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A Developmental Scale of Mental Computation with Part-Whole Numbers

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In this article, data from a study of the mental computation competence of students in grades 3 to 10 are presented. Students responded to mental computation items, presented orally, that included operations applied to fractions, decimals and percents. The data were analysed using Rasch modelling techniques, and a six-level hierarchy of part-whole computation was identified. This hierarchy is described in terms of the three different representations of part-whole reasoning – fraction, decimal, and percent – and is elaborated by a consideration of the likely cognitive demands of the items. Discussion includes reasons for the relative difficulties of the items, performance across grades and directions for future research.

Introduction

This article brings together two areas of mathematics education research that individually have received considerable attention but that together have been virtually ignored: mental computation and part-whole numbers, specifically fractions, decimals, and percents. Mental computation research has focused on the four operations with whole numbers, whereas part-whole number research has focused on conceptual understanding, concrete models, students' strategies, and teaching intervention in the classroom. This study sought to provide a developmental scale of increasing proficiency in dealing with part-whole number operations in the context of performing mental computations.

One of the justifications for interest in this study comes from the increasing attention given to proportional reasoning based on part-whole numbers at the middle school level. Traditional links between proportional thinking and other parts of the mathematics curriculum have included measurement, similarity in geometry, trigonometry, and basic probability. More recently the links to chance and data have been highlighted in terms of the relationship of samples to populations (Saldanha & Thompson, 2002) and in the context of probability sampling and comparing data sets of different sizes (Watson & Shaughnessy, 2004). Further, the focus on quantitative literacy across a more eclectic school curriculum with higher level numeracy requirements (e.g., Madison & Steen, 2003) leads to specific statements of the need to apply proportional reasoning and calculations, often undertaken mentally, based on fractions, decimals and percents in many social and life-skills contexts (Steen, 2001; Watson, 2004).

The prominence of mental computation has waxed and waned in the school curriculum over the past 150 years. The advent of calculators initiated debate about the necessity to have mental computation skills and even brought into question the necessity for some of the paper-and-pencil algorithms that flourished throughout the 20th century (McIntosh, 1990). By 2000, however, a balanced view on computation was advocated in the National Council of Teachers of Mathematics' [NCTM] *Principles and Standards* (NCTM, 2000). Computing fluently is a goal at all grades, and the methods to be employed include mental computation, estimation, calculators or computers, and paper-and-pencil. Similarly in Australia, the Australian Education Council's [AEC] curriculum profile for mathematics (AEC,1994) includes mental computation as one of its seven strand organisers for the Number Strand, and this provides a second justification for the study.

Mental computation with part-whole numbers is not ignored in curriculum documents. In the NCTM's *Principles and Standards* (NCTM, 2000), for example, the edict concerning choice among methods to compute fluently with fractions and decimals appears in the Number and Operations standard for grades 6-8. In the Australian profile (AEC, 1994) under Mental Computation, simple fractions appear at level five; at level six students should "estimate and calculate mentally with whole and fractional numbers, including finding frequently used fractions and percentages of amounts" (p. 104). There is, at present, no detailed research base on which to decide whether these expectations are reasonable, and this presents a third justification for the study.

Previous Research

The growing initial interest in mental computation has focused on addition and subtraction of whole numbers (e.g., Heirdsfield, 2001; Hope, 1986; Reys, Reys, & Hope, 1993), followed by multiplication and division of whole numbers (e.g., Heirdsfield, Cooper, Mulligan, & Irons, 1999). As the numbers involved become larger, issues related to place value become more significant, as does the capacity to hold information in the mind (Callingham & McIntosh, 2002). Hopkins and Lawson (2004) have shown that the time taken to retrieve an answer to addition problems, as opposed to computing the answer, gets longer as the problem size increases. Strategies used during mental computation with whole numbers have been documented (e.g., McIntosh, De Nardi, & Swan, 1994; Threlfall, 2000), as have errors that occur during the process (McIntosh, 2002). Research, however, has been slower to focus on fractions, decimals, and percents. Caney and Watson (2003) considered the successful strategies of 24 students interviewed while solving part-whole number operations mentally. They found parallels to the strategies described for whole numbers by McIntosh et al. (1994) and provided examples illustrating the strategies employed.

