Asia Pacific Education Review
2006, Vol. 7, No. 1, 51-61.

Cultivating Divergent Thinking in Mathematics through an Open-Ended Approach

Oh Nam Kwon **Jung Sook Park** **Jee Hyun Park**
Seoul National University
Korea

The purpose of this study was to develop a program to help cultivate divergent thinking in mathematics based on open-ended problems and to investigate its effect. The participants were 398 seventh grade students attending middle schools in Seoul. A method of pre- and post-testing was used to measure mainly divergent thinking skills through open-ended problems. The results indicated that the treatment group students performed better than the comparison students overall on each component of divergent thinking skills, which includes fluency, flexibility, and originality. The developed program can be a useful resource for teachers to use in enhancing their students' creative thinking skills. An open-ended approach in teaching mathematics suggested in this paper may provide a possible arena for exploring the prospects and possibilities of improving mathematical creativity.

Key Words : mathematical creativity, divergent thinking, open-ended approach

What is common in a lot of middle-school mathematics problems is that they are supposed to have only one solution. Pehkonen (1995) defined this type of problem as a closed problem. He suggested that these particular problems, which do not allow divergent thinking, are not able to enhance the quality of education, even after the introduction of new approaches such as small-group cooperative learning. If the goal of mathematics education is to realize an individual's potential of mathematical creativity, it is necessary to break the habit of "knowledge delivery" from the teacher to the student, which is the conventional instructor-oriented teaching method. Mathematics education should be focused on the development of creative thinking where students are free to try their own original possible solutions. It means avoiding the traditional teaching method that emphasizes 'convergent thinking,' in which a student memorizes existing mathematical rules and theorems and then applies them to problems with great adroitness in order to find one exclusive solution. Since these closed problems do not encourage students to adopt divergent thinking and reasoning, it is necessary to introduce new contexts that allow them to respond positively and participate actively in the learning process.

For this reason, many new approaches have been set forth. A recently highlighted method, Realistic Mathematics Education, claims that a constructive mathematics education should provide challenging tasks and an environment in which students try to construct their own concrete and informal problem solution strategies by exploring experientially real and mathematically real context problems (Gravemeijer & Doormann, 1999). Silver (1997) also tried to combine a student's originality and problem-solving ability by giving a problem-posing assignment to them. Brown (2001) used a "What-if-not" strategy to help students invent new ideas as possible solutions. What they have in common is

Oh Nam Kwon, Jung Sook Park, and Jee Hyun Park, Department of Mathmatics Education, Seoul National University, Seoul, Korea.
Support for this paper was funded by the Korea Research Foundation under grant No. KRF-2002-030-B00055. The opinions expressed do not necessarily reflect the views of the foundation
Correspondence concerning this article should be addressed to Oh Nam Kwon, Department of Mathmatics Education, Seoul National University, Gwanak-gu, Seoul, 151-742, Korea. E-mail: onkwon@snu.ac.kr.

that they provided challenging assignments and regarded students not as simple knowledge receivers, but as agents of knowledge creation. Therefore, what was important to them was not only the introduction of contexts that encouraged students to fully participate, but also how to lead those contexts effectively.

In Japan in the 1970s, Shimada with other researchers provided an assignment of an 'incomplete problem,' not only to open many possible avenues for different solutions, but also to discover new approaches by combining previously learned knowledge (Becker & Shimada, 1997). An incomplete problem is a problem which does not define clearly what the question asks for, therefore allowing many possible solutions. This problem is also called an open-ended problem, and this method is called "the open approach" or "the open-ended approach," which is the basis of this research. In other words, the open-ended approach is a pedagogical strategy that aims to produce creative mathematics activities that stimulate the students' curiosity and cooperation in the course of tackling problems.

Nohda (1995) held the view that open-ended problems are un-typified problems which should have two prerequisites. First, they should suit every single student by using familiar and interesting subjects. This implies that students realize it is necessary to solve the problems, feel it possible to solve them with their own knowledge and have a sense of achievement after solving them. Therefore the problems should be sufficiently flexible to take into account the students' different mathematical abilities. Second, open-ended problems should be suitable for mathematical thinking and should also be able to be generalized into new problems. They should also allow various solutions at diverse levels, include mathematical expressions. This approach can be fully realized whereby open-ended problems serve as a tool to bring out a fundamental change of class structure (Nohda, 2000). Since ill-structured or open problems can be used for multiple purposes and allow many different solutions due to a learner's different interpretation of those problems, they are regarded as contributing to the development of a student's originality and flexibility (Silver, 1997). This is why open-ended problems are widely used to measure a student's creativity.

