Making Pythagoras count

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Typically, we introduce students to Pythagoras' theorem in the junior years of secondary school. Students consolidate their understanding of the theorem by using it for finding missing sides of triangles and for checking whether a given triangle has a right angle. But the topic often seems to dry up rapidly once these few practical applications are exhausted.

Mathematicians tend not to confine their interest to the practical applications but imagine new questions and seek new results following on from the original idea. Indeed, it was Fermat's consideration of the Diophantine version of $x^2 + y^2 = z^2$ that led to his "last theorem" and subsequently to several hundred years of development in algebra and number theory. The purpose of this article is to suggest some ways to enliven the topic of Pythagoras' theorem, at least for senior secondary students.

It must seem mysterious to students that mathematics teachers are able to come up with a plethora of right-angled triangles with integer sides as practice examples while such triangles that the students might propose for themselves, with integer legs, will usually not have integer hypotenuses.

The ancient Greeks knew how to find an endless supply of right-angled triangles with integer sides and these objects have fascinated mathematicians since. To begin, recall that solutions (a, b, c) in positive integers of the equation $x^2 + y^2 = z^2$ are known as *Pythagorean triples*. My own fascination with Pythagorean triples led me to think about right-angled triangles that are close to being isosceles.

To see that there are infinitely many Pythagorean triples, take any two positive integers p and q and verify the identity $(2pq)^2 + (p^2 - q^2)^2 \equiv (p^2 + q^2)^2$. The three squared quantities in this identity correspond to x^2, y^2 and z^2 in the Diophantine equation.

A good student would be able to follow the extra reasoning needed to show that not only does the identity give a triple for every p and q, but also it generates *all possible* Pythagorean triples.

Triples whose elements have no common factors, called *primitive* Pythagorean triples, are generated when p and q are chosen so that p is strictly greater than q, p and q have opposite parity, and p and q have no common factors. All other triples are obtained from primitive triples by multiplying by some factor.

Here is a list starting with all numbers p and q that add up to 3, then all p and q that add up to 5, then all p and q that add up to 7, and so on. (To avoid having p and q with the same parity, I have omitted pq pairs that sum to an even number.)

The fact that the triples can, in principle, be tabulated in this way shows that there is a countable infinity of them.

ps and qs			**triples**		
$p + q$	p	q	$x = 2pq$	$y = p^2 - q^2$	$z = p^2 + q^2$
3	2	1	4	3	5
5	4 3	1 2	8 12	15 5	17 13
7	6 5 4	1 2 3	12 20 24	35 21 7	37 29 25
9	8 7 5	1 2 4	16 28 40	63 45 9	65 53 41

If we continue to generate the list of triples explicitly we find that occasionally the two smaller numbers in the triple only differ from one another by one. For example: (3, 4, 5), (20, 21, 29), (119, 120, 169)…

The triangles these triples represent look more and more like right-isosceles triangles, as the numbers get bigger. The next one is (696, 697, 985). Are there any more? How close to isosceles can these triangles get? How must p and q be related in such triples?

The problem of finding all *nearly isosceles* right triangles can be solved using a Pell equation (Dye, 1998) and this connection is worth investigating too. But the approach I take here is accessible to anyone with a high school toolkit.

If x and y differ by 1, we must have, according to the identity

$$\left|\left(p^2 - q^2\right) - 2pq\right| = 1$$

Solving the implied quadratic equations for positive p, we find

$$p = q + \sqrt{2q^2 + 1} \quad \text{or} \quad p = q + \sqrt{2q^2 - 1}$$

Now, the square root part of the first of these solutions can only be an integer if q is even — because of the remainders obtained when odd or even squares are divided by four (1 and 0 respectively) — and the square root part of the second solution cannot be an integer unless q is odd.

So, we can express the dependence of p on q by constructing a function P. The domain of P may as well be restricted to a yet to be determined, possibly empty, subset of the natural numbers that will make the function integer valued.

$$P(q)=\begin{cases} q+\sqrt{2q^2+1} & \text{if } q \text{ is even} \\ q+\sqrt{2q^2-1} & \text{if } q \text{ is odd} \end{cases} \qquad \ldots(1)$$

If this function has an integer value, it must be an odd number when q is even and an even number when q is odd.

By experiment we find that $q = 1$ gives $p = 2$, then $q = 2$ leads to $p = 5$, then $q = 5$ gives $p = 12$, and then $q = 12$ leads to $p = 29$, and so on.

In general, suppose $q = q_e$, an even integer, satisfies P so that $P(q_e)$ is an odd integer. Then we find in turn that

$$\begin{aligned} P(p_0) &= p_0+\sqrt{2p_0{}^2-1} \\ &= p_0+\sqrt{2\left(q_e+\sqrt{2q_e{}^2+1}\right)^2-1} \\ &= p_0+\sqrt{6q_e{}^2+4q_e\sqrt{2q_e{}^2+1}+1} \end{aligned}$$

Although it might not at first look like one, the expression under the outer square root sign should be a square — otherwise the solutions found experimentally would be hard to explain. That it is a square can be seen more easily if the above expression for $P(p_0)$ is rewritten as

$$p_0+\sqrt{4q_e^2+4q_e\sqrt{2q_e^2+1}+2q_e^2+1}$$

So,

$$\begin{aligned} P(p_0) &= p_0+\sqrt{\left(2q_e+\sqrt{2q_e{}^2+1}\right)^2} \\ &= 2p_0+q_e \end{aligned}$$

Thus, $P(p_0) = p_e$ is an integer if $P(q_e) = p_0$ is.