Research into proportional reasoning more generally, related to the use of fractions, decimals, and percents, has been on the mathematics education agenda for a long time. An excellent recent summary of work principally with fractions is provided by Thompson and Saldanha (2003). Bana, Farrell and McIntosh (1997) considered test items reflecting fraction and decimal concepts rather than computation. They assessed number sense in terms of distinguishing larger from smaller numbers, part from whole, and top from bottom in fractions. The connections between proportional reasoning and ratios were illustrated in the developmental work of Noelting (1980a, 1980b) with consequent links to concepts in probability contexts (Shaughnessy, 2003; Watson, Collis, & Moritz, 1997). Building conceptual understanding associated with proportional reasoning has been the focus of many studies across contexts that constantly reflect back to the requirement for multiplicative rather than additive features (Cobb, 1999; Harel & Confrey, 1994; Thompson & Saldanha, 2002). Related to this are the difficulties associated with decimal representations, both on their own (e.g., Stacey & Steinle, 1998) and in association with fractions (e.g., Watson, Collis, & Campbell, 1995). Part of the focus of this body of work has been on conceptualising the representation of fractions, decimals and percents, and part has been on performing operations on quantities represented in these fashions. In following students' difficulties, however, there has been an exploration of student explanation through interviews and through responses to paper-and-pencil instruments (e.g., Steinle & Stacey, 2002) but not on mental computation specifically. Although closely related to fractions and decimals, working with percents does not seem to have attracted as much research attention (e.g., Lembke & Reys, 1994). Dole (1999) suggested use of a proportional number line as an appropriate computation strategy for percent, but again not in relation to mental computation.

The interest in part-whole number operations in this study grew out of a research project on mental computation more generally, which included the four operations on whole numbers. A particular interest was in establishing evidence concerning the hierarchical nature of performance in relation to fractions, decimals, and percents. The fact that curriculum documents (e.g., AEC, 1994) imply a developmental progression in difficulty in referring to mental computation with different types of numbers, reinforces the desire to document this progression. The purposes of this article are:

- To describe a developmental scale of part-whole number mental computation proficiency identified from the responses to the relevant items in the instruments used in the study;
- To report performance against the scale across grades 3 to 10; and
- To begin to explain the relative difficulties of items across the three representations of part-whole numbers, fraction, decimal, and percent.

Methodology

Instruments and Procedures

Students undertook tests of mental computation that included whole numbers as well as part-whole numbers: fractions, decimals, and percents. These latter items were restricted to commonly used equivalents, such as half, quarter, and third, and single operations including *of*, *percent of* and the four key operations of addition, subtraction, multiplication, and division. More complex numbers, such as mixed fractions and fractions expressed as ratios, decimals to more than two places, and percents that could not easily be expressed as fraction equivalents were not included in the instruments, since it was considered that these lay outside what could reasonably be expected to be undertaken mentally. The language used in the tests was standardised: the addition operation was always expressed as *add*, subtraction was *subtract*, multiplication was *times*, and division was *divided by*. Decimals were given as *zero point five …* fractions as their usual fraction names. Thus the item '$\frac{1}{2}$ of 0.7' was spoken as 'one half of zero point seven'.

The items were organised into tests that had overlapping items so that they could be linked using Rasch (1960/1980) modelling techniques (Kolen, 1999). Over the three-year period of the project, six tests were developed for each adjacent year group: grades 3/4, grades 5/6, grades 7/8, and grades 9/10. Each test contained whole number computations, as well as some involving fractions, decimals, and percents. The part-whole items of fractions, decimals and percents only are the focus of this report.

Teachers administered the tests in their own classrooms, using a CD that could be used in a standard CD player. The CD included all instructions to the students to ensure a standardised testing environment. Students responded by writing their answers only in the appropriate place on the answer sheet; no paper and pencil calculations were allowed. Items were of two kinds: *short* items had a three-second delay and *long* items a 15-second delay to allow for students' responses. Most of the fraction, decimal, and percent items were *long* items (Callingham & McIntosh, 2001, 2002).

