test are shown in Table 1.

Table 1
Criteria for Assessing "Understanding Scores" for the Four Interview Questions

	Question: Solve the Equation …			
Score	$(x-3)(x-5)=0$	$x^2-x=12$	$x^2=9$	$2x^2=10x$
0	Does not know how many solutions to expect, and does not comprehend the instruction "solve the equation".	Does not know how many solutions to expect, and does not comprehend the instruction "solve the equation".	Does not know how many solutions to expect, and does not comprehend the instruction "solve the equation".	Does not know how many solutions to expect, and does not comprehend the instruction "solve the equation".
1	Knows there could be up to two solutions. Applies an appropriate procedure (e.g., equate each bracket to 0). But does not know *why* the procedure works, and cannot carry it out accurately.	Knows there could be up to two solutions. Applies an appropriate procedure (e.g., writes $x^2-x-12=0$, factorises, and then equates each bracket to 0). Does not know why the procedure works, and does not apply it accurately.	Knows there could be up to two solutions, Applies an appropriate procedure (e.g., writes $x^2-9=0$, factorises and equates each factor to 0, or writes $x=\pm\sqrt{9}$). Cannot explain why the procedure works and does not apply it accurately.	Knows there could be up to two solutions. Applies an appropriate procedure (e.g., writes $2x^2-10x=0$, factorises and equates each factor to 0). Cannot explain why the procedure works and does not apply it accurately.
2	Accurately carries out an appropriate procedure (minor careless error permitted). Cannot state the null factor law or relate answers to the original equation.	Accurately carries out an appropriate procedure (minor careless error permitted). Cannot relate answers to the original equation.	Selects, and accurately carries out, an appropriate procedure (minor careless error permitted). Cannot state a mathematical reason for using the procedure, or relate answers to the original equation.	Selects and accurately carries out, an appropriate procedure (minor careless error permitted). Cannot state a mathematical reason for using the procedure, or relate answers to the original equation.
3	Accurately carries out an appropriate procedure (minor careless error permitted). Can relate answers to the original equation, or, if asked to do so, can check solutions meaningfully, but cannot do both.	Accurately carries out an appropriate procedure (minor careless error permitted). Can relate answers to original equation, or if asked to do so, can check solutions meaningfully, but cannot do both.	Accurately carries out an appropriate procedure (minor careless error permitted). Can relate answers to original equation, or, if asked to do so, can check solutions meaningfully, but cannot do both.	Accurately carries out an appropriate procedure (minor careless error permitted). Can relate solutions to the original equation, or, if asked to do so, can check solutions meaningfully, but cannot do both.

4	Accurately carries out an appropriate procedure. Knows what solutions mean in relation to the original equation. If asked to do so, can check solutions meaningfully.	Accurately carries out an appropriate procedure. Knows what solutions mean in relation to original equation. If asked to do so, can check solutions meaningfully.	Accurately carries out an appropriate procedure. Knows what solutions mean in relation to original equation. If asked to do so, can check solutions meaningfully.	Accurately carries out an appropriate procedure. Knows what solutions mean in relation to original equation. If asked to do so, can check solutions meaningfully.

Inter-rater reliability in assessing understanding. In order to check inter-rater reliability, the extent to which two qualified persons independently allocated the same understanding score to the same interview transcript was ascertained. Since each of the 18 students answered the four interview questions in both the pre- and post-teaching interviews, altogether 144 interview units were assessed.

The first author assessed the 144 Thai-language units of transcript, and the second author independently assessed the corresponding 144 English-language translations of the transcripts. The Pearson-product moment correlation coefficient for the 144 pairs of scores for the interview units was 0.97. The two assessors gave the same score for 130 of the units, and differed by 1 on the other 14 units. Whenever different scores were given, a consensus score was reached, through discussion. Most differences in initial scores arose as a result of linguistic factors. Post-grading discussion led to refinement of criteria, and the final criteria were those shown in Table 1.

Results

Performance Trends

Figure 2 shows the percentages of the 231 students who gave correct answers, at the pre- and post-teaching stages, to each of the 18 questions on the *Quadratic Equations Test*. In every case, more students gave a correct answer after the lessons than before. It can be seen that at the pre-teaching stage about half of the students correctly solved the equations for Questions 7 and 14, probably because each of those equations had one solution only and that solution could be obtained fairly easily by substitution. At the pre-teaching stage, relatively few students correctly solved any of the other equations.

