Mathematics Education Research Journal 2005, Vol. 17, No. 3, 91–112

Students' Understanding of Trigonometric Functions

Keith Weber
Rutgers University

In this article students' understanding of trigonometric functions in the context of two college trigonometry courses is investigated. The first course was taught by a professor unaffiliated with the study in a lecture-based course, while the second was taught using an experimental instruction paradigm based on Gray and Tall's (1994) notion of procept and current process-object theories of learning. Via interviews and a paper-and-pencil test, I examined students' understanding of trigonometric functions for both classes. The results indicate that the students who were taught in the lecture-based course developed a very limited understanding of these functions. Students who received the experimental instruction developed a deep understanding of trigonometric functions.

Introduction

Relevant literature

Trigonometry is an important course in the high school curriculum. Understanding trigonometric functions is a pre-requisite for understanding topics in Newtonian physics, architecture, surveying, and many branches of engineering. Further, as trigonometry is one of the earliest mathematics topics that links algebraic, geometric, and graphical reasoning, it can serve as an important precursor towards understanding pre-calculus and calculus. Unfortunately, the initial stages of learning about trigonometric functions are fraught with difficulty (Blackett & Tall, 1991). Trigonometric functions are operations that cannot be expressed as algebraic formulae involving arithmetical procedures, and students have trouble reasoning about such operations and viewing these operations as functions (e.g., Breidenbach, Dubinsky, Hawk, & Nichols, 1992). Students also need to relate diagrams of triangles to numerical relationships and to manipulate the symbols involved in such relationships (Blackett & Tall, 1991), and many high school and college students are not accustomed to this type of reasoning.

Despite the documented difficulties with learning trigonometric functions, the educational research literature in this area is sparse. Empirical studies on this topic typically compare two groups of students who were taught in different ways. For instance, Blackett and Tall (1991) examined two groups of students; the first group took part in an experimental course with a computer that allowed students to explore numerical and geometrical relationships in an interactive manner, while the second group was instructed at the same school by a teacher who regularly taught trigonometry. Blackett and Tall's findings were that the experimental students significantly outperformed the control students on a post-test in both standard and non-standard tasks. In a large-scale study, Kendal and Stacey (1997) found that students who learned trigonometric functions in the context of a right triangle model performed better on a post-test than those who learned about the subject using a unit circle model. The remaining literature on trigonometry largely consists of teaching techniques that can be used to supplement or replace standard instruction of the topic (e.g., Barnes, 1999; Miller, 2001; Searl, 1998). While these pedagogical suggestions are interesting and potentially valuable, they generally are not based on research or theory and their efficacy is rarely assessed.

There is a broad consensus among mathematics education researchers that the goal of mathematics courses is not only for students to memorise procedures and acquire reliable methods for producing correct solutions on paper-and-pencil exercises; rather students should learn mathematics with understanding (e.g., Davis, 1992; National Council of Teachers of Mathematics [NCTM], 2000; Skemp, 1987). In particular, students should be able to explain why the procedures they apply are mathematically appropriate and justify why mathematical concepts have the properties that they do (cf. Skemp, 1987). There have not been systematic investigations into how trigonometry courses are taught. However, several researchers have noted that current teaching practices in trigonometry classrooms do not seem geared toward developing students' understanding of trigonometric functions. For instance, Kendal and Stacey (1997) observed that many approaches to teaching trigonometry, such as "the right triangle" approach, primarily stress procedural skills and such approaches do not allow students to understand sine and cosine as functions. Hirsch, Weinhold, and Nichols (1991) contended that current instruction of trigonometry is not consistent with the goals laid out in the NCTM Standards (NCTM, 1989, 2000) and further argued that we need to move away from trigonometric instruction that encourages "memorisation of isolated facts and procedures and proficiency with paper-and-pencil tests [and move towards] programs that emphasise conceptual understanding, multiple representations and connections, mathematical modelling, and problem-solving" (p. 98).

Theoretical perspective

In this article, Gray and Tall's (1994) theoretical notion of *procept* was used to analyse students' understanding of trigonometric functions. Gray and Tall defined an elementary procept to be "the amalgam of three components: a process that produces a mathematical object, and a symbol which is used to represent either process or object" (p. 121). For example, the symbol 8^2 is a procept as it can be viewed either as a prompt for a student to engage in the process of squaring (i.e., to multiply eight by eight) or as representing the result of squaring (in this case, 64). Gray and Tall argued that "proceptual thinking", that is, the ability to think of mathematical operations and objects as *procepts*, is critical to successfully learning mathematics. Although their discussion of procepts primarily occurred within the context of elementary arithmetic, Gray and Tall stressed that their construct can be applied to concepts throughout students' mathematics curricula, and specifically made the point that trigonometric functions are procepts. Below I explain how trigonometric functions can be understood as mathematical procepts and argue why thinking about trigonometric functions in this way is essential for understanding them.

Suppose that a student were asked to provide an estimate for the value of the sine of 20°. What type of reasoning or justification would a mathematics educator like to see? Desirable reasoning might include:

- Mentally (or physically) constructing a standard right triangle containing an angle of 20°, and then comparing the length of the side of this triangle opposite to the 20° angle with the length of its hypotenuse, noting that the former length is considerably shorter than the latter, perhaps by a ratio of 1 to 3. Then, realising that sine is equal to opposite over hypotenuse, approximating sin 20° to be about $\frac{1}{3}$.
- Mentally (or physically) constructing a unit circle on the Cartesian plane and then constructing a ray emanating from the origin of the plane that forms a 20° angle with the positive half of the x-axis. Then, focusing on the point of intersection between the ray and the unit circle, noting that the y-value of this intersection will be a small positive number, such as 0.3.