Sample

The student sample was taken from the responses of students in grades 3 to 10 who were in schools that had a focus on developing mental computation competence. Schools were from the Tasmanian Government and Catholic sectors and the ACT Government sector, and included primary schools (grades K to 6), high schools (grades 7 to 10) and composite schools (grades K to 10). The sample distribution across grades for the subsample of students used in this study is shown in Table 1. The number of students was lower in grades 3 and 4, and these students were only presented with a limited number of fractions items. Although this was a convenience sample of students associated with a large collaborative industry research project, the size of the sample and its representation of government and private sectors in two geographical locations lend credence to the belief that it is generally representative of students in grades 3 to 10 across Australia.

Table 1
Sample Distribution Across Grades

Grade	n	Percentage of sample
3	381	6.88
4	342	6.18
5	800	14.45
6	879	15.88
7	940	16.98
8	867	15.66
9	736	13.30
10	590	10.66
Total	**5535**	**100.00**

Data Collection

Data were collected at three different points in time but, for the purposes of this study, the data have been combined into a single data set of 7235 responses to 374 different items. When students had not undertaken any of the fraction, decimal or percent items, an ability measure of their performance could not be estimated. These students were eliminated from the data set, as were the students for whom there were missing background data, such as grade level. The final sample, as shown in Table 1, consisted of 5535 valid responses to 122 items addressing only computations involving fractions, decimals, and percents.

If students had been presented with the items, missing data were coded as incorrect; if the items had not been presented, data were coded as missing. Data were entered into a spreadsheet to allow for error analysis to be undertaken. The items were scored dichotomously, using a 1/0 score for correct/incorrect, and these data were subsequently used for Rasch analysis.

Data Analysis

Rasch (1960/1980) modelling was used to place all items and all students onto the same measurement scale using *Quest v2.1* Rasch modelling software (Adams & Khoo, 1996). Callingham and McIntosh (2001) demonstrated that Rasch modelling was an appropriate technique to use to develop a scale of mental computation competence because the characteristics of the Rasch model allow a large number of items to be used in several tests that can be linked to bring all items, and the students who undertake them, into alignment on a single measurement scale. The scale so produced provides a means of describing progress against the underlying construct. Rasch measurement has been used extensively to describe developmental levels (Wilson, 1992, 1999), and the progress that students make along the scale is concomitant with their development of competence (Bond & Fox, 2001). The measure of proficiency thus obtained is termed the *ability estimate*. In this study the target construct is proficiency at mental computation with part-whole numbers. The items are placed on the same scale based on their measured difficulty. Both ability and difficulty measures are expressed in logits, the logarithm of the odds of success.

The scale produced by the *Quest v2.1* Rasch modelling software (Adams & Khoo, 1996) is presented as a *variable map*. Figure 1 shows a diagram of a variable map of 20 items presented to 50 persons to illustrate the components of the map. Logit values are shown on the extreme left hand side of the map. Each person is shown by X, and the position on the scale indicates the ability measure. Items are shown at the right hand side of the display, clustered by measured difficulty level. Items are distributed about a mean of 0 logits. It can be seen in Figure 1 that there are discontinuities, or jumps in difficulty, between some items, such as that between It4 (Item 4) and the two higher items, It7 and It11. These discontinuities indicate some change in the demands of the items, and can be used to identify groups of items that form clusters or bands along the variable of similar cognitive complexity. By identifying groupings of items that have similar demands or complexity, a profile of development can be identified (Bond & Fox, 2001; Griffin, 1990). This process is used in many Australian state tests to indicate development in mathematics and literacy (e.g., NSW Department of Education and Training, 2002).

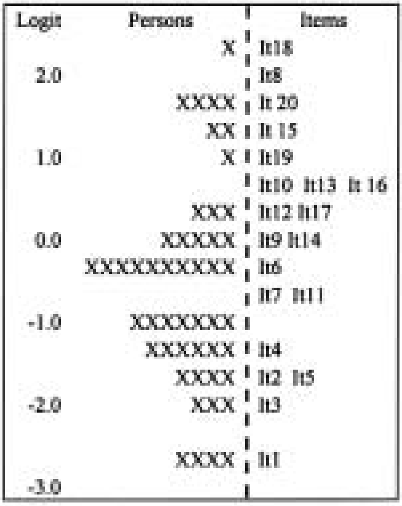


Figure 1. Components of a *variable map*.