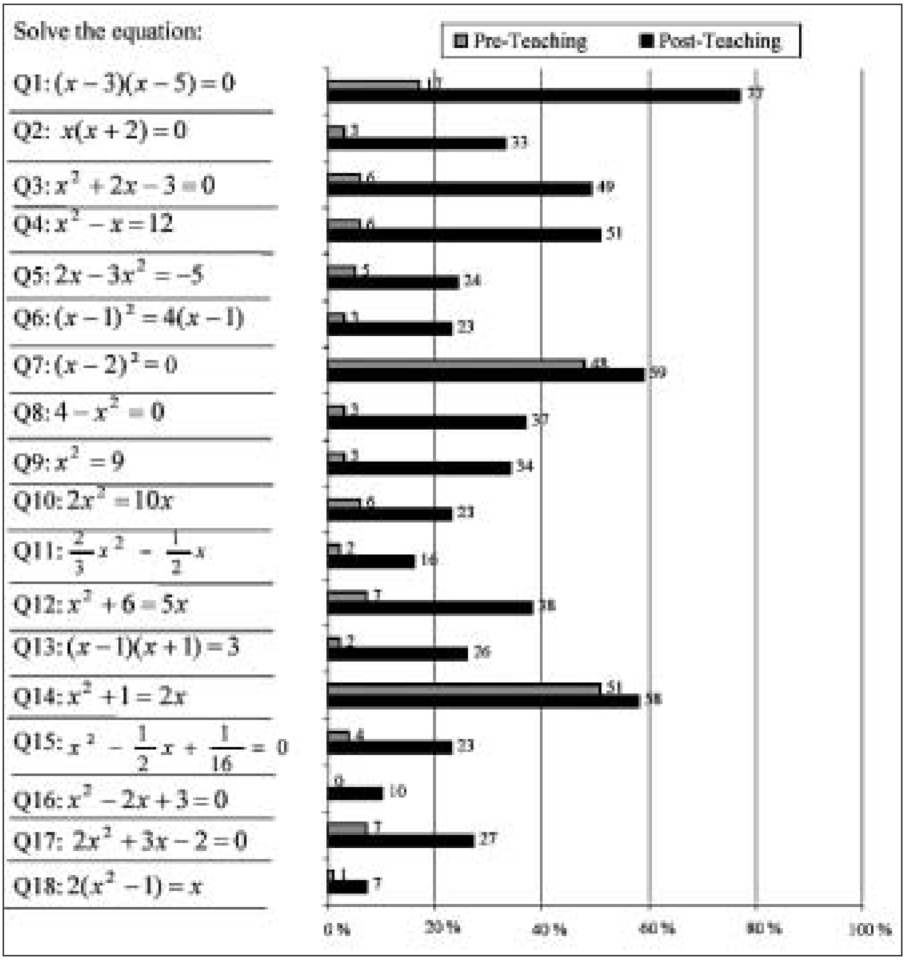

Figure 2. Pre- and post-teaching percentages correct for the 18 equations comprising the *Quadratic Equations Test* (231 students).

Table 2 summarises pre- and post-teaching test means and standard deviations for the six classes, and shows Cohen's *d* effect sizes (Cohen, 1992) for pre-/post-teaching comparisons of performance.

Table 2
Performance Mean Scores, Standard Deviations, Effect Sizes (Quadratic Equations Test)

Class	Mean Pre-Test Score / 18	Mean Post-Test / 18	Standard Deviation Pre-Test	Standard Deviation Post-Test	Pre-Post Effect Size (Cohen's *d*)
School X					
Grade 9/1	1.75	9.49	2.17	4.82	2.07
Grade 9/2	1.08	4.90	1.00	4.02	1.30
Grade 9/3	0.73	2.67	0.91	3.08	0.85
School Y					
Grade 9/4	3.70	7.91	3.29	4.58	1.05
Grade 9/5	2.18	4.45	2.89	4.11	0.64
Grade 9/6	0.55	6.66	0.83	4.89	1.74
Total	1.75	6.17	2.41	4.83	1.16

The high effect sizes might suggest that the lessons were "effective". We would argue, however, that such a conclusion is not warranted because it ignores the fact that for five of the six classes the post-teaching means on the *Quadratic Equations Test* were less than 9.0 (i.e., 50% of the maximum possible score on the test). The overall post-teaching mean for the 231 students was 6.17 (out of a possible 18).