In both cases, estimating the value of sin 20° involved anticipating the result of a geometric process, a process that involved both construction and measurement. In this sense, sin 20° may be thought of as an elementary procept; this symbol refers both to a geometric process and to the mathematical object that is the result of that process. Viewing trigonometric functions as procepts is not only useful for estimating the numerical values of trigonometric expressions, but also for justifying why trigonometric functions have the properties they do. Justifying why sin 270° = –1 involves reasoning about a process that could be used to produce the value of sin 270°. Likewise, justifying why 2 is not in the range of sin θ involves understanding why the process of computing a sine could never yield a value of 2, no matter what the input for this process. In short, when reasoning about the properties of the numeric values of trigonometric expressions, one often has to refer back to the geometric processes used to obtain those values.

Research questions

The purposes of the research reported in this article were to address the following questions:

1. How do students in a college trigonometry course understand trigonometric functions? In particular, do these students understand trigonometric functions as processes and/or procepts? If so, what

types of processes do students think of when reasoning about trigonometric functions?

- Can students in a college trigonometry course use their understanding of trigonometric functions to justify why trigonometric functions have the properties they do?
- Is it possible to design instruction that can lead students to understand trigonometric functions as procepts? Can students who receive such instruction justify why trigonometric functions have the properties they do?

Research methods

Participants

Two groups of students participated in this study. The first group was enrolled in a college trigonometry course at a regional university in the southern United States in the fall of 2002. This class was taught by a professor who was not affiliated with the study. Thirty-one students were enrolled in the class. The professor had thirty years experience teaching at the university where this study took place. Through self-report, the professor claimed he taught the course using "traditional methods" and "mostly straight from the textbook". In particular, his course was lecture-based and he would spend most of the class time explaining to the students how to do particular exercises and then illustrating how to complete the exercises with illustrative examples. From this point, his instruction is referred to as "standard instruction".

The second group of students was enrolled in a separate section of college trigonometry taught during the same time period at the same university. The class was taught by the author of this article using experimental instruction based on the theoretical perspective presented in the preceding section of this article. A description of this instruction is provided in the following subsection. Forty students were enrolled in this class. There was no a priori reason to expect there to be systematic differences between the two populations of students. The textbook used in both classes was College Algebra and Trigonometry (Lial, Hornsby, & Schneider, 2001).

Experimental instruction

Based on a large body of research, Tall et al. (2000) proposed a learning trajectory for how a student can successfully construct an understanding of a mathematical procept – see, also, Dubinsky and McDonald (2001). In Tall et al's theory, a student can go through the following three stages to understand a procept:

- *Procedure*. The student first learns how to apply an operation as a procedure, or as a step-by-step algorithm. At this stage, the procedure is highly mechanical and may be relatively meaningless to the student.

- *Process*. If the student applies the procedure multiple times and is given the opportunity to reflect upon it, he or she may come to view the procedure as a process, or a meaningful method designed to accomplish a particular mathematical goal.
- *Procept*. A student who understands an operation as a process can begin to anticipate the results of this process without applying each of its steps. He or she can also reason about what properties the output of the process must have, based upon properties of the process itself. When this occurs, the student may see the symbol for this operation as simultaneously representing the process itself and the result of the operation.

The experimental instruction was based on this learning trajectory and covered the following five concepts and procedures:

- computing sine and cosine using the unit circle model;
- computing tangents using a Cartesian graph;
- computing sines, cosines, and tangents using right triangles;
- computing sines, cosines, and tangents using reference angles (based on the unit circle); and
- graphing the sine, cosine, and tangent functions.

Each concept/procedure was taught over two to three class periods. In the first class period, students worked in small groups to complete in-class activities. The following class periods consisted of a classroom discussion that summarised the results from these activities and a lecture that introduced procedural skills based on these activities. Both the activities class period and the lecture class period are described in more detail below.

Activities class period. In the activities period, students were asked to work in groups of three or four to complete in-class activities. The activities consisted of four parts:

1. Students were first shown how to execute a procedure to accomplish a specific trigonometric task (e.g., evaluate sine of a given angle). In each case, this procedure required the student to construct a geometric figure using a protractor and a ruler. The students were then required to measure parts of the figure that they constructed. Often these figures were constructed on graph paper with a unit circle drawn on x- and y-axes, where 10 "tics" on the graph paper constituted one unit. Frequently this obviated the need for students to use a ruler, as they were permitted to round to the nearest tenth. After students read instructions describing how to execute the procedure, each group of students was asked to apply the procedure once and then present their work to the instructor. The instructor would then verify that the students executed the procedure correctly, or demonstrate to the group of students where

they went astray. This was done to ensure that all groups of students could apply the procedure correctly before attempting the rest of the activities.