This study aims to investigate the cognitive consequences of redesigning an open-ended mathematics unit to enhance mathematical creativity. To prove its effectiveness and applicability, it will compare the results of pre- and post-tests on students' fluency, flexibility and originality on the basis of a crucial factor in mathematical creativity known as divergent thinking.

Background

Mathematical creativity

Creativity is the high-dimensional human ability or skill to think up something new, and has drawn much attention in every field of education that attempts to cultivate the fundamental power of thinking and reasoning. In this age of information, the importance of creativity becomes more sharply recognized than ever before, even though it is known by other various terms. The same is true when it comes to mathematical creativity. A significant aspect of creativity in mathematics emerges from consideration of the process of thinking involved in problem-solving. This has often been seen as involving four stages: preparation, incubation, illumination and verification (Poincare, 1948; Hadamard, 1945).

Another trend in mathematical creativity focuses on creative problem solving as Ervynck (1991) proposed, mathematical creativity being the ability to formulate mathematical objectives and find their innate relationships. He claimed that mathematical creativity is the capacity to solve problems according to the appropriateness of integrating both the nature of logic-deduction in mathematics education and its evolved concepts into its core. Haylock (1987), who mentioned the difficulty of trying to define mathematical creativity, divided it into two types of ability; one to overcome the fixation of thoughts or conventional mentalities and the other to show various creative responses to open-ended mathematics contexts or problems. Krutetskii's research (1976), which analyzed the elements of mathematical ability in children with great mathematical talent, is not directly related to mathematical creativity although one of the elements suggested by him is closely related to it. Among the necessary mathematical abilities enumerated in his information-gathering stage, information-processing stage and information-grasping stage, one of the abilities required in the information-processing stage is "flexibility in thought, rapid and free conversion of thought directions, and the power to reconstruct thoughts," which is very similar to mathematical creativity.

In summary, there are two major definitions of mathematical creativity – the creation of new knowledge and flexible problem-solving abilities. Instructional approaches in school mathematics greatly depend on which definition is emphasized. However, what is really crucial from a teacher's point of view in this field is choosing an appropriate task for students and their contextual knowledge. It all depends on a well-chosen task in which students can be guided towards generalizing their learning assignment and create new knowledge, not to mention improving their problem-solving abilities. Therefore, this study developed a program using open-ended problems to which every single student can offer his/her own solution according to his/her own abilities, because those problems may provide the best means of cultivating and enhancing students' creativity.

Open-Ended Problems

In general, an open-ended problem is a problem open to many different solutions (Becker, 1997; Nohda, 2000). Its definition, however, varies between scholars. Pehkonen (1995) first defined closed problems. A closed problem possesses a very clear start and objectives and doesn't allow the possibility of divergent thinking. Therefore, a problem that is 'open' with respect to either its introduction or objectives, and is by corollary open to divergent thinking, can be considered an open-ended problem. In this sense, an open-ended problem is defined as a problem that may have a very clear starting context but is open to many different possible solutions.

Feedman (1994) stated that open-ended problems enable students to use high-dimensional thinking skills by employing various writing methods. London (1993) defined some traits of open-ended problems. For example, they require the three stages of problem recognition, trial, and perseverance. They should have divergent solutions and they can assess students. Finally, they should be solvable by every single student and should take some time to be solved. Sawada (1997) enumerated the five advantages of open-ended problems. First, students take part in class more actively and express their ideas more freely. Second, students can have a greater opportunity to use their mathematical knowledge and skill more widely. Third, all students can answer the problem in their own meaningful ways. Fourth, classes with open-ended problems provide students with rational experience. Finally, students are offered the chance to feel the fulfillment of discovery and the approval of other students.