After the 11 lessons on quadratic equations most participating students found it difficult to solve most of the questions on the *Quadratic Equations Test*. The effect was not uniform across the classes, however. For example, although most Grade 9/1 students benefited from Teacher A's lessons on quadratic equations, his lessons were not nearly so successful with Grade 9/2 and Grade 9/3 students. By comparison with the two medium-stream classes, and with the other low-stream class, students in the low-stream Grade 9/6 did well at the post-teaching stage.

Summary of Performance Data for the 18 Interviewees

Table 3 shows pre- and post-teaching scores of the 18 interviewees on the *Quadratic Equations Test*. The selection of the interviewees was based on the total of the pre-teaching scores on the *Language of Equations and Inequalities Test* and the *Linear Equations and Inequalities Test*. With respect to Table 3 it should be noted that Students 1, 4, 7, 10, 13, and 16 were high-performing students *relative to their classmates*, and Students 2, 5, 8, 11, 14, and 17 were medium-performing students. The other interviewees were low-performing students. The percentile ranks for the 18 interviewees (within the sample of 231 students), based on the sum of their pre-teaching scores on *Test 1* and *Test 2*, are shown in the second column.

Although the 18 interviewees were not selected randomly from the total sample of 231 participating students, it can be seen from the percentile ranks in the third column of Table 3 that the sample covered the full spectrum of performance. The mean percentile rank of the 18 interviewees was 48.2.

Table 3
Performances of 18 Interviewees on 3 Tests (Language of Equations and Inequalities, Linear Equations and Inequalities, and Quadratic Equations)

School, and Grade	Interviewee	Percentile Rank of Interviewee (Within Total Sample of 231 Students)	(Pre-T) Language of Equations and Inequalities Test Score / 27	(Pre-T) Linear Equations and Inequalities Test Score / 27	Pre-Teaching Quadratic Equations Test Score / 18	Post-Teaching Quadratic Equations Test Score / 18
School X	Student 1	99	23	20	1	12
Grade 9/1	Student 2	75	18	4	1	6
(high-stream)	Student 3	24	7	2	0	3
Grade 9/2	Student 4	79	18	6	1	11
(medium-	Student 5	48	11	3	1	3
stream)	Student 6	00	0	0	0	6
Grade 9/3	Student 7	57	15	2	0	7
(low-stream)	Student 8	19	6	2	1	3
	Student 9	04	3	2	0	1
School Y	Student 10	100	24	25	15	17
Grade 9/4	Student 11	81	16	10	5	11
(high-stream)	Student 12	19	8	0	3	3
Grade 9/5	Student 13	95	21	14	11	12
(medium)	Student 14	41	10	2	2	3
-stream	Student 15	01	0	1	0	0
Grade 9/6	Student 16	88	17	12	1	14
(low-stream)	Student 17	28	6	4	2	7
	Student 18	01	3	0	1	2

Entries in Table 4 reveal that 11 of the 18 interviewees obtained a post-teaching *mean understanding* score per question score less than 2.0. That is to say, after the lessons they did not understand, relationally, much of what they had been taught.

From Table 4 it can be seen that the mean understanding gains *per person per question* (based on the sums of the pre- and post-teaching assessments for the four interview questions) can be compared for the high-, medium- and low-performing interviewees. Mean "per person per question" understanding gains for the three groups were 1.54, 0.58, and 0.42, respectively. Thus, high performers started off understanding more than other students and after the lessons their understanding had increased relative to the other students.

Student 10 was clearly an outlier. Even at the pre-teaching stage she demonstrated a strong understanding of the mathematics associated with each of the four interview tasks. If her data were to be removed from the analysis of pre-teaching scores then the mean would reduce from 3.39 to 2.65 and the standard deviation form 3.78 to 2.15, both very substantial effects. It was decided, however, that since Student 10 was included in the interview sample on the basis of a selection procedure that specifically included the highest performing student in each of the six classes, it would be inappropriate to exclude her interview data from the analyses.