2. The students applied the procedure in a number of cases, usually about five or six. By repeated application of the procedure, it was hoped that students would begin to think about the procedure as a coherent whole and acquire the ability to estimate the result of the procedure without executing each of its steps. During this stage, the instructor circled the room and made certain that students were obtaining reasonable answers.
3. Students were asked questions that required them to reason about the result of the procedure without actually executing the procedure. For example, when learning the procedure for computing sines using the unit circle, students were asked the question, "Without going through the calculations, which number is bigger: sin 23° or sin 37°?" Here students would be expected to draw rough approximations of a 23° and a 37° reference angle on the Cartesian plane and see where the angles intersected the unit circle. The students would then need to notice that the 37° angle had a greater sine since its intersection with the unit circle was "higher up". Students were similarly asked: "Is sin 145° positive and why?" The goal of these activities was for students to realise that the procedure can be reasoned about without rigorously applying each of its steps.
4. Finally, students were asked to reason about the procedure itself. This was primarily accomplished in three ways. First, students were asked what properties the output of the process must have, regardless of its input. In the case of sine, one question could be, "explain why sin θ can never equal 2". Second, students were asked questions requiring them to reverse the procedure or to compose it with another procedure (e.g., "Describe how I can find a θ such that sin θ = 0.3?). Third, students were asked if they could use the nature of the procedure to justify mathematical laws (e.g., "Why is tan 90° undefined?").

An example of a classroom activity is shown in Appendix A. After this class session, the students were given homework assignments that asked them to further their investigations of these procedures.

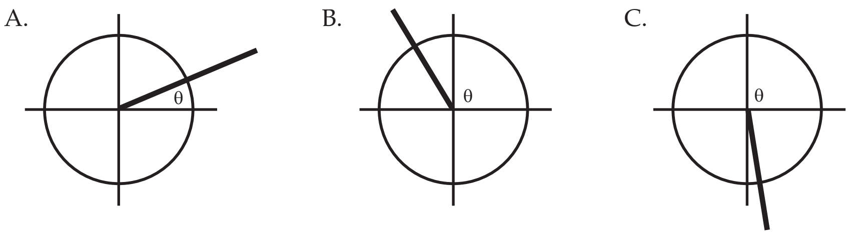
Classroom Discussion. In the following period(s), the instructor and students first discussed the activities that they had recently completed. Following this, the instructor gave a lecture in which he stated declarative facts (e.g., sec x is the reciprocal of cos x) and demonstrated procedural skills (e.g., after learning about computing sines by using right triangles, the instructor demonstrated how the sine of an angle can be computed if an appropriate right triangle is provided). After these lessons, the students were asked to complete standard homework exercises from the course textbook.

Post-test questions

The questions for the post-test are presented in Table 1.

Table 1
Post-test questions

1. Approximate a. sin 340° and b. cos 340°. Explain your work.
2. I draw three angles A, B, and C below. (Each angle is drawn as a standard reference angle. For instance, angle A is about 30°).



   a. Which of the three angles above has the greatest sine? Which has the least sine? (Note if your answers to these questions are A and C, you are saying that sin A > sin B > sin C)
   b. Which of the three angles has the greatest cosine? Which has the least cosine?
   c. Which of the three angles above has the greatest tangent? Which has the least tangent.
3. What is sin 270°? Why does sin 270° have this value? (You can describe this in terms of the unit circle, or anything else that you like).
4. For what values is sin x decreasing? Why is it decreasing for these values? (Please try and explain your answer in terms of the function of sin x, not in terms of a mnemonic device like "All Students Take Calculus").
5. Why does $\sin^2\theta + \cos^2\theta = 1$?

The first question on the post-test (Table 1) asked students to estimate the output of trigonometric functions. The ability to form reasonable estimates is frequently cited as evidence of understanding an operation while a student's failure to form reasonable estimates is evidence of not having a strong understanding of the operations being applied (e.g., NCTM, 2000, p. 32–33). One reason this question was included was to provide an indication of how students think about the sine and cosine function and whether they can view these functions as operations that are applied to angles.

The purpose of the last three questions was to see if students could recall (or derive) specific properties of trigonometric functions and justify why trigonometric functions have these properties. The ability to justify why concepts have the properties they do is evidence a concept is understood; a failure to form justifications for properties of concepts indicates that students may have memorised "rules without reasons" (cf., Skemp, 1987). In the theoretical perspectives section of this article, it was argued that understanding trigonometric functions as procepts is critical for justifying why they have the properties they do. Finally, the second question examined students' ability to link graphical representations of angles to values of their sines, cosines, and tangents. This ability, while important, was not central to the main goals of study reported in this article and is discussed only briefly in the results section.

Procedure

During the second class meeting, the students who received the experimental instruction, but not the students who received the standard instruction, completed the pre-test presented in Table 2. During the first six weeks of the course, both instructors covered sections 6.1 through 6.6 of the textbook. The topics covered included defining trigonometric functions in terms of right triangles, finding missing lengths of right triangles using trigonometric functions, reference angles, defining trigonometric functions in terms of the unit circle, and graphing trigonometric functions.

Table 2
Pre-test given prior to experimental instruction

What is the range of sin x? Why?
What is sin 90°? What is cos 270°? Why?
In what quadrants is cos x positive? Why?
In what quadrants is cos x increasing? Why?

After six weeks, both classes of students were asked to complete the post-test presented in Table 1. The test was not announced in advance and students were given 30 minutes to complete it. After students' responses to the test were coded, four students from each class were interviewed about their understanding of the sine function. When considering which students to interview, two factors were taken into account. First, of the four students chosen, one student's performance was exceptional (relative to his or her classmates), one student's performance was above average, one student's performance was below average, and one student's performance was poor. Second, these students' responses on the post-test were representative of responses of some of the other students in their class. The purpose of these interviews was to probe students' understanding of the sine function. The interviews were semi-structured and based around the following four questions:

1. Describe sin x for me in your own words.
2. Why is sin x a function? (If students did not know what a function was, they were told that an operation was a function if each input had a unique output).

3. What does the sentence, sin 40˚ = 0.635 mean to you?
4. What can you tell me about sin 170˚? Will this number be positive or negative? How do you know? Can you give me an approximation for this number?