When Rasch (1960/1980) analysis of the data set was undertaken, the overlapping items among the different instruments used allowed all tests to be equated and calibrated in a single operation (Kolen, 1999). A single analysis, hence, linked all tests together and established the difficulty estimates of each item, and the ability estimates of each student. The approach provided a large item pool in which every item was connected to every other item (Linacre, 1999). In this way, every item addressing fraction, decimal, or percent computation was placed on a scale, and students' performances on these items were converted into the same metric and placed on the same scale. The unit of measure used was the logit, the natural logarithm of the odds of success (Adams & Khoo, 1996). The model was evaluated by consideration of infit mean square values (that is, the mean squared differences between observed and expected values weighted by the variance) for both items and persons. Generally accepted levels of fit lie between 0.77 and 1.3 logits (Keeves & Alagumalai, 1999) and have an ideal value of 1.0 logit. In addition, item and case separation reliabilities (Wright & Masters, 1982) were used to provide a measure of the internal consistency of the scale. These have a theoretical value of 1.00 and values less than approximately 0.6 are generally considered low. These statistics were available from the Quest output.

Results

Model Evaluation

When the summary infit mean square values were considered, for all items ($IMSQ_1$ = 1.01) and for all persons ($IMSQ_p$ = 0.97), fit values were within the specified limits, indicating that the items worked together consistently to define a unidimensional scale of part-whole reasoning, and that the students' responses were also consistent. Item and person separation reliabilities were high (R_1 = 1.00; R_p = 0.82), indicating that the items produced a reliable measure of students' performance of mental computation of fraction, decimal, and percent problems.

The scale produced by the *Quest* software is shown as a variable map in Figure 2. The map shows the distribution of items on the right hand side and persons on the left. For convenience, in this map where the operation is 'divided by', the operation is shown as d, and fractions are shown with $\frac{a}{b}$. Thus '$\frac{3}{4}$ d $\frac{1}{4}$' represents the item presented as 'three quarters divided by one quarter'. Items shown in boldface type were *short* items; all others were long items.

The horizontal lines on the map indicate the points at which there appeared to be some change in the cognitive demands of the items for a correct response. These points were identified by agreement between the authors by considering the content of the items, the skills required to compute the items successfully, and apparent discontinuities in the difficulty levels. It should be noted that, because of printing limitations on the display, these discontinuities do not always appear clearly on the map. As an example, consider the items $\frac{1}{2} + \frac{4}{8}$ (Difficulty = -0.13 logit) and $1 - \frac{1}{3}$ (Difficulty = -0.20 logit) that appear at the boundary between Level 3 and Level 4. The difference in difficulty between these two items was small but greater than the differences between each of these items and the items next to them in the overall scale (e.g., 0.24 + 0.76, Difficulty = -0.11; 0.1 x 10, Difficulty = -0.22). When the content was considered, although these were both fraction items, the easier item, $1 - \frac{1}{3}$, required recognition only of unit fractions and one-whole, and could be completed in a single step, whereas $\frac{1}{2} + \frac{4}{8}$ required recognition of a simple non-unit fraction that was equivalent to one-half, and the addition operation of ‘one half add one half’. Each item where a discontinuity was identified was similarly considered, and the ultimate placement of the horizontal lines was agreed to by the two authors. The six groupings of items thus identified, as shown in Figure 2, were labelled Level A to Level F for convenience.

Figure 2. Variable map showing all fraction, decimal, and percent items.

Description of Identified Levels

As can be seen in the variable map in Figure 2, the only items appearing at Level A involve the fraction one-half. The fraction 'half' has always had a special place in the world of numbers. Hart (1981) gave it the title of "honorary whole number" (p. 216), and Thornton (1985) considered the halving operation to be so fundamental that it could be considered as an operation beside addition, subtraction, multiplication, and division. It is of interest that both uses of half, as number ($\frac{1}{2} + \frac{1}{2}$) and operator ($\frac{1}{2}$ of ...) appear at Level 1. In essence, the cognitive demands of this level are limited to the recognition of one half as a fraction.