Table 4
Summary of Understanding Scores for the 18 Interviewees

Interviewee Student Number	Pre-Teaching Score on Question Concerned with …				Sum of Pre-T Scores	Post-Teaching Score on Question Concerned with …				Sum of Post-T Scores
	$(x-3)(x-5)=0$	$x^2-x=12$	$x^2=9$	$2x^2=10x$		$(x-3)(x-5)=0$	$x^2-x=12$	$x^2=9$	$2x^2=10x$	
(School X, Grade 9/1)										
1	2	2	1	1	6	3	3	3	3	12
2	1	1	1	1	4	3	3	3	1	10
3	0	1	1	1	3	2	1	2	2	7
(School X, Grade 9/2)										
4	0	0	1	0	1	4	3	4	4	15
5	2	1	1	1	5	1	2	4	0	7
6	0	0	0	0	0	0	1	0	0	1
(School X, Grade 9/3)										
7	0	0	1	0	1	2	2	4	2	10
8	0	1	1	0	2	2	2	2	1	7
9	0	0	0	0	0	0	0	1	0	1
(School Y, Grade 9/4)										
10	4	4	4	4	16	4	4	4	4	16
11	1	1	1	1	4	2	1	1	1	5
12	0	0	1	0	1	2	1	1	1	5
(School Y, Grade 9/5)										
13	2	2	1	0	5	2	2	2	3	9
14	2	0	1	0	3	1	1	1	1	4
15	0	0	0	0	0	0	0	0	0	0
(School Y, Grade 9/6)										
16	2	2	1	0	5	2	2	4	1	9
17	2	2	1	0	5	2	1	1	0	4
18	0	0	0	0	0	0	0	0	0	0
Total	18	17	17	9	61	32	29	37	24	122
Mean	1.00	0.94	0.94	0.50	3.39	1.78	1.61	2.06	1.33	6.78
SD	1.19	1.11	0.87	0.99	3.78	1.26	1.14	1.51	1.37	4.80

English-Language Translations of Excerpts of Thai Interview Transcripts

Excerpts from English-language composite transcripts of two interviewees, in relation to two of the four interview questions, are reproduced in Table 5. Excerpts appear in pairs, with a pre-teaching interview excerpt being visually accompanied by the corresponding post-teaching interview excerpt. Both interview excerpts chosen could be associated with "understanding scores" that approximated the average scores for the 18 interviewees before the lessons and after the lessons were given. By reading the transcripts the reader should get a "feel" for how much "typical" students knew and understood about the question before and after the lessons. The symbol [...] is used to indicate where sections of an excerpt have been omitted.

Excerpts of Interviews with Student 11 in Relation to "(x – 3)(x – 5) = 0"

Student 11, who was involved in the interview that generated the transcript in Table 5, was a medium-performing student in the high-stream Grade 9/4. He gave the same correct answer, "$x = 3, 5$", to $(x-3)(x-5) = 0$ on both the pre- and post-teaching administrations of the *Quadratic Equations Test*. At the pre-and post-teaching stages, he scored 5 and 11, respectively, out of a possible 18 on the *Quadratic Equations Test*, thereby achieving a gain of 6 on the *Test*.

Although Student 11 managed to obtain correct solutions to $(x-3)(x-5) = 0$ at both stages he could not describe a correct method for checking his answers, and he thought that the two x's in the equation represented different numbers. At the post-teaching stage he still used a substitution method and still thought that the two x's represented different numbers. However, at that stage he knew how to check his answers and how his solutions related to the original equation.

Table 5
English-Language Translations of Excerpts of Transcripts of Pre- and Post-Teaching Interviews with Student 11 for the Question: "Solve the equation (x – 3)(x – 5) = 0"

Pre-Teaching Interview		Post-Teaching Interview	
Interviewer:	Are those x's the same variable?	***Interviewer:***	Are those x's the same variable?
Student:	No, they aren't.	*Student:*	No. They are different.
Interviewer:	They aren't the same variable? So, what method will you need to use to solve the problem?	*Interviewer:*	So, what do you need to do to solve this equation?
		Student:	Use the substitution method.
Student:	The substitution method. It is an easy method.		