Despite the advantages of open-ended problems, it is not easy to devise different categories for them. For this reason, the problems with the following three traits will also be regarded as open-ended problems in this study.

First, the starting point of the problem is relatively clear but the solutions for its objectives can vary.

Second, students can choose their own appropriate approach and explain the reason for their choice.

Third, they are problems in which students can use high-dimensional thinking skills and employ divergent thinking in the pursuit of their own solutions.

Open-ended problems can give students a sense of achievement and fulfillment because it is possible even for students with less mathematical ability to set forth their own solutions within their own ability. Furthermore, it offers students a chance to feel what it is to be a real mathematics learner in the course of creating their own problems. Here, what is significant is that both teachers and students recognize the learner's contribution in all learning processes and that every learner has confidence in their ability to find their own answers.

One characteristic of creative thinking is divergent thinking, which Guilford (1967) defined as an act of pursuing diversity in solving a problem without one fixed answer or thinking in a different perspective. He also suggested that divergent thinking increases the possibility of arriving at original thoughts. Because it encourages diverse thoughts, an open-ended problem contributes towards boosting divergent thinking. In the course of searching for diverse solutions and various approaches, students can put forward many ideas freely (fluency), and make other attempts to devise new strategies to tackle the problem where others fail (flexibility), and think up very clever and unexpected ideas (originality). In a word, an open-ended problem is very effective in cultivating mathematical creativity.

Method

Participants

The participants for this study were seventh-grade students in 13 intact classes at five middle schools in Seoul, Korea. Most of the students were from families of lower middle to middle socioeconomic status and were generally of average scholastic ability. For this study, classes in the field were held for eight months from April to November of 2003 and three teachers participated in the treatment groups. Eight classes from public middle schools served as treatment groups while five classes from two of the middle schools were selected as comparison groups. The five schools are located in the same school district so that they are all at approximately the same level with regard to students' scholastic ability. Although 433 students took the pre-test and 419 students took the post-test, the statistical analyses were based on test results from 398 seventh grade students who were present at all pretesting and post-testing sessions. The teachers for the treatment group had incorporated components of new instructional strategies using open-ended problems. The comparison classes in this study were taught with traditional instruction methods from a textbook.

Procedure

This study designed a total of twenty sessions of teacher's guidelines and student's worksheets which were composed of many types of open-ended problems tailored for the first year of middle school mathematics. In constructing the worksheets, this study tried to cover as wide a range as possible in the curriculum. That is, the first ten sessions derived from the contents of stage 7-Ga* and the remaining ten sessions covering the materials and relevant areas of stage 7-Na**. The worksheets used in class were made to encourage students to produce as many answers as they could in order to observe their fluency. They were designed not only to observe students' flexibility by introducing various types of responses but also to determine their originality by encouraging them to produce responses different from each other. Each worksheet was also made to fit within a 45-minute class.

There was one class every week consisting of both individual learning and small-group cooperative learning. By maintaining a free discussion atmosphere, teachers made an effort just to guide the class, not to give fixed solutions to the problems. The treatment groups participated in the program for an hour either in mathematics classes or CR classes once a week. In contrast, the comparison groups were given their own CR classes devised by each school and they were also taught within the curriculum even in their mathematics classes. The textbook and traditional instructional practices formed the core of the comparison classes. Thus the comparison classes in this study were taught traditional instruction methods from a textbook.

Following Nohda's guidelines (1995), we developed, in conjunction with two researchers and three school teachers, a unit that could be integrated with the curriculum of middle-school mathematics classes. Since this unit aimed to provide a variety of assignments and learning environments, it was mainly created for average class contexts, not for special or in-depth classes for mathematically talented students or those with good academic performance. Therefore, the program contents were designed in the same order as the curriculum of middle-school mathematics textbooks and also chosen from the relevant materials closely related to it. In the spring term, the program covers numbers and arithmetic, letters and expressions, and patterns and functions, while the program for the autumn term deals with probability, statistics and diagrams. The types of open-ended problems are as follows: (1) overcoming fixations, (2) multiple answers, (3) multiple strategies, (4) strategy investigation, (5) problem posing, (6) active inquiry tasks, and (7) logical thinking. Figure 1 shows an example of a "multiple-answer" type of open-ended problem.