At Level B the decimal and percent items appear to reflect the earlier understanding of half. There is, however, an implicit understanding of more general part-whole relationships, since decimals that sum to a small whole number (e.g., 0.6 + 1.4) and simple partitioning (e.g., $\frac{1}{3}$ of 15) appear in this level.

Items appearing at Level C involve the earlier concepts with larger or more complex numbers (e.g., $\frac{1}{3}$ of 120; 4.5 – 3.3). The prominence of one-digit decimal sums (e.g., 0.3 + 0.7) appears to be parallel to the sum of unit fractions with the same denominators (e.g., $\frac{2}{7} + \frac{3}{7}$). Notions such as bonds to 10 (e.g., using 4 + 6 = 10 to help with 0.4 + 0.6) applied to decimal numbers, and the application of understanding of multiples and factors to fractions and percents (e.g., seeing 25% of 80 as 80 divided by 4) could be inferred from the nature of many of the items in this level.

At Level D, equivalence appears to become an important idea, with fraction items such as $\frac{1}{2} + \frac{4}{8}$, and percent items such as 20% of 25 appearing in this level. The understanding of multiples and factors is sufficiently well developed to be used for simple fraction division (e.g., $\frac{3}{4}$ d $\frac{1}{4}$). Place value ideas are beginning to emerge in decimal items (e.g., 0.25 x 10).

Level E items use less familiar numbers and representations. In this level, many of the earlier concepts appear to be consolidated and applied to more complex items. In decimals, for example, the idea of bonds to 10 and place value must be combined for success on items such as 0.18 + 0.2. In fractions, the ideas of multiples and equivalence are used to solve 5 x $\frac{2}{5}$, for example, and place value and equivalence ideas can be seen in 90% of 40. Implicit in these items is the need to draw on structures underlying the number representations.

The highest level of the variable map, Level F, contains the most difficult items presented to this group of students. Students at this level would appear to have a sufficiently good grasp of the number structures to be able to apply them successfully to a range of less familiar numbers and more complex operations (e.g., 30% of 80; $\frac{1}{2} + \frac{1}{3}$; $\frac{20}{0.5}$).