Interviewer: What number will you substitute? *Student:* The first *x* is three and the second *x* is five. *Interviewer:* Why do you use that method? *Student:* I substituted numbers and got three minus three equals zero, five minus five equals zero. Zero multiplied by zero equals zero. It is a true sentence. *Interviewer:* What are the answers to this question? *Student:* Three and five. *Interviewer:* All right! Please show me your working. *Student:* [*Wrote, on a piece of paper:* (3 – 3)(5 – 5) = 0] *Interviewer:* What are your answers? *Student:* Three and five.	***Interviewer:*** You will use a substitution method. What are the numbers? *Student:* Three and five. *Interviewer:* Please show me your working. *Student:* [*Wrote, on a piece of paper:* 3 – 3 = 0, 5 – 5 = 0, 0 x 0 = 0] Three minus three equals zero. Five minus five equals zero. Zero multiplied by zero equals zero. It is a true sentence. *Interviewer:* What is your answer? *Student:* Three and five.
Interviewer: Can you check your answers? *Student:* It equals zero. It is a true sentence. *Interviewer:* How did you do it? *Student:* I substituted the answers for x here. *Interviewer:* Where? *Student:* I substituted them for x in the question. [*Pointed to the line he had written.*]	***Interviewer:*** Can you check your answers? *Student:* Yes. I substitute the first answer, three, for x. *Interviewer:* All right! Please show me your working. *Student:* [*Wrote, on a piece of paper:* (3 – 3)(3 – 5) = 0] Three minus three equals zero. Three minus five equals negative two. Zero multiplied by negative two equals zero. *Interviewer:* Have you finished your check? *Student:* The other one, five. [*Wrote the next line:* (5 – 3)(5 – 5) = 0] Five minus three equals two. Five minus five equals zero. Two multiplied by zero equals zero. *Interviewer:* Why did you use this method to solve the equation? *Student:* It is an easy method.
Interviewer: Please tell me your answers again. *Student:* Three and five.	***Interviewer:*** Please tell me your answers again. *Student:* Three and five

The authors agreed that Student 11 should receive a pre-teaching understanding score of 1, even though he got both correct solutions. Student 11 substituted 3 for one of the x's and 5 for the other, and thought that the variable had different values within the two sets of parentheses. Although he obtained correct solutions the procedure he used was not "appropriate", and he could not meaningfully relate solutions to the original equation. Both authors therefore gave his response a score of 1, not 2. For the post-teaching interview, both authors wondered whether Student 11 should receive a score of 1 or 2. Although he still said the two x's stood for different variables, when checking his solutions he related solutions to the original equation. Both authors tentatively allocated a score of 2, and after subsequent discussion agreed that 2 should remain as the score.

For the 18 interviewees, the means of the pre- and post-teaching scores for the "$(x - 3)(x - 5) = 0$" task were 1.00 and 1.78, respectively.

Excerpts of Interviews with Student 8 in Relation to $x^2 - x = 12$

Student 8, who was involved in the interviews that generated the excerpts shown in Table 6, was a medium-performing student in the low-stream Grade 9/3. On the pre-and post-teaching administrations of the *Quadratic Equations Test* he scored 1 and 3 respectively, out of a possible 18. On the pre-teaching administration he gave an incorrect answer ("$7^2 - 2$") to the "$x^2 - x = 12$" question, but his answer on the post-teaching administration, "$x = 4, -3$", was assessed as correct.

During the pre-teaching interview Student 8 tried to use a substitution method to solve $x^2 - x = 12$. He got one correct solution, and could check that answer. During the post-teaching interview, he correctly solved the equation by factorisation. However, although he used the null factor law he thought the x's stood for different variables and did not know how to check his answer.

The authors agreed that the interview data indicated that Student 8 should receive a pre-teaching understanding score of 1, and a post-teaching score of 2 for this question. For the 18 interviewees, the mean of the pre-teaching understanding scores awarded for the "$x^2 - x = 12$" question was 0.94, and the mean of the post-teaching scores was 1.61.