Goals of analysis

The goal of this research study was to gain insight into students' understanding of trigonometric functions after being instructed in both a lecture-based and innovative classroom. To avoid misinterpretation, let me stress that the purpose of the study was not to demonstrate that my experimental instruction was superior to the instruction used in a typical trigonometry classroom. As Schoenfeld (2000) argued, demonstrating the superiority of one form of instruction over another is often not possible. In particular, the methodology used in this study would not permit me to make such a claim. For this reason, comparisons between the two groups of students are intentionally avoided. The most appropriate unit of analysis in this study was each of the two classes in its entirety. As such, the data presented in this article can essentially be viewed as two case studies. Hence, the results from the study are meant to be suggestive and illustrative; any conclusions drawn are tentative.

Results

Evaluation of the standard instruction

Performance on the post-test

Table 3 presents students' quantitative performance on the post-test. The data in Table 3 only distinguish whether students' responses were correct or incorrect, but do not examine students' justifications for their answers. A more detailed analysis of students' responses and justifications is provided below.

Table 3
Performance on the post-test questions of students who received standard instruction

Question	Correct student responses (N=31)
1a.*	5 (16%)
1b.*	6 (19%)
2a.	10 (32%)
2b.	6 (19%)
2c.	7 (23%)
3.	16 (52%)
4.	9 (29%)

* Responses to question 1a (approximate sin 340˚) were judged to be "correct" if they were between 0 and –0.5. Responses to 1b (approximate cos 340˚) were judged to be correct if they were between 0.5 and 1.

Question 1. Approximating a. sin 340° and b. cos 340°

When asked to approximate sin 340° and cos 340°, five and six students respectively gave responses that were coded as correct. In each case, three of the responses that were coded as correct had a specified range for the possible values of sin 340° and cos 340° (e.g., $-0.5 < \sin 340° < 0$) but did not give a specific number. Most of the students who did not give correct responses appeared not to know how to approach the task. Seven students drew a right triangle with angles of 20° and 70° and recognised that this was the reference triangle for an angle of 340°. However, since they were unable to approximate sin 20° and cos 20°, they could not make further progress. One other student, whose thought processes appeared to be indicative of these students, wrote, "I know sin = y/r and cos = x/r and if the angle had a common angle for a reference angle I could find the sin and cos but I don't know how to on this one." Three students drew a 340° angle in the Cartesian plane, but did not know how to proceed from there. Fifteen students either offered no answer at all, or wrote answers that were wildly off (e.g., sin 340° = 0.96 or cos 340° = $2\sqrt{3}/3$).

Question 3. What is sin 270° and why?

Seven of the 16 students who correctly stated that sin 270° = –1 justified this assertion by illustrating that the line along the lower-part of the y-axis intersected the unit circle at the point (0, –1). The other nine students did not offer satisfactory justifications.

Question 4. When is sinθ decreasing and why?

Of the nine students who answered this question correctly, six offered convincing explanations, using a unit circle diagram to show how as one proceeded counter-clockwise around the circle, the y-values of the circle's points decreased when θ was between 90° and 270°. The other three did not offer justifications.

Question 5. Why is $\sin^2\theta + \cos^2\theta = 1$?

Only four students offered mathematically valid explanations for why this identity was true. All did so by drawing an arbitrary right triangle and used the Pythagorean theorem to establish this statement. Five other students justified this identity by looking at specific cases (e.g., $\sin^2\theta + \cos^2\theta = 1$ because $\sin^2 90° + \cos^2 90° = 1^2 + 0^2 = 1$). The remaining 22 students did not provide a justification.

Student interviews

Describe sin x for me in your own words.

Two of the four students interviewed described the sine function as ratios of given right triangles. Marcie offered the following response:

I: Describe sin x for me in your own words.

Marcie: I think of sine as opposite over hypotenuse or y over r. To find sine, you look at a right triangle. If sine is a bigger value [sic], you find its reference angle and take the sine of that. And the answer will either be that or negative of that, depending what quadrant your angle is in.

I: OK, when you take… I should say when you want to find sine of θ, you look at a right triangle with an angle θ and divide the length of the opposite with the length of the hypotenuse?

Marcie: Yes.

I: How do you get this triangle so you can divide?

Marcie: That's a good question [laughs]… I don't know. That triangle has always been given to us, unless we were looking at a special angle.

The other two students described the sine function in terms of multiple algorithms. These algorithms all involved algebraic calculations, and were seemingly unrelated to one another. The response of Steve is given below:

I: Describe sin x for me in your own words.

Steve: To find sine, it would depend on the problem that was given to me. If I was given a triangle, I would divide y and r. If I were given one of the special angles, like 30°, 45°, or 60°, I would have this number memorised. There are other problems which can be solved by reference angles, or using formulas, like sin θ + cos θ = 1 [sic]. How you find the answer depends on how the problem is worded.

Why is sin x a function?

Perhaps not surprisingly, none of the four students knew the definition of function. For each student, the interviewer rephrased the question to be, "how do you know that sin x can only have one value for a given x?" No student provided a meaningful answer to this question. Two students expressed certainty that sin x was a function, but could not explain why.

What does the sentence, sin 40° = 0.635 mean to you?

Erin offered the following response:

Erin: Well it means that the sine of 40 is equal to .635 and … it means to me that it's just the one output you are going to get for the sine of 40. It's basically just the equation that sin 40 equals 0.635.