> Among the following numbers, choose a number which is different from the others.
> If possible, try to find many possible cases or answers.
> 1 2 4 6 8 12

Figure 1. An example of "multiple answers"

If 1 is chosen for the answer, the reason is that it is the only odd number. 12 can be chosen as an answer because it is the only two-digit number, unlike all the other one-digit numbers. 8 can also be an answer because it is not a divisor of 12, unlike the others. Some students may choose 2 as their answer because it is the only prime number. Of course there were more possible answers generated by students.

Because the users of this program were teachers who played a guiding role in the "comparison and discussion" and "new ideas/problems/problem posing" tasks in class, this program prepared and supplied them with class guidelines that mentioned the noteworthy points in assessment, teaching strategies, and learning activities. This program also prepared the necessary reference data at the end of the worksheets for the situation whereby the teachers needed some form of amendment or modification.

Instruments

The items used in the pre- and post-tests were chosen or modified from those in Becker and Shimada's *The Open-ended Approach: A New Proposal for Teaching Mathematics* (1997) and Marilyn Burns's *Problem-solving Lessons* (1996) after consulting with the teachers and researchers. The tests are composed of four descriptive questions each of which requires diverse responses and which are also suitable for measuring divergent thinking, a factor of original problem-solving skills. The Cronbach alpha's for the pretest was 0.53.

* 7-Ga is equivalent to the first semester of 7th grade curriculum.

** 7-Na is equivalent to the second semester of 7th grade curriculum.

The post-test was conceptually the same and essentially an alternative version of the pre-test. The Cronbach alpha's for the post-test was 0.57. Examples of test items can be found in the Appendix A.

Scoring rubrics

The pre-test was given to the treatment participants comprising of 8 classes and the comparison groups comprising of 5 classes in April 2003. After the program was applied to the treatment groups, a post-test was given to the same groups in December 2003. Both tests took 45 minutes, which implies students were supposed to allot approximately 10 minutes to each question with an indication of the lapse of time.

When marking the questions, the responses to each were analyzed to produce a standard marking table. Three teachers, who didn't participate in the treatment, conducted the marking after the purpose of the tests was sufficiently explained beforehand. They also had training on grading to help them fully understand the standards of marking by exchanging their opinions of a few sample questions. They were able to mark the students' responses with objectivity because they hadn't been involved with the treatment at all.

Points were calculated by adding up the scores for flexibility, fluency and originality. Fluency was measured by the number of correct responses, while flexibility by the number of categories with or close to correct responses. In measuring originality, scarcity and utility were kept in mind and each category was given a score from 0 to 3. In this case, even though there were many categories for one score, a point was only assigned to them once. The maximum score for originality was 6 in questions 1, 2 and 3, and 12 in question 4. Appendix B shows the standard marking table for the post-test.

Results and Discussion

The treatment program was aimed at cultivating students' divergent thinking to solve open-ended problems in mathematics. The results of the scores for fluency, flexibility and originality in the pre- and post-tests on both treatment and comparison groups are shown below.

Analysis of covariance (ANCOVA) was used to compare the treatment and comparison groups. Table 1 illustrates the results of ANCOVA for total fluency points.

An ANCOVA was conducted on the post-test fluency scores using pre-test fluency scores as a covariate and testing the effects of group. In the results of ANCOVA, there was a statistically significant difference in the fluency scores between the treatment and comparison groups. The effect size is 0.42, which may be seen as a medium effect (Cohen, 1988). These results can be interpreted as evidence that the program

Table 1. *Posttest Scores of Treatment and Comparison Groups (Fluency)*

Fluency	Treatment Group (*N*=225)		Comparison Group (*N*=173)			
	Mean (*SD*)	Adj. Mean	Mean (*SD*)	Adj. Mean	*F*	*p*
	18.20 (7.31)	17.98	15.69 (6.01)	15.97	10.565	.001

Table 2. *Post-test Scores of Treatment and Comparison Groups (Flexibility)*

Flexibility	Treatment Group (*N*=225)		Comparison Group (*N*=173)			
	Mean (*SD*)	Adj. Mean	Mean (*SD*)	Adj. Mean	*F*	*p*
	5.29 (1.89)	5.22	4.17 (1.66)	4.26	30.108	.000

given to the treatment groups was effective in encouraging fluency, particularly with the questions of high difficulty.