Table 6
English-Language Translations of Excerpts from Pre- and Post-Teaching Interviews with Student 8 in Relation to the Question: "Solve the equation $x^2 - x = 12$"

Pre-Teaching Interview		Post-Teaching Interview	
Interviewer:	Are both *x*'s the same variable?	***Interviewer:***	Are the *x*'s in the first and second terms the same variable?
Student:	Yes.	*Student:*	No, they aren't.
		Interviewer:	Which *x* do you need to find?
		Student:	That *x* [*Pointed to the second term of the question.*]
		Interviewer:	You will find the second term. Will you find the first term, too?
		Student:	The first term is a square number. I can factorise it.
Interviewer:	What do you need to do to solve this question?	***Interviewer:***	What method will you need to use to solve this question?
Student:	[*Quiet.*]	*Student:*	Change … change twelve to the other side so that one side of the equation equals zero.
Interviewer:	What method do you need to use?	*Interviewer:*	Why do you do that?
Student:	Find a number that equals twelve.	*Student:*	I want to find the value of *x*.
Interviewer:	How do you do that?	*Interviewer:*	Can you find the value of *x* without making one side of the equation equal to zero?
Student:	Find two numbers which, when you subtract, equals twelve.	*Student:*	No.
Interviewer:	Do it, and show me your answer.	*Interviewer:*	What do you do next?
Student:	[*He thought for about three minutes*]	*Student:*	I factorise into two brackets.
Interviewer:	What did you get?	*Interviewer:*	That's your method. Please show me your working.
Student:	I got four squared minus four. [*Wrote:* $= 4^2 - 4$]	*Student:*	[*Wrote, on a piece of paper:* $x^2 - x - 12 = 0$ $(x - 4)(x + 3) = 0$ The solutions of the equation are 4, – 3]
Interviewer:	What is your answer?	[…]	
Student:	Four.	*Interviewer:*	How did you get the answers?
		Student:	Like … may I write it down? [*Wrote the following:* $(x - 4)(x + 3) = 0$ $x - 4 = 0$ or $x + 3 = 0$ $x = 4, x = -3$.]
		Interviewer:	Why does *x* minus four equal zero or *x* plus three equal zero?
		Student:	The two brackets are multiplied.
		Interviewer:	Why could you separate the question into two equations?

		Student:	I found the product of the two numbers, which equals twelve.
		Interviewer:	Can you explain to me why you could get two equations?
		Student:	No, I can't.
		Interviewer:	You just remember how to do it?
		Student:	Yes.
Interviewer:	Can you check your answer?	***Interviewer:***	Can you check your answer?
Student:	Yes, I can.	*Student:*	[*He thought for about half a minute.*] No, I can't.
Interviewer:	How do you do it?		
Student:	Four squared equals four multiplied by four, which equals sixteen. Sixteen minus four equals twelve.		

Relationships Between Performance and Understanding

Entries in Table 7 summarise the 18 interviewees' pre- and post-teaching *Quadratic Equations Test* performances (out of 18, in each case) and their understanding scores (out of 16) with respect to the four interview questions.

In particular, Table 7 shows:

- The interviewees' performance scores on the pre-teaching administration of the *Quadratic Equations Test* (out of a possible 18);
- The interviewees' performance scores on the post-teaching administration of the *Quadratic Equations Test* (out of a possible 18);
- The sum of the understanding scores of each interviewee for the four interview questions at the pre-teaching stage (out of a possible 16);
- The sum of the understanding scores of each interviewee for the four interview questions at the post-teaching stage (out of a possible 16).

Table 8 shows Pearson product-moment correlation coefficients between four variables (pre-teaching performance score, pre-teaching understanding score, post-teaching performance score, and post-teaching understanding score) defined in the above dot points. Only one correlation coefficient (between X_1 and Y_2) is less than 0.5.

Table 7
Performances of 18 Interviewees on the Quadratic Equations Test, and Understanding Scores for the Interview Questions (Pre- and Post-Teaching)

Interviewee Student Number	Pre- and Post-Teaching Performance Scores on the *Quadratic Equations Test* (Max. Possible = 18)		Total of Pre- and Post-Teaching Understanding Scores on the Four Interview Questions (Max. Possible = 16)	
	Pre-T	Post-T	Pre-T	Post-T
(School X, Grade 9/1)				
1	1	12	6	12
2	1	6	4	10
3	0	3	3	7
(School X, Grade 9/2)				
4	1	11	1	15
5	1	3	5	7
6	0	6	0	1
(School X, Grade 9/3)				
7	0	7	1	10
8	1	3	2	7
9	0	1	0	1
(School Y, Grade 9/4)				
10	15	17	16	16
11	5	11	4	5
12	3	3	1	5
(School Y, Grade 9/5)				
13	11	12	5	9
14	2	3	3	4
15	0	0	0	0
(School Y, Grade 9/6)				
16	1	14	5	9
17	2	7	5	4
18	1	2	0	0
Total	45	121	61	122
Mean	2.50	6.72	3.39	6.78
S. D.	4.08	4.98	3.78	4.80