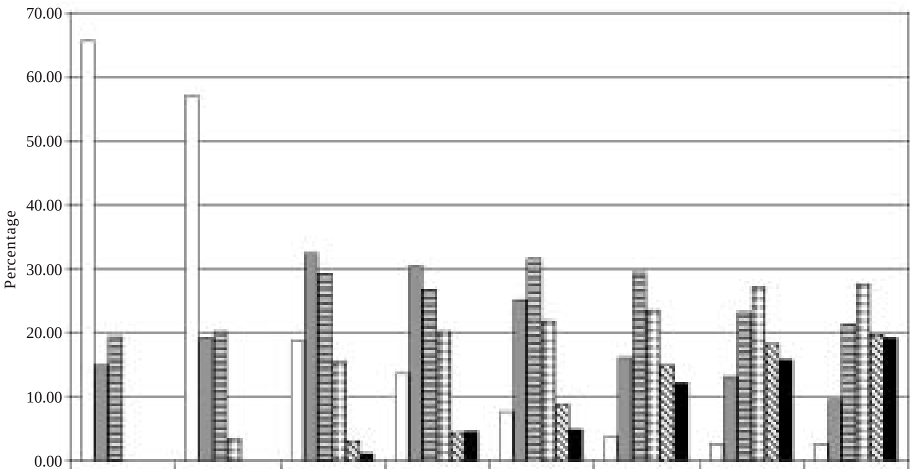
Table 2
Summary of Identified Levels

Level	Fractions	Decimals	Percent	Cognitive Demands
F	Operations involving $\frac{1}{3}$ or other non-unit fractions including addition of unit fractions with different denominators.	Multiplication of two decimals (1 d.p.); decimal division where multiples of the digits are involved.	Percentages involving less familiar fraction equivalents (e.g., $12\frac{1}{2}$%).	Uses underlying structures for computation with less familiar numbers or fractions with non-equal denominators.
E	Addition or subtraction with unequal denominators; whole numbers multiplied by non-unit fractions where cancelling is possible.	Multiplication by powers of 10 (0.1, 10, 100); division of small whole number by 0.1; decimal sum of numbers with unequal decimal places.	10% and 90% of small 2-digit whole numbers.	Draws on underlying structures, such as equivalence or place value, for straightforward computations.
D	Use of equivalent fractions for $\frac{1}{2}$, unit fraction 'of' whole number multiples, or $\frac{1}{2}$ of non-even decimal; division with $\frac{1}{2}$ or $\frac{1}{4}$.	Division by the same decimals (answer = 1); sum to one whole with regrouping; one and two digit whole numbers multiplied by 0.5; familiar decimal multiplied by 10 or 100.	Percentage equivalents of quarter/three quarter of whole number multiples of 4; 150% of even two digit numbers less than 30.	Understands and uses ideas of equivalence in all notations, and is beginning to draw on place value concepts with decimals.
C	Addition or subtraction of fractions with the same denominator or multiple (2, 4); $\frac{1}{2}$ or $\frac{1}{2}$ 'of' relate to multiples of 10.	Addition with the same number of decimal places; subtraction of small whole number from a decimal (1 d.p.).	Only instances of 25% with multiples of 10 less than 100; 10% of a small multiple of 10.	Understands the notion of one whole and additional parts, and is beginning to develop ideas of equivalence.
B	Mainly unit fractions with denominators of 2, 3 or 4; 'of' operation used with unit fractions of small whole numbers or $\frac{1}{2}$ of single digit even numbers.	Equal addends of the most basic decimal equivalents (of $\frac{1}{2}$ and $\frac{1}{4}$).	Only instances of 50% and 100%; equivalence to a half; knowledge of a whole.	Understands part-whole ideas and can use this with fraction notation to partition simple whole numbers.
A	Concept of one-half; half of an even whole number; no regrouping.			Recognises the meaning of one-half in fraction notation.

The levels appear to show increasing understanding of the structure of part-whole numbers, and the application of number knowledge learned in whole number contexts, such as factors and multiples, and place value. These six levels are summarised in Table 2.

Performance across Grades

Figure 3 shows the percentage of students in each grade whose mental computation ability, as measured on this scale, lay within each level. There is a progression of increasing competence with part-whole reasoning in mental computation across the grades, as would be expected.



	G3	G4	G5	G6	G7	G8	G9	G10
Level A	65.62	57.02	18.75	13.65	7.66	3.81	2.58	2.71
Level B	14.96	19.30	32.38	30.38	25.11	16.16	13.18	9.49
Level C	19.42	20.18	29.25	26.73	31.60	29.41	23.23	21.36
Level D	0.00	3.51	15.38	20.14	21.91	23.53	27.04	27.63
Level E	0.00	0.00	3.13	4.44	8.83	14.88	18.21	19.66
Level F	0.00	0.00	1.13	4.66	4.89	12.23	15.76	19.15

Figure 3. Distribution of mental computation ability across grades.

Students in grades 3 and 4 have had little opportunity to develop competence in mental computation with numbers other than whole numbers, and more than half of the students in these two grades are in Level A. On the other hand, students in grades 9 and 10 have generally met fractions, decimals, and percents through their school experiences. Despite this experience, over ten percent of students in the later grades can only deal with the most limited part-whole calculations, such as those dealing with half and its decimal and percent representations, which appear in Levels A and B. Of particular interest is the performance in the middle years, from grades 5 to 8. There does appear to be an increase in competence across these grades, and this may reflect curriculum influences, with an emphasis on teaching fractions, decimals and percents in the middle years of schooling.

Discussion

The discussion considers the results from the perspective of the demands of the items at various levels, of the performance of students across grades 3 to 10, and of reasonable expectations for the classroom, together with the implications for teaching. Directions for future research are suggested.

