CONVEX LATTICE POLYGONS

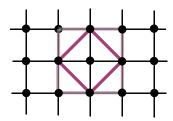**PAUL SCOTT**

The problem

It was 1976. I had been looking at my lecture notes on Pick's Theorem, when the thought suddenly came to me. Suppose you have a lattice polygon P with a fixed number I of interior points, and you insist that the polygon be convex. Now try to increase the number of boundary points B, while keeping I fixed, and retaining the convexity. Might we not expect there to be some bound on the number B?

A *convex* polygon is one with no re-entrant angles. Alternatively we can use the standard convexity definition, asserting that for any two points of the convex polygon, the line segment joining them is contained completely within the polygon.
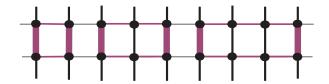

$I = 1$, $B = 4, 5, 6, 7, 8$.

It turns out that there are two cases to consider:

(a) convex lattice polygons with $I = 0$.
(b) convex lattice polygons with $I \geq 1$.

Suppose we look at case (a). We quickly find the set of polygons:

for which $I = 0$, and $B = 4, 6, 8...$ This means that for $I = 0$, the number of boundary points B can be made arbitrarily large; thus there is no upper bound on B.
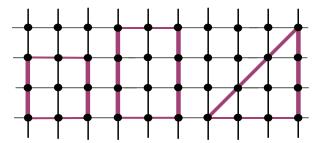
Let us now look at case (b). See if you can find convex lattice polygons having the values of I, B listed in the following table. Remember, for given I we are looking for the maximum value of B. Also, can you find a simple inequality involving B and I (and maybe some constants) which will hold for all convex lattice polygons with $I \geq 1$?

I	1	1	1	1	1	1	...	1	2	3	4	...
B	4	5	?	?	?	?	...	8	?	?	?	...

Answering the above question, you should have found convex lattice polygons with the following values of I, B.

I	1	1	1	1	1	1	...	1	2	3	4	...
B	4	5	6	7	8	9	...	8	10	12	14	...

The pairs at right come from a sequence of rectangles of width 2, like the first two polygons below.

In this sequence, there seem to be six end boundary points, and each interior point is associated with two further boundary points. This suggests $B = 2I + 6$, and we might have conjectured $B \leq 2I + 6$, were it not for the surprise triangle at right. Did you find it? So, our conjecture becomes $B \leq 2I + 7$. In fact we have:

Convexity theorem

If P is a convex lattice polygon with B boundary points and I interior points, then

$$B \leq 2I + 7 \text{ if } I = 1,$$
and
$$B \leq 2I + 6 \text{ if } I > 1.$$

We observe that this result will remain true for any planar lattice, since it just depends on the numbers B and I. This result first appeared in Scott (1976).

A partial solution

I find it amazing that this simple result could have remained undiscovered for so long. We will not give the whole proof here, but we can make some observations.
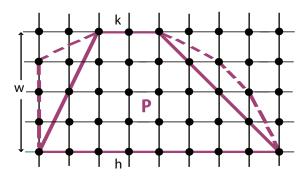
Suppose we set $g(P) = B - 2I$.

The statements of the Convexity Theorem now become $g(P) \leq 7$ and $g(P) \leq 6$ respectively. We can rewrite $g(P)$ in different ways using Pick's Theorem (refer to *AMT 62*(3)) which states

$$A = \tfrac{1}{2} B + I - 1$$

to obtain:

$$\tfrac{1}{2} g(P) = B - A - 1 = A - 2I + 1.$$

Now, how would we begin to prove the Convexity Theorem? Suppose our (convex) polygon P contains the lattice trapezium pictured below with parallel sides of length h and k placed distance w apart.

Then $\qquad A(P) \geq \tfrac{1}{2} w(h + k)$

and $\qquad g(P) = 2B(P) - 2A(P) - 2$
$\qquad\qquad\quad \leq 2(h + k + 2w) - w(h + k) - 2$
$\qquad\qquad\quad = (h + k - 4)(2 - p) + 6.$

It follows that $g(P) \leq 7$ if:
- $p = 2$, or $h + k \leq 4$; or
- $p = 3$ and $h + k \leq 3$; or
- $p \leq 4$ and $h + k \leq 3$.

We see that in order to establish our theorem, we now only need consider lattice polygons which do not satisfy these conditions. Details of the rest of the proof can be found in Scott (1976).

Further results

Various other results are known or conjectured when some restriction is placed on the size or shape of the lattice polygon.

For example, suppose P is a convex lattice pentagon. What is the smallest number of interior points it can have? Try some examples! Ehrhart (1955) gives the answer to this (in French!); the answer is 1.

In fact, it is relatively easy to obtain a sequence of results of the form:

A convex lattice n-gon contains at least $I(n)$ interior points.

In 1978, Coleman independently proved the Convexity Theorem. For convex polygons with $I > 0$, he proposed the relatively weak inequality $B \leq 9I$ (best possible for $I = 1$, but $9I > 2I + 7$ for $I > 1$).

He then makes the interesting conjecture: if a convex lattice polygon has n sides, then $B \leq 2I + 10 - n$. Setting $n = 3$ we see that the bound is best possible (can you see why?); but even if it is true, it is probably much too large for large values of n.

Coleman also establishes a result in the other direction for convex lattice n-gons:

$$I \geq \left[\frac{n-3}{2} \right]$$

and for $n \geq 7$, $\qquad I \geq \left[\frac{n-2}{2} \right]$

Here the square brackets denote the integral part of the number inside; e.g., [3.2] = 3. The proof is rather involved, but we can easily check the first part.

For $n = 3$ or 4, $\qquad \left[\dfrac{n-3}{2} \right] = 0$
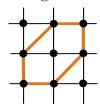
and clearly $I \geq 0$.

For $n = 5$ or 6, $\quad \left[\dfrac{n-3}{2}\right]=1$

and $I \geq 1$ is correct (for example, using Ehrhart's result).

Arkinstall (1980) incorporated Ehrhart's result and extended it as follows.
For a convex lattice n-gon P:

- $n = 4$ and P has no pair of parallel edges $\Rightarrow I \geq 1$.
- $n \geq 5 \Rightarrow I \geq 1$.
- $n = 6$ and $I = 1 \Rightarrow P$ is a centrally symmetric hexagon,
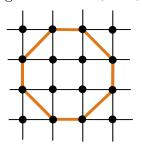

- $n > 6 \Rightarrow I \geq 2$.

Rabinowitz (1989) later used some computer help to list all convex lattice polygons in the plane (up to equivalence) with at most one interior lattice point.

We can improve the Convexity Theorem inequality if we add further constraints to the lattice polygon. For example suppose we insist that all the angles in the polygon P be acute. Then $B \leq 2I + 4$, and this result is best possible; but this is not very interesting because in this case P must be an acute-angled triangle.

Of more interest is when we insist that every angle of P be obtuse. What do you think the inequality might be in this case? Find an example where equality holds. The proof of this result is given in Scott (1979).

Simpson (1990) suggests looking for convex lattice n-gons which have minimal area for given n. He shows that for $n \geq 3$

$$a(n) = g(n) + \frac{n}{2} - 1$$

where $a(n)$ denotes the minimal area, and $g(n)$ the minimal number of interior points for the given value of n. He gives values of $a(2n)$ for low values of n.

Honsberger (1978) observes that an $n \times n$ square S can be placed in the plane with its sides parallel to the coordinate axes to cover $(n + 1)^2$ lattice points. He then asks if S can ever cover a greater number of lattice points if it is just tossed onto the lattice plane; for the answer, see Honsberger (1978).

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