Table 8
Pearson Product-Moment Correlation Coefficients Between Two Performance and Two Understanding Variables for 18 Interviewees (at the Pre- and Post-Teaching Stages)

Variable	X_1	X_2	Y_1	Y_2
X_1: Pre-T Scores on Quadratic Equations Test / 18	1.00			
X_2: Sum of Pre-T Scores on 4 Interview Questions / 16	0.78	1.00		
Y_1: Post-T Scores on Quadratic Equations Test / 18	0.64	0.71	1.00	
Y_2: Sum of Post-T Scores on 4 Interview Questions / 16	0.45	0.64	0.75	1.00

It is not being argued, here, that entries in Table 7 and Table 8 support the proposition that "understanding" *caused* high performance on the *Quadratic Equations Test*, or "high performance" caused "understanding". Rather, the evidence supports the view that understanding and performance went "hand in glove".

A Fundamental Misconception

At the post-teaching stage, two of the 18 interviewees (Students 4 and 10) convinced the interviewer, by the quality of their reasoning in interviews, that they had achieved a strong relational understanding of quadratic equations. They gave correct solutions to the four "interview questions" and generally understood how the null factor law could be applied in the context of quadratic equations. They did not think that the two x's in $(x – 3)(x – 5) = 0$ represented different variables. When checking solutions, they did not substitute $x = 3$ into $(x – 3)$ and $x = 5$ into $(x – 5)$ and did not conclude that since 0 x 0 is equal to 0 their solutions were correct.

It is reasonable to assume that four of the 18 interviewees (Students 6, 9, 15 and 18) learned virtually nothing about quadratic equations from the lessons. These four did not give correct solutions to any interview question, did not know how to factorise quadratic trinomials, and did not know how to check whether any of the "solutions" that they obtained were correct.

The other 12 interviewees gave correct answers to at least two of the "interview questions", but that did not mean that they achieved a relational understanding of quadratic equations. Most obtained 3 and 5 as solutions to $(x – 3)(x – 5) = 0$. But when asked to check their solutions, substituted $x = 3$ into $(x – 3)$ and $x = 5$ into $(x – 5)$ and concluded that since 0 x 0 = 0 their solutions were correct. They believed that the two x's in $(x – 3)(x – 5) = 0$ represented different variables and should take different values. This misconception found expression not only with $(x – 3)(x – 5) = 0$, but also with $x^2 – x = 12$. Ten of the 12 students rearranged $x^2 – x = 12$ to $x^2 – x – 12 = 0$ and $(x – 4)(x + 3) = 0$. They then equated $x – 4$ and $x + 3$ to zero and got the correct solutions. But, they thought that the x in the x^2 term in the original equation represented a different variable from the x in the other x term in the same equation.

When asked to check their solutions, seven of the 12 students said they did not know how to check. Three students "checked" in the $(x - 4)(x + 3) = 0$ form of the equation, putting x equal to 4 in $(x - 4)$ and x equal to (–3 in $x + 3$). After noting that 0 x 0 = 0 was true, they concluded that their solutions were correct. The other two students substituted into $x^2 - x - 12 = 0$, but let x equal one "solution" with x^2, and x equal the other "solution" for "$- x$." They got 16 + 3 – 12 and wondered why this was not zero.

The authors believe that if all 231 students could have been interviewed, the misconception that equations like $(x - 3)(x - 5) = 0$, $x^2 - x = 12$, and $2x^2 = 10x$ have two variables, not one, would have been shown to be guiding the thinking of many, perhaps a majority, of the students. The misinterpretation could have arisen from teacher statements, often made in lower-secondary algebra classes, that expressions like $2x^2$ and $10x$ are "*unlike* terms". It could also have arisen from students misinterpreting their teachers' statements that quadratic equations can "have two different solutions". In students' minds this could mean that if two x's appear in an equation then they *should* take different values. That could explain why, even at the post-teaching stage, relatively few students got both solutions to $x^2 = 9$. In the words of one of the interviewees, "in that equation x appears only once, and therefore there is only one solution".