Relative Difficulty of Items

It should be remembered that the arrangement of the items in Figure 2 reflects student performance in a timed mental computation test, not in a context where students could write on paper, drawing pictures or carrying out known algorithms. In general terms, fraction items were easier than decimals and percents, as shown by the appearance of similar items in different representations.

It appears that familiarity with fractions, especially *half*, is the first requirement in the part-whole number arena at Level A. The extent to which this finding reflects the school curriculum, which introduces fractions before decimals and percents, or mirrors common usage of the language of half and quarter, which may develop students' facility with simple fraction computations, is unknown. The presence of equivalent tasks for decimals and percents at Level B may reflect their continued basic nature but later introduction. Throughout the higher levels for fractions, the denominators become larger, non-unit fractions appear, and denominators in sums and differences are not the same. Although unit fractions *of* another number occur at Level C, division by a fraction does not appear until Level D. Multiplication by non-unit fractions appears at Level E and operations resulting in denominators not in the original question appear at Level F.

The different representations of the numbers appear to provide additional complicating aspects of decimal mental computation, shown by items appearing at a higher level than their fraction equivalents. Multiplication by powers of 10 is an issue for decimal operations that is not central for fraction operations. These products begin to appear at Level D. Addition problems with decimals reflect increasing demands on place value understanding (e.g., regrouping) with level. That the only two problems involving the multiplication of two decimals are found at Level F may reflect the later introduction of this type of problem and the interference of rule learning without understanding in terms of where to put the decimal point.

All percent questions in this study were of the straightforward direct calculation *X% of Y* type, where Y is a whole number. This reflects the authors' reasonable expectations of mental computation outcomes and predicted use of them in actual life situations. The increased difficulty of the items is apparently related to the movement from percent equivalents of the fractions half and quarter, to multiples of 10% and the use of whole numbers, where fraction-equivalents of percent values are not straight multiples or divisors (e.g., 150% of 24, 90% of 40, or 30% of 80).

In general, multiplication and division operations with part-whole numbers are more difficult for students than addition and subtraction, mirroring findings from whole numbers (Callingham & McIntosh, 2001). Fraction multiplication in its simplest form (e.g., 3 x $\frac{2}{3}$) does not appear until Level E, after multiplication of a simple decimal by a power of 10 (e.g., 0.01 x 100), which appears at Level D. In terms of both language (three times two thirds, or zero point zero one times one hundred) and mathematical complexity, these kinds of fraction and decimal problems appear similar. The consistency of the appearance of fraction multiplication problems at higher levels than decimal multiplication problems, when the opposite is observed for addition and subtraction, suggests that there may be some additional, as yet unexplained, complexity.

Dealing with single digits was generally easier than dealing with two digits in decimal addition, as expected from research indicating that the complexity of a problem increased its difficulty (Hopkins, 2004), whereas multiplication problems were much more difficult. There were, however, some anomalies. The item 0.25 + 0.25 appeared at Level B, whereas the apparently easier item 0.5 + 0.5 was found at Level C. This may be explained by a consideration of common errors. The common strategy of *treat the decimals as whole numbers and insert a decimal point in front* would lead to a correct solution of 0.50 if applied to 0.25 + 0.25, but an incorrect solution of 0.10 if applied to 0.5 + 0.5. This explanation was supported when the error patterns were examined, with 0.10 accounting for over 80% of the errors for the item 0.5 + 0.5.

Performance across Grades

As seen in Figure 3, the performance in terms of levels improves with grade. There is little change in distribution across the levels between grade 3 and grade 4 or between grade 9 and grade 10. In the middle years, however, there does appear to be a grade to grade change in performance from grade 5 to grade 8.

For grade 3, all students are at Level C or below, whereas at grade 10, roughly 20 percent or more of students are at each of Levels C to F. It is clear that teachers need to deal with a greater range of performance from the middle school level upward. Although this is not surprising, the distribution of students in the top four levels, with the mode at Level D, suggests that realistic expectations for grade 10 may not include Level E and F performance for a majority of students. It also indicates that more effort must be placed on mental computation skills and strategies if the goals of the curriculum are to be met. The Victorian Curriculum and Standards Framework (CSF) II standard for Level 5.1, for example, is “Extend the use of basic number facts to mentally compute operations on fractions and decimals, and squares and square roots” (Board of Studies, 2000, p. 139). The results from this study suggest that less than half of all students in grades

9 and 10 can deal well with mental multiplication of a decimal by powers of ten, casting some doubt on the achievability of the CSF II outcome quoted.