The difference in the scores of the treatment and comparison groups was significant not only in the scores of response numbers but also in their quality. In question 4, where the subjects described the characteristics of number arrangements, the simple explanation of response type 1 increased in the number of responses in the post-test, compared to the result of the pre-test. While the comparison group responded to the question with such answers as “each arrangement of horizontal and vertical lines is different” or “each line is an arrangement of multiples of a certain number,” the treatment group explained not only the traits shown in both horizontal and vertical lines but also explained the relationships between horizontal and vertical lines or the relationship in the quantity of associated numbers. For example, they discovered a certain rule in each line by adding such extra conditions as “all the numbers except 1 in the vertical line 5 are multiples of 5” or “all the numbers except 1 in the vertical line 7 are multiples of 7.” In short, the treatment group demonstrated more creative responses in quality even to questions in the same category. This result can be said to prove that the program was valid in fluency.

The scores for flexibility were calculated by the number of correct categories after classifying students’ answers into categories. A higher score was given to a category with diverse approaches and solutions, demonstrating greater flexibility. To show differences between the treatment and the comparison groups following the program, ANCOVA was applied to the adjusted mean between them after removing the statistical influence of its parameters. The results of ANCOVA for both groups are shown in Table 2.

The bottom line of Table 2 shows the mean post-test results for each group, as observed and as adjusted on the basis of pre-test score. An ANCOVA was conducted on the post-test flexibility scores using the pre-test flexibility score as a covariate and testing for the effects of group (treatment and comparison). There was a statistically significant difference in the total scores. The effect size is 0.67, which is considered a medium effect in terms of Cohen (1988). The total scores showed a significant difference at the .001 level. These results show that the open-ended approach improves students’ flexible thinking in solving mathematical problems.

To maintain the consistency in the pre- and post-tests, the same classification of the correct answers in terms of flexibility was applied to both tests. Nevertheless, it is obvious that the students in the treatment group did not adhere to one method or to one simple description but attempted to answer the questions in various ways, given that their answers were diverse. For example, in question 1 in the pre-test, where students were requested to obtain a certain number using given numbers, many, in both treatment and comparison groups, attempted to obtain their solution by using less than 4 numbers and 4 basic operations. However, in the post-test, an increased number of students in the treatment group used more than 5 numbers or used different operations such as involution to obtain their solutions.

In question 3 where students were requested to calculate the number of figures, students in the treatment group used diverse approaches rather than a simple way of counting directly. For instance, they tried to obtain solutions by ‘bundling in a certain rule,’ ‘using the area of figures’ or ‘applying a generalized formula.’ Some students in the treatment group achieved a very high mark in flexibility even though they had a low mark in fluency. This result can be interpreted as proof that the program had a positive effect on the students with regard to the enhancement of their flexibility.

The scores for originality in this study depended on the uniqueness of ideas, the value and meaning of students’ responses, and their usefulness and scarcity. Table 3 summarizes the results of ANCOVA for the corrected averages between the treatment and comparison groups after

Table 3. Post-*test Scores of Treatment and Comparison Groups (Originality)*

Origin-ality	Treatment Group (*N*=225)		Comparison Group (*N*=173)			
	Mean (*SD*)	Adj. Mean	Mean (*SD*)	Adj. Mean	*F*	*P*
	1.89 (2.01)	1.85	.54 (1.23)	.59	54.451	.000

removing the influence of its parameters. The originality scores in both pre-and post-tests were higher in the treatment group than in the comparison group.

In the results of ANCOVA, there was a statistically significant difference between the post-test originality scores of the treatment and comparison groups. As a whole, the post-test originality scores of the treatment group were significantly higher than those of the comparison group at the .001 level. The effect size is 1.1, which is interpreted as a large effect according to Cohen (1988).

The originality of students in the treatment group was particularly outstanding in questions 2 and 4. In question 2 in which they had to draw a figure, both treatment and comparison groups in the pretest created simple diagrams by using one basic diagram (1). While the comparison group showed the same phenomenon in the post-test, the treatment group in the post-test created more complex figures using more than 2 diagrams (2), using a middle point (3) in the question, or using curved lines (4). In particular, the number of students using curved lines was much higher in the treatment group than in the comparison group.