From her use of the word "output", it seems as if she may have been aware that 0.635 was the result of a mathematical process. However, when the interviewer asked her to clarify what she had meant, she was unable to elaborate on her response. The other three students responded by phrasing the sentence in a different way (e.g., "if you take the sine of 40 degrees, you will get 0.635") and by saying "it's just an equation". When considering the sentence sin 40° = 0.635, no student alluded to a process for computing a sine.

What can you tell me about sin 170°? Can you give me an approximation for this number?

Erin was able to use reference angles to make a reasonable approximation for sin 170°, but was unable to justify why her estimation was reasonable. Her response is given below.

Erin: So let's see. By reference angles, what we're looking for will um… be sin 10° and it will be positive since 170 is in quadrant two. sin 10°? I'm sorry, I don't know how we could find what that would be.

I: You don't need to give me an exact value. All I want you to do is to give me your best approximation.

Erin: [pause]… I would guess about 0.2.

I: That is a good guess. Why did you choose that number?

Erin: I know that it is between 0 and .5, because that is the sin 0° and the sin 30°.

I: How did you know that sin 10° would be positive?

Erin: I think of it like, "All Students Take Calculus", to tell me when sine, cosine, and tangent will be positive… you know what I mean?

I: Yes, I'm familiar with that. Could you explain why that expression works? How do you know that sines are always positive in quadrant two?

Erin: Oh, [The teacher] explained that to us in class, but right now I can't remember what he said.

The other three students were unable to make progress on this question. From their responses, these students appeared to argue that they were given insufficient information to answer this question. Steve's representative response is given below:

Steve: I know that sine is y over r. I would need to see a triangle and know what y and r are to answer this question.

I: Even if you didn't have this triangle, could you try to make a guess at what sin 170° would be?

Steve: I don't… I would need the triangle. Maybe if you told me what some other value of sin would be, or like what cos 170° would be, I could find what sine is. Otherwise, I would need to use a calculator.

Discussion

The students who received the standard instruction did not appear to develop a strong understanding of trigonometric functions. These students were generally unable to justify why trigonometric functions had the properties that they did and they were unable to form reasonable estimates for what the output of a trigonometric function should be. Students' interviews illustrated that they did not understand trigonometric expressions as procepts. These interviews further suggested two other ways in which their understandings were limited.

The first limitation in students' understanding concerns the role that geometric figures played in their understanding of these functions. Clearly relating trigonometric functions to appropriate geometric models is important for understanding these functions. To the students in this study, however, it appeared that trigonometric functions could not exist independently from their geometric models. When asked to approximate $\sin \theta$ for specific values of θ, many students argued that they were not given sufficient information to complete this task; some argued that the task could only be completed if they were given an appropriately labelled triangle. What these students seemed to lack was the ability or inclination to mentally or physically construct geometric objects to help them deal with trigonometric situations.

The second limitation concerned the level of control that students felt they had when operating with the sine function. Breidenbach et al. (1992) distinguished between two ways that an operation could be perceived. They said that an operation is external to an individual if the individual finds the operation to be a mechanical step-by-step prescription that is relatively meaningless to them and can only be applied in response to an external cue. An operation was internal to an individual if the individual felt control over the operation. Such an individual would view the operation as a means of accomplishing a mathematical goal and would be capable of applying the operation flexibly as a problem-solving tool. The sine operation appeared to be external to the undergraduates in this study. None of the undergraduates interviewed spoke of sine as a meaningful or goal-directed process, but as a step-by-step prescription (or in some cases, a set of prescriptions) that were to be applied to an external cue. When asked to describe $\sin x$, some students spoke of sine in terms of "finding an answer" and described how cues in problem situations would dictate which sine algorithm they would apply. In some cases, the problem situations required a labelled triangle with numbers provided for them. In the absence of such a diagram, many students did not know how to proceed.

Evaluation of experimental instruction

Quantitative data

The students' performance on the pre-test was generally poor. They collectively answered only 21 (out of a possible 160) questions correctly and no student was able to justify a single correct answer. All justifications, when they were offered at all, relied either upon a mnemonic device or responses such as "it's just a mathematical law". Twenty-six students answered every question on the pre-test incorrectly. The analysis that follows focuses on the performance of two groups: the students as a whole and the subset of the students who scored a zero on the pre-test. The latter group was included to determine whether any observed effects were due to students in the experimental group entering the course with prior knowledge of trigonometry.

Table 4 presents both groups' quantitative performance on the post-test. The data in Table 4 only distinguish whether students' responses were correct or incorrect, but do not examine students' justifications for their answers. A more detailed analysis of students' responses and justifications is provided below.

Table 4
Performance of students who received experimental instruction on the post-test questions

Question	Students as a whole (N=40)	Students who scored zero on the pre-test (N=26)
1a.*	32 (80%)	20 (77%)
1b.*	37 (93%)	23 (88%)
2a.	35 (88%)	21 (81%)
2b.	34 (85%)	20 (77%)
2c.	23 (58%)	17 (65%)
3.	36 (90%)	23 (88%)
4.	34 (85%)	23 (88%)

* Responses to question 1a (approximate sin 340˚)were judged to be "correct" if they were between 0 and –0.5. Responses to 1b (approximate cos 340˚)were judged to be correct if they were between 0.5 and 1.

Question 1. Approximating a. sin 340˚ and b. cos 340˚

Nearly all students made their approximations using a unit circle. To approximate sin 340˚ and cos 340˚, most students drew a rough sketch of a Cartesian plane, a unit circle, and a 340˚ angle and then approximated the x- and y-coordinates of the intersection.

Question 3. What is sin 270˚ and why?