The conjecture that curriculum emphasis on part-whole numbers in the middle years may explain the growth in performance is borne out, to some extent, by an examination of some of the common errors. In addition to the appearance of *treat decimals as a whole number and insert the decimal point* misconception, the fraction addition error of *add the tops and add the bottoms* rarely appeared before grades 7/8. The item $\frac{1}{4}+\frac{1}{4}$, for example, was presented to all grades. In grades 3/4, 60.0% of the students correctly calculated the answer, and this percentage rose in grades 5/6 (83.2%), grades 7/8 (86.4%) to 90.2 % in grades 9/10. In the lower grades the most common incorrect response was 1, accounting for 20.0% of the errors observed in grades 3/4. There was a wide range of incorrect answers in these grades. The second most common response was 30 (8.6% of errors) and the same proportion wrote 8. Incorrect responses were often expressed as whole numbers. In grades 5/6, there was also a variety of incorrect responses with the most common being 1, $\frac{3}{4}$ and 49 (11.1% of errors each). Only one student answered $\frac{2}{8}$, which is a commonly expected incorrect response to fraction addition problems. In grades 7/8, however, $\frac{2}{8}$ became the most common incorrect response (18.2% of the observed errors) and $\frac{1}{4}$ accounted for a further 13.6% of errors, which may have arisen from students reducing the incorrect response of $\frac{2}{8}$ to its simplest form. This pattern was similar in grades 9/10, with 1 (44.6% of errors), $\frac{2}{8}$, and $\frac{1}{4}$ (33.3% of errors) the most common incorrect responses. Notably, no incorrect response in the higher grades was a whole number, apart from 1, whereas whole number responses were common in the lower grades. This observation suggests that students are becoming increasingly familiar with part-whole numbers as they move through school, but that they are also affected by partially understood rules. Such findings have implications for teaching.

Future Research

There are several directions in which future research on mental computation with part-whole numbers can proceed. As part of a larger study including whole number operations, this study used a limited range of items, sometimes with very few for a particular type of operation (e.g., multiplying two decimals or combining operations with two of the types of numbers). It will be instructive to expand the item set, while at the same time remembering that mental computation is intended to involve realistic tasks that one might be expected to perform in contexts outside the mathematics classroom. Using only part-whole numbers and working with middle school students would allow a more detailed analysis of what Watson, Kelly, and Callingham (2004) have called partial number sense in relation to the errors observed. This may in turn lead to the use of the partial credit Rasch model (Masters, 1982) as a method of analysis, in which common errors are recognised and scored as partial understanding, and more detailed developmental pathways identified.

It is also of interest to consider the strategies that students report using while completing mental computation tasks, and interviews with students are likely to be helpful in this regard. Caney and Watson (2003), for example, have begun this task by interviewing 24 students, sometimes discovering unexpected paths to correct answers. This is a potentially fruitful area for research that could lead to recommendations for teaching approaches.

Comparisons of levels of performance with other students from different systems, states, and countries will also be useful in developing policy about appropriate teaching sequences to be employed in relation to mental computation with part-whole numbers. This is particularly true in light of the observation in this study that many students appear not to be meeting the standards in current Australian curriculum documents.

It may also be possible to coordinate the study of mental computation competence with other numeracy skills that are considered important across the rest of the school curriculum. There is some evidence that teachers implicitly expect mental computation in many different classroom situations (Callingham, 2003). The interaction of mental computation skills with these other facets of mathematics performance is likely to be a productive area for future research.

This study has highlighted the progression of part-whole mental computation proficiency across grades 3 to 10, and has identified apparent difficulties of some forms of part-whole representation in the context of mental computation. If the goals of the mathematics curriculum with regard to mental computation are to be met, there is a considerable research agenda needed concerning the strategies that students use and the trends in errors that they make.

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