In question 4 where students were requested to determine the rule of the arrangement of numbers, students in the treatment group showed great originality by answering with descriptions such as the generalization of the n-th number or line, instead of just a simple description. Sometimes, they were able to deduce a peculiar quality in the arrangement such as a binominal theorem, which greatly contributed to their high mark of originality.

This implies that the program developed by this research had a positive effect on the treatment students' generating new ideas in solving open-ended problems in mathematics.

Therefore, this result suggests the program based on open-ended problems produced a significant difference in all three factors of divergent thinking – fluency, flexibility and originality, shown by the results of ANCOVA between the treatment and the comparison groups. The effect size of fluency, flexibility and originality are 0.42, 0.67, and 1.1 respectively, which implies that the unit had a more positive effect on the treatment groups in terms of originality. This result indicates that open-ended problems with diverse answers and problem-solving strategies are effective in the improvement of divergent thinking. Furthermore, mathematics classes which utilize highly active mathematical thinking skills with open-ended problems are also effective in cultivating creative problem-solving ability, unlike many traditional classes which focus on the closed problems.

Conclusion

Most questions in mathematics have only one answer. Therefore they are highly likely to discourage students from exploring diverse ideas. However, open-ended problems can make up for this disadvantage because they allow various answers or various approaches. They can also contribute to cultivating the ability of mathematical communication in the course of discussing their diverse inferences and solutions. Another advantage of open-ended problems is that every student, whether they excel or struggle in mathematics, can try and find their own answers to the problems within their own scope and range of abilities. This is why open-ended problems can be easily adopted for differentiated classes. Ultimately, classes which use open-ended problems make it easy to evaluate students' high-ordered thinking and also to obtain very precise information about how well students understand what they have learned.

The program in this study was designed on the basis of the stages of middle school 7-Ga and Na, consisting of twenty sessions. This study explored the applicability and effect of the program. To show its effects, this study carried out pre- and post-tests on both treatment and comparison groups to show the difference in their achievement. In the results, the treatment participants of this program showed a significantly higher mean score in the post-test than those in the comparison group. In the analysis of each question, the treatment group showed the effect of the program in all the questions, except for the first, possessing a low degree of difficulty, with regard to all three divergent thinking factors of fluency, flexibility and originality. This implies that the program had a positive effect on cultivating divergent thinking skills of mathematical creativity.

This study did not focus on the sensitivity and elaboration (Guilford, 1976) components of creativity, but

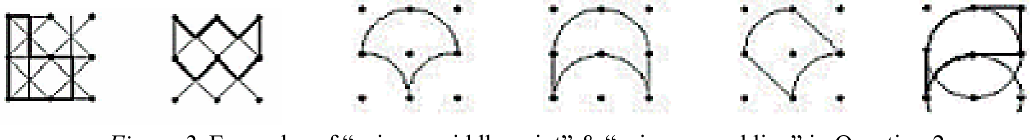


Figure 2. Examples of "using a middle point" & "using curved line" in Question 2

rather, only focused on certain components of creativity such as fluency, flexibility and originality. Sensitivity and elaboration were excluded because the period of this study was not long enough to explore them. Therefore, it is necessary to perform a complementary study to consolidate the effects of the developed program in terms of mathematical creativity. Likewise, it may be possible to make a more sophisticated classification of open-ended problems by using the students' responses to each question within the tests. Although this study could not give a great deal of attention to the affective domain, it could be effective in stimulating students' curiosity and creativity, judging from the increase in their questions during the course of the program, as well as their positive attitude shown in the interviews with the teachers and other supervisory activities. Although the mathematics education reform recommendations have been broadly endorsed, there is a need to build a research base concerning the cognitive and affective consequences of learning via reform-based methods such as the open-ended approach. This study contributes to that research base by exemplifying how to implement one of the basic mathematics reform goals – namely cultivating student divergent thinking in problem solving.