The authors believe that at the post-teaching stage a minority of students in the six classes grasped the concept of a variable in the context of quadratic equations. Because of that fundamental misconception, many did not really understand the null factor law, or how "solutions" to quadratic equations could be checked.

Conclusions and Final Comments

The pre-teaching analyses of understanding scores indicated that before the lessons on quadratic equations most interviewees had little understanding of the mathematics associated with quadratic equations. That was to be expected. After the lessons, the mean understanding scores for interview questions increased, but for each question the mean of the 18 post-teaching understanding scores was still less than 2.0. A score of 2.0 was assumed to correspond to an accurate instrumental understanding of related concepts and skills but an absence of relational understanding.

Although the lessons on quadratic equations helped most students to perform better on the *Quadratic Equations Test* at the post-teaching stage, the post-teaching test mean for the 231 students was only slightly more than 6 out of a possible 18. Analysis suggested that improvement was related to a moderate increase in understanding, but the extent of improvement in performance and understanding was disappointing.

With the obvious exception of Student 10 (the outlier) – curiously, there is no evidence that Student 10 learned anything about quadratic equations from the lessons that she did not already know before the lessons – the high-performing interviewees tended to improve their understanding the most. The failure of most medium-performing and low-performing interviewees to develop relational understanding was reflected in relatively low mean performance gains of students in Grade 9/2, Grade 9/3, and Grade 9/5. Arguably, the failure of most students in those three classes to develop relational understanding for most questions meant that any improvement in test performance was based on rote-learned knowledge and skills.

Analysis of interview data revealed that many interviewees who obtained correct solutions actually had serious misconceptions about what quadratic equations actually are. Their answers were correct but, from a mathematical point of view, they did not know what they were talking about. Getting correct answers to quadratic equations on traditional pencil-and-paper tests merely served to reinforce their misconceptions about the nature of a variable within a quadratic equation.

The issues raised in this article present a real challenge to mathematics teachers and researchers. Are there realistically feasible forms of teaching that will result in students, and not just high-achieving students, learning quadratic equations, and other mathematics topics, in a relational way? Although that issue was not a primary focus of this report, a few concluding comments in relation to it might be in order.

In school education, traditionally accepted sequences of content, and teaching approaches, die hard. In many countries – perhaps most countries – traditional elicitation/exposition approaches to teaching quadratic equations are still widely used and preferred by teachers (Lim, 2000). Some teachers, and education researchers, believe that a teaching approach which places the study of equations, including quadratic equations, within the study of functions – the so-called "functions" approach – is far more likely than traditional elicitation/exposition approaches to solving equations to induce relational understanding within students. The same teachers and education researchers believe that the functions approach is especially likely to be successful if it is enriched by the use of modern technology, like graphics calculators (e.g., Drijvers & Doorman, 1996; Kirshner & Awtry, 2004; Mourão, 2002; Schwarz & Hershkowitz, 1999; Zazkis, Liljedahl, & Gadowsky, 2003).

Perhaps at the beginning of this new millennium, the power and accessibility of graphics calculators and computer algebra systems – technology which can readily generate graphs of even complicated functions, and permit the solutions of equations to be quickly linked to graphs of associated functions – make it more likely that a functions approach will win the hearts and minds of most secondary mathematics teachers (and their students). Some studies have provided tentative support for that view (see, e.g., Schwarz & Hershkowitz, 1999).

The now well-documented failure of many – almost certainly, most – middle-secondary students, across the world, to cope with the mathematical demands of quadratic equations suggests that curriculum designers should delay the inclusion of quadratic equations into curricula until Grade 10, Grade 11, or Grade 12. At this time of flux, when technology is raising questions about what mathematics should be studied, and when, we leave that issue to future researchers. We would comment, in closing, though, that the analyses for this present study, and for Lim's (2000) study, suggested that most middle-secondary students who participate in lessons on quadratic equations taught in traditional ways fall well short of acquiring a relational understanding of what they are taught.

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