Of the 36 students who correctly stated that sin 270˚ = –1, 34 gave a valid mathematical justification explaining why this was so. All justifications could be described as typical explanations given in terms of the unit circle.

Question 4. When is sin θ decreasing and why?

Thirty-four students correctly answered that sin θ decreased when $90^\circ < \theta < 270^\circ$. Thirty-two of these students respectively offered adequate explanations of why the sine function was decreasing for these values. Most explanations referred to a diagram of the unit circle illustrating how when one moves counter-clockwise from the top to the bottom of the circle, the y-values of the points on the circle decrease.

Question 5. Why is $\sin^2\theta + \cos^2\theta = 1$?

Fifteen students gave a mathematically valid explanation for why this identity was true. Thirteen students showed that this was true using a right triangle model, while two students pointed out that (cos θ, sin θ) was on the unit circle, thus ensuring that the identity would be true. Five students verified the identity for only a single case (e.g., $\sin^2 90^\circ + \cos^2 90^\circ = 1^2 + 0^2 = 1$). The other 20 students offered no response.

Student interviews

Describe sin x for me in your own words.

Of the four students who were interviewed, all described sin x in terms of a procedure that was shown to them in class. Three students described sin x in terms of a geometric construction using the unit circle. John's response is presented below.

I: Describe sin x for me in your own words.

John: Well, on the unit circle, its uh… in direct relation to the y-value, if you draw an angle on the unit circle, it will be the y-value.

I: OK, can you describe it for me in more detail?

John: Such as, where the angle… if you draw the sine of an angle, where it crosses the unit circle, the y-value is equal to the sine of that unit circle. I mean, the y-value where the angle crosses the unit circle is equal to the sine of that angle.

The remaining student, Ben, described sin x in terms of right triangles.

Ben: sin x is y over r, where y is the length of the height of a right triangle and r is the length of the hypotenuse.

I: How do you get this right triangle and the lengths so you can divide y by r?

Ben: Um, like we did in class, I guess. We can draw the right triangle. Basically, you can start with the angle that you are working with, make a right triangle out of it, and measure the lengths of its sides.

Hence all students associated the sine function with a process that they could perform for any given angle.

Why is sin x a function?

None of the four students knew what constituted a function. In each case, the interviewer informed them that an operation was a function if each input had a unique output. After this explanation, three of the interviewed students explained that sin x was a function because the geometric construction used to compute sin x would always yield only one answer. (Ben, the weakest student of the four, was unable to answer this question). John's response, which was typical, is presented below.

John: [pause]… uh… cause [laughs]… it just is. I don't know why, it just is.

I: Maybe this will help. A function is an operation so that for every input you give it, the output will be unique.

John: Because if you, no matter what you put in for the sin x, you only get one answer. It couldn't be positive or negative, it can only be one or the other. You can't get two answers, like a positive or negative number.

I: Why will that be the case?

John: Because for uh each angle, there's… going back to the unit circle, if you put something in for the sin x, its only gonna cross at one point. Each angle is going to be one angle, and that one angle is going to cross the unit circle at one point. That one point is going to have a y-value. It will have one and only one y-value.

Hence three of the students could use the nature of the process of the sine function to justify one of its important properties. It is interesting to note that Blackett and Tall (1991) were concerned that students who viewed trigonometric operations as constructions may fail to distinguish between approximate and exact answers. Based both on my experience in the classroom and the responses above, this did not seem to be a difficulty for most of my students.

What does the sentence, sin 40˚ = 0.635 mean to you?

Three of the interviewed students indicated that the above sentence expressed the fact that 0.635 was the result of applying a geometric construction. Two representative responses are given below:

John: It says that, when you draw an angle of 40º, if you take the sin of 40º, if you draw an angle of 40º and see where this angle intersects the unit circle, this crossing will have a y-value of 0.635.

Erica: The y-value of the point where a 40˚ line intersects the unit circle equals 0.635.

These responses indicated that these students were able to think of sin x as a procept, that is, as representing both a geometric construction and the result of that geometric construction. The remaining student, Ben, responded by reading the equation out loud, and when asked to elaborate, was unable to do so.

What can you tell me about sin 170˚?
Can you give me an approximation for this number?

Each interviewed student estimated that sin 170˚ was a number between 0.1 and 0.4 and was able to justify why his or her estimation was reasonable by imagining performing appropriate geometric constructions on a unit circle. It should be noted that these students had practiced completing this type of task in their classroom exercises. One representative response is given below.

Erica: The answer would, oh, be, I'd say, 0.1.

I: That is a good guess. How did you get that answer?

Erica: I pictured making a 170˚ angle with a protractor and seeing where the angle intersected the circle.

I: I see. And how did you know it would intersect at 0.1?

Erica: [Drawing a diagram] Well, it would intersect right there [pointing to the point of intersection].

These students were able to use their understanding of sin x as a process to estimate sin 170˚ and justify why it had some of the properties that it did. In Erica's case, she was able to do so mentally, and could anticipate the result of the sine function without physically applying any of its steps.

Discussion

The results of the paper-and-pencil test indicate that most of the students in this teaching experiment were able to approximate the values of basic trigonometric expressions, determine properties of trigonometric functions, and justify why these functions have the properties that they do. The interviews with students suggest that they thought of trigonometric functions as meaningful processes and that they were able to use this conception of trigonometric functions to accomplish the previously listed tasks.