References

Becker, J. P., & Shimada, S. (Eds.) (1997). *The open-ended approach: A new proposal for teaching mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Brown, S. (2001). *Reconstructing school mathematics.* New York: Peter Lang Publishing.

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences*. Hillsdale, NJ: Lawrence Erlbaum.

Ervynck, G. (1991). Mathematical creativity. In D. Tall. (Ed.), *Advanced mathematical thinking* (pp.42-53). Netherlands: Kluwer Academic Publishers.

Freedman, R. (1994). *Open-ended questioning: A handbook for educators.* Don Mills, OH: Addison-Wesley.

Hadamard, J. (1945). *An essay on the psychology of invention in the mathematical field.* New York: Dover Publication.

Haylock, D. W. (1987). A framework for assessing mathematical creativity in school children. *Educational Studies in Mathematics, 18*, 59-74.

Haylock, D. (1997). Recognizing mathematical creativity in schoolchildren. *Zentralblatt fur Didaktik der Mathematik, 27*(2), 68-74.

Gravemeijer, K., & Doorman, M. (1999). Context problem in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*, 111-129.

Guilford, J. (1967). *The nature of human intelligence*. New York: McGraw-Hill.

Krutetskii, V.A. (1976). *The psychology of mathematical abilities in school children*. The Univ. of Chicago Press.

London, R. (1993). A curriculum of nonroutine problems. *Paper represented at the Annual Meeting of the American Educational Research Association,* Atlanta, GA, April. (ERIC Document Reproduction Service No. ED359213)

Marilyn Burns Education Associates. (1996). *Problem-solving lessons*. Sausalito: Math Solutions Publications.

Nohda, N. (1995). Teaching and evaluating using "open-ended problems" in classroom. *Zentralblatt fur Didaktik der Mathematik, 27*(2), 57-61.

Nohda, N. (2000). Teaching by open-approach method in Japanese mathematics classroom. *Proceeding of the 24th conference of the international Group for the Psychology of Mathematics Education*, Hiroshima, Japan, July 23-27, volume 1-39-53.

Pehkonen, E. (1995). Using open-ended problem in mathematics. *Zentralblatt fur Didaktik der Mathematik, 27*(2), 67-71.

Poincaré, H. (1948). *Science and method*. New York: Dover.

Sawada, T. (1997). Developing Lesson Plans. In J. Becker, & S. Shimada (Eds.), *The open-ended approach: A new proposal for teaching mathematics.* (p. 23-35). National Council of Teachers of Mathematics.

Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *Zentralblatt fur Didaktik der Mathematik, 27*(2), 68-74.

Received September 14, 2005
Revision received April 3, 2006
Accepted June 8, 2006

Appendix A: Assessment Instrument

<Pretest 1>

Make an expression to obtain 30 after calculating by the following rules.

< Rules >

1. Use all or some of the following numbers.
2. You are allowed to use any kind of mathematical symbols.
3. Use the given numbers only once in one expression.

   e.g.) 10+20 (○) , 10+10+10 (×)

< Given Numbers >

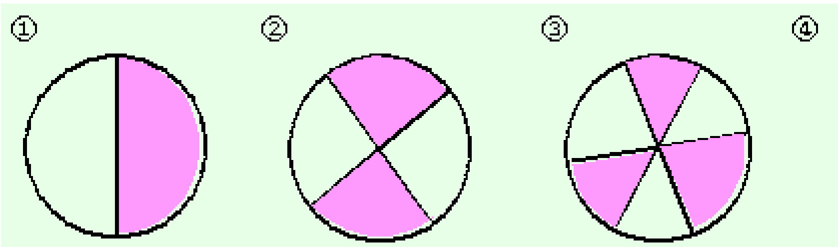
18 2 10 12
40 48 3 20
90 15

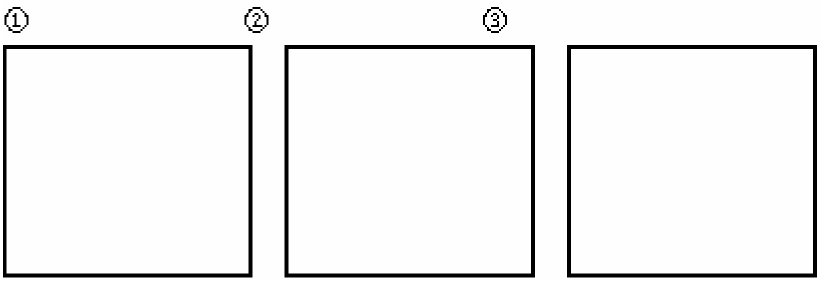
<Pretest 2>

Divide the given square in various ways to make the areas equal between the colored and the un-colored as shown in the example.