In a similar way, suppose $q = q_0$, an odd integer, satisfies P so that $P(q_0) = p_e$ is an even integer. Then

$$\begin{aligned} P(p_e) &= p_e+\sqrt{2p_e{}^2+1} \\ &= 2p_e+q_0 \end{aligned}$$

This also is an integer. So, $P(p_e) = p_0$ is an integer when $P(q_0) = p_e$ is.

So, iterating the function P generates an infinite sequence of integer pairs $(q, P(q))$. In fact, q and $P(_q)$ are successive terms in a sequence defined by $t_n = 2t_{n-1} + t_{n-2}$. The first two terms in the sequence are $q_0 = 1$, $p_e = 2$ so the sequence goes 1, 2, 5, 12, 29, 70…

It follows that the qp pairs from this sequence can be used to generate a sequence of Pythagorean triples that represent the almost isosceles triangles: (4, 3, 5), (20, 21, 29), (120, 119, 169), (696, 697, 985), (4060, 4059, 5741)…

Have we found *all* the nearly isosceles Pythagorean triples? Yes. We solved the original quadratics for p, but we could just have easily solved them for q and then constructed a function Q analogous to (1):

$$Q(p)=\begin{cases}-p+\sqrt{2p^2+1} & \text{if } p \text{ is even}\\ -p+\sqrt{2p^2-1} & \text{if } p \text{ is odd}\end{cases} \qquad \ldots(2)$$

In fact, Q is the inverse function of P. We can use it to show that the process encapsulated in the sequence (t_n) works in reverse and leaves no room for qp pairs other than the ones already found.

Going through the same reasoning as above but using the function Q instead of P, we find that $Q(q_e) = q_0$ is an integer when $Q(p_0) = q_e$ is, and $Q(q_0) = q_e$ is an integer when $Q(p_e) = q_0$ is. This chain reverses the previous situation and allows us to work back from larger ps to smaller qs.

Suppose there is a p that lies between two terms t_k and t_{k+1} of the sequence (t_n), such that $Q(p)$ is an integer. Then $Q(p)$ lies between t_{k-1} and t_k. Then $Q(Q(p))$ lies between t_{k-2} and t_{k-1} and $Q(Q(Q(p)))$ lies between t_{k-3} and t_{k-2} and so on. Eventually we see that there must be an integer lying between $t_2 = 2$ which, of course, is not the case. So the original assumption that there is a p between two terms of t_n must have been false.

Pairs of terms from the sequence (t_n), then, generate all possible nearly isosceles Pythagorean triples when we let $q = t_k$ and $p = t_{k+1}$.

If we rescale the triangles obtained from these nearly isosceles triples, so that the $2pq$ side always has length 1, the sequence of hypotenuses goes like this:

$$\frac{5}{4}, \frac{29}{20}, \frac{169}{120}, \frac{985}{696}, \frac{5741}{4060}\ldots$$

with general term

$$\frac{p^2+q^2}{2pq}$$

Intuition suggests that this sequence is converging to the limiting value $\sqrt{2}$. To confirm this, write the hypotenuse as

$$\frac{\sqrt{(2pq)^2+\left(p^2-q^2\right)^2}}{2pq}=\sqrt{1+\left(\frac{p^2-q^2}{2pq}\right)^2}$$

But $p^2 - q^2$ is always 1 more or less than $2pq$. So the expression becomes

$$\sqrt{1+\left(\frac{2pq\pm 1}{2pq}\right)^2}=\sqrt{1+\left(1\pm\frac{1}{2pq}\right)^2}$$

Because the sequence of ps and qs increases without limit, the fraction

$$\frac{1}{2pq}$$

becomes arbitrarily small and the conclusion follows:

$$\frac{p^2 + q^2}{2pq} \to \sqrt{2}$$

Where does this story lead? I suspect it could be extended indefinitely and could have all sorts of offshoots useful in teaching high school mathematics. Recently I gave my Year 9 students an exercise in the use of their scientific calculators in which they had to evaluate the expression

$$\frac{\sqrt{2}}{4}\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right]$$

for $n = 1, 2, 3, 4, 5$. After fumbling with the use of brackets and the "to-the-power-of" button, they found the corresponding values: 1, 2, 5, 12, 29. Next I asked what the rule would be for getting each new term from the previous terms, dramatising the fact that there *was* a rule by quickly working out the next few terms: 70, 169, 408. A few got it. The expression defines explicitly the sequence (t_n) that was defined by the recurrence relation $t_n = 2t_{n-1} + t_{n-2}$.

A similar recurrence relation $t_n = t_{n-1} + t_{n-2}$ produces the Fibonnaci sequence. What happens when the terms of the Fibonnaci sequence are used for the *p*s and *q*s in generating Pythagorean triples? Can we approximate $\sqrt{5}$ using this, in a manner similar to the way in which $\sqrt{2}$ was approximated? Are there other families of Pythagorean triples that approximate whole number ratios, other than 1:1 or 1:2? What about right angled triangles in which the *hypotenuse* and another side differ by 1?

The questions keep coming.

References

http://mathworld.wolfram.com/PythagoreanTriple.html

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