Students' successful performance on the post-test appeared to be due, in part, to their propensity to reason about trigonometric functions using a unit circle model. I do not believe that all forms of trigonometry that stress the unit circle model will lead to successful student learning. On the contrary, Kendal and Stacey (1997) found that students who were taught trigonometry using a unit circle model learned less than students who were taught using a right triangle model. For the students who received the experimental instruction in my study, understanding the *process* used to create unit circle representations of trigonometric expressions appeared to be an integral part of their understanding of these functions.

Conclusions

The students who received the standard instruction performed poorly on the post-test and the students' responses to the interview questions indicated that they did not have a strong understanding of trigonometric functions. By itself, these results *suggest* that the type of instruction that these students received may be ineffective for developing students' understandings of trigonometric functions. The data reported here are consistent with other studies reported in the research literature. For instance, Blackett and Tall (1991) found that high school students in a controlled trigonometry classroom had difficulty completing many basic tasks. Kendal and Stacey (1997) gave 178 high school students a trigonometry test one year after they completed a unit in which they studied trigonometry; 172 of these students scored a zero on this test. Parish and Ludwig (1994) claimed that students frequently make algebraic errors that indicate that they have little conceptual understanding of trigonometric functions. In short, it appears that the performance of the students who received the standard instruction was not uncommon.

The students who received the experimental instruction were able to demonstrate a strong understanding of trigonometric functions. Most of the students were able to use their understanding of trigonometric functions to recall and derive their properties and justify why they have the properties they do. The interviewed students were able to articulate the process of computing a sine and were able to use that process to approximate the value of trigonometric expressions. They were also able to use their understanding of that process to form justifications for why the sine function had the property it did. Further, the interviewed students explained that the expression $\sin 40^\circ = 0.645$ implied that 0.645 was the result of applying the sine process to 40°. This indicates that these students viewed trigonometric expressions as procepts. The data indicate that the style of instruction described in this article can be effective and, more generally, that students can construct a strong understanding of trigonometric functions within a six week period. Investigating whether similar forms of instruction can be effective in high school classrooms would be an interesting topic for future research.

Acknowledgments

I would like to thank Arthur Powell, Suzanne Reynolds, Iuliana Radu, Elizabeth Uptegrove, and the reviewers for helpful comments on earlier drafts of this paper.

References

Barnes, J. A. (1999). Creative writing in trigonometry. *Mathematics Teacher, 92*(6), 498–503.

Blackett, N., & Tall, D. O. (1991). Gender and the versatile learning of trigonometry using computer software. In F. Furinghetti (Ed.), *Proceedings of the 15th conference of the International Group for the Psychology of Mathematics Education* (Vol.1, pp. 144–151). Assisi, Italy: PME.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*, 247–285.

Davis, R. B. (1992). Understanding "understanding". *Journal of Mathematical Behavior, 11*, 225–241.

Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study.* (pp. 273–280). Dordrecht, The Netherlands: Kluwer.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A proceptual view of elementary arithmetic. *Journal for Research in Mathematics Education, 26*(2), 114–141.

Hirsch, C. R., Weinhold, M., & Nichols, C. (1991). Trigonometry today. *Mathematics Teacher, 84*(2), 98–106.

Kendal, M., & Stacey, K. (1997). Teaching trigonometry. *Vinculum, 34*(1), 4–8.

Lial, M. L., Hornsby, J., & Schneider, D. I. (2001). *College algebra and trigonometry*. Menlo Park, CA: Addison Wellesley.

Miller, S. (2001). Understanding transformations of periodic functions through art. *Mathematics Teacher, 94*(8), 632–635.

National Council of Teachers of Mathematics. [NCTM] (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics. [NCTM] (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Parish, C. R., & Ludwig, H. J. (1994). Language, intellectual structures, and common mathematical errors: A call for research. *School Science and Mathematics, 94*(5), 235–239.

Searl, J. (1998). Practical activities. *Mathematics in School, 27*(2), 30–31.

Schoenfeld, A. (2000). Purposes and methods of research in mathematics education. *Notices of the AMS, 47*, 641–649.

Skemp, R. R. (1987). *The psychology of learning mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Tall, D. O., Thomas, M., Davis, G., Gray, E., & Simpson, A. (2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior, 18*(2), 1–19.

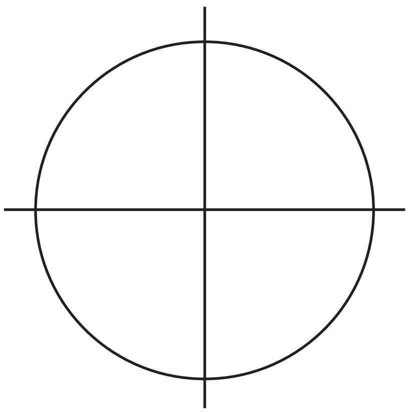
Author

Keith Weber, Rutgers University, 10 Seminary Place, Graduate School of Education, New Brunswick, NJ 08901, USA. Email: khweber@rci.rutgers.edu

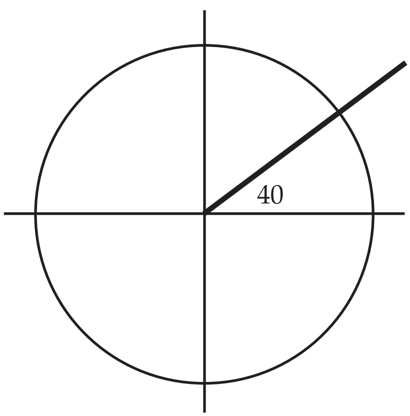
Appendix A.