<Example>

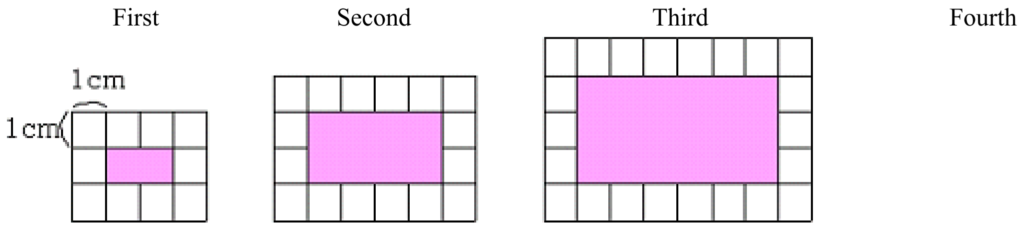
It might be possible to divide a circle like this.





<Posttest 3>

Let’s arrange 1 centimeter squares of paper as shown in the following pictures.



(1) When we arranged paper squares like the above, how many paper squares are required in the fourth picture?

(2) There may be other possible ways to count the paper squares in the fourth picture. Can you explain those ways?

<Posttest 4>

Let’s arrange the numbers by a certain rule as follows.

	C(1)	C(2)	C(3)	C(4)	C(5)	C(6)	C(7)	C(8)
A(1)	1							
A(2)	1	1						
A(3)	1	2	1					
A(4)	1	3	3	1				
A(5)	1	4	6	4	1			
A(6)	1	5	10	10	5	1		
A(7)	1	6	15	20	15	6	1	
A(8)	1	7	21	35	35	21	7	1

Write as many facts or characteristics found in the table as possible.

Appendix B. Example of Scoring Rubrics

Division	Fluency	Flexibility	Originality
Question number	Response number	Type number	Relatively rare, original and useful answers
1	Number of the answers with correct rules	Ⅰ. Use less than 4 numbers Ⅱ. Use over 5 numbers Ⅲ. Use involutions Ⅳ. Use other than four basic operations or particular mathematics symbols – respectively	① Ⅱ type -1points ② Ⅲtype -2points ③ Ⅳtype -3points
2	Number of the correct answers with correct rules	Ⅰ. Use 1 basic diagram. Ⅱ. Use more than 2 diagrams Ⅲ. Use a middle point Ⅳ. Use curved lines *Regard the diagrams as one diagram if they are identical by rotation or symmetry.	① Among the answers using a diagram, a type which is rare – 1 point ② Ⅲ type – 2 points ③ Ⅳ type – 3 points
3	Numbers of the answers with correct rules and expressions	Ⅰ. Many ways to count paper squares directly by drawing the fourth picture. Ⅱ. Use the rules of progression Ⅲ. Change them into certain patterns or groups Ⅳ. Use averages easy to calculate the numbers or the areas and use the rules of square area. Ⅴ. Use a generalized formula to find the n-th term. Ⅵ. Other peculiar methods - respectively	① a peculiar approach in Ⅰ, Ⅱ, & Ⅲ - 1 point ② Ⅳ type - 2 points ③ Ⅴor other original methods - 3 points
4	Numbers of the answers with correct rules and expressions	Ⅰ. Simple techniques Ⅱ. Symmetry by horizontal lines Ⅲ. Find a rule on the basis of diagonal lines Ⅳ. Find peculiar qualities or rules.	① General qualities (the n-th horizontal line has n numbers or line symmetry on the basis of the middle number) – 1 point ② The k-th number in the n-th line is the sum of the k-th and k+1th number in the n-1th line –1 point ③ Generalization of the n-th number or line. – 2 points ④ Peculiar qualities (Binominal theorem) – 3 points