Activity to learn the process of computing sines and cosines using the unit circle

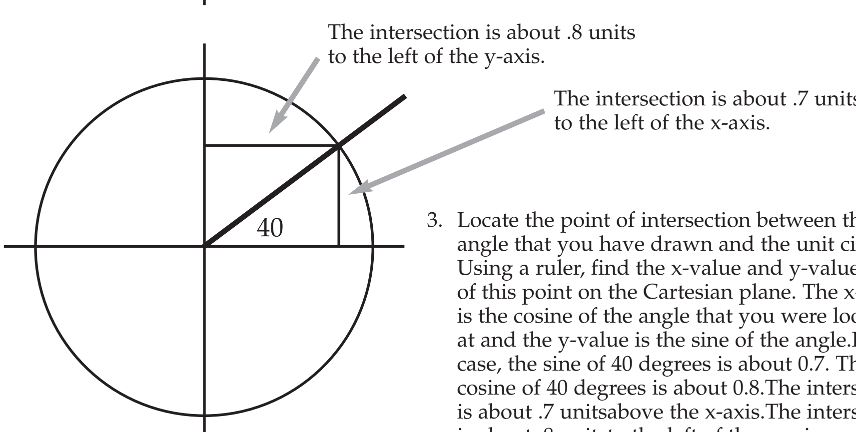
1.1 How to compute sines and cosines



1. Start with a unit circle drawn on a Cartesian graph. A unit circle is a circle with a radius of 1 and a center at the origin.



2. Use your protractor to make an angle with respect to the positive part of the x-axis. A 40 degree angle is drawn on the right.



3. Locate the point of intersection between the angle that you have drawn and the unit circle. Using a ruler, find the x-value and y-value of this point on the Cartesian plane. The x-value is the cosine of the angle that you were looking at and the y-value is the sine of the angle.In this case, the sine of 40 degrees is about 0.7. The cosine of 40 degrees is about 0.8.The intersection is about .7 unitsabove the x-axis.The intersection is about .8 unitsto the left of the y-axis.

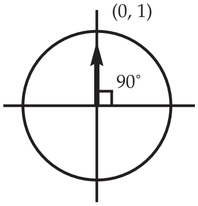
1.2 Classroom exercises

1. Compute the following sines and cosines using a protractor and unit circle:

   a. sin 30° and cos 30°
   b. sin 170° and cos 170°
   c. sin 120° and cos 120°
   d. sin 260° and cos 260°
   e. sin 80° and cos 80°
   f. sin 325° and cos 325°

2. Compute the following sines and cosines using a protractor and unit circle:

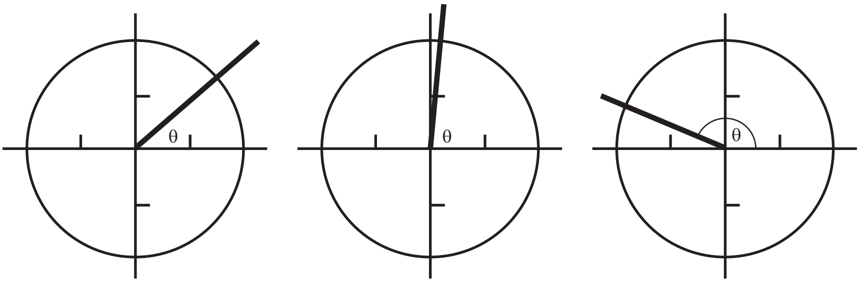
   a. Find sin 90°

   A 90° is a right angle. I draw a right angle inside a unit circle. It intersects the circle at the top of the circle. This point is (0, 1). So cos 90° is 0 (it is not to the left or the right of the y-axis) and sin 90° is 1 (the point of intersection is 1 unit above the x-axis).

   b. Find sin 0° and cos 0°
   c. sin 180° and cos 180°
   d. Find sin 270° and cos 270°
   e. sin 360° and cos 360°

3. Approximate the sine and cosine of the θ angle in the diagram below:



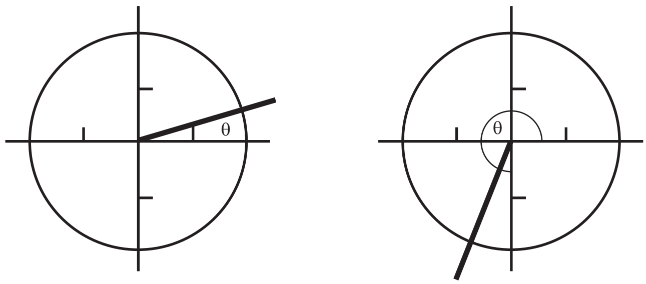
4. Without doing computations, answer the following questions. Justify your answer:

   a. Is sin 140° a positive number or a negative number? (*Hint:* Draw a unit circle and approximate a 140° angle.)
   b. Is cos 200° a positive or negative number?
   c. Which number is bigger: sin 23° or sin 37°?
   d. Which number is bigger: cos 300° or cos 330°?

1.3 Homework exercises

1. Compute the following sines and cosines using a protractor and unit circle:
   a. sin 50° and cos 50°
   b. sin 127° and cos 127°
   c. sin 200° and cos 200°
   d. sin 300° and cos 300°

2. Approximate the sine and cosine of the angle θ in the diagrams below:



3. Without doing computations, answer the following questions. Justify your answer:
   a. Is sin 240° a positive number or a negative number? (*Hint:* Draw a unit circle and approximate a 140° angle.)
   b. Is cos 300° a positive or negative number?
   c. Which number is bigger: sin 130° or sin 147°?
   d. Which number is bigger: cos 30° or cos 230°?

4. In what quadrants will sin θ be positive? In what quadrants will cos θ be positive?