

# Coin on a CHESSBOARD

## A rich problem

If you roll a coin across a chessboard and it comes to rest on the board, what is the probability that it covers some corner of one of the grid squares?

The online magazine *Plus* (2004) posed this problem for students to solve. It is a useful problem for several reasons: it introduces the idea of probability in a continuous sample space, it has a historical background worth exploring and it can be simulated easily using a computer program.

## Le jeu de franc-carreau

The coin-and-chessboard problem is a variant of the problem of finding the odds in an old game, *Le jeu de franc-carreau*, in which bets were laid concerning whether a small coin dropped on a square-tiled floor would come to rest touching a crack between the tiles. Georges Louis Leclerc, Comte de Buffon analysed this game in 1733.

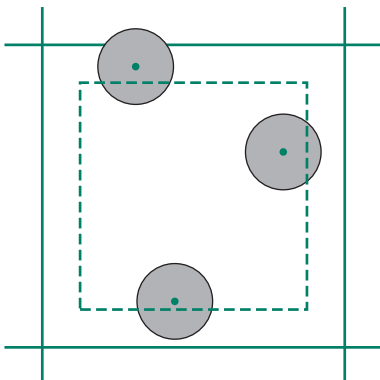


Figure 1

Buffon saw that the coin would not touch a boundary if its centre landed within a smaller

square inside a tile such that the side of the inner square was smaller than the side of the tile by an amount equal to the diameter of the coin. And he came to the important realisation that the associated probability was just the ratio of the areas of the smaller and larger squares.

In this way, Buffon made the leap from probabilities calculated by counting — in what are now called discrete sample spaces — to probabilities involving measuring. He was the first to apply geometry and the newly invented tools of calculus to the study of probability.

In the same memoir, Buffon went on to explain what would happen if the coin were to be replaced by a baguette or stick. This is the origin of the famous Buffon needle problem:

What is the probability that a needle dropped on a grid of equally spaced parallel lines will come to rest touching a line?

Buffon eventually published his solution to the needle problem in 1777 in his *Essai d'arithmétique morale*. He showed that the theoretical probability for this experiment is

$$\frac{2l}{\pi d}$$

where  $d$  is the distance between the lines and  $l$  is the length of the needle (with the restriction  $l < d$ ). We note that the number  $\pi$  appears in the solution.

## Coin and chessboard

The coin-and-chessboard problem is easier to solve than Buffon's needle problem but the two have a common feature. It happens that the number  $\pi$  is involved in the theoretical probability in both. This is because in the

coin-and-chessboard problem the area of the circular coin comes into the calculation.

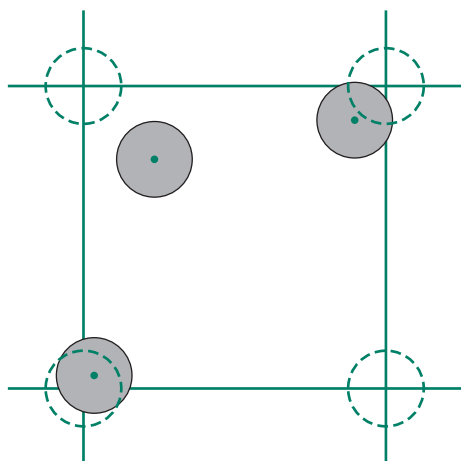


Figure 2

It is not hard to see that the coin covers a corner of a grid square whenever its centre lands within a circle with the same radius as the coin, centred on the corner.

We should require the *centre* of the coin to land on the board. Then, to simplify the situation, we can allow the coin to land on just one square of the 8×8 grid instead of the full 64 — the probability will be the same.

For a small enough coin, there is a quadrant at each corner of a square such that the corner will be covered if the centre of the coin lands within the quadrant.

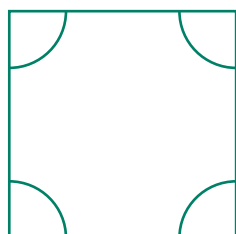


Figure 3

Assuming the radius  $r$  of the coin is no more than half the side  $s$  of the square, the probability  $P$  that the coin covers a corner is just the ratio of the total area of the quadrants at the corners to the area of the whole square.

So, if  $C$  is the event that the coin covers a corner, we have

$$P(C) = \frac{\pi r^2}{s^2}, \quad 0 < r \leq \frac{s}{2}$$

It is an interesting although not immediately useful exercise to work out the probability, when  $r$  is between  $\frac{s}{2}$  and  $\frac{s}{\sqrt{2}}$

To do this, divide up a quarter of the tile as shown, and then find the areas of the two triangles and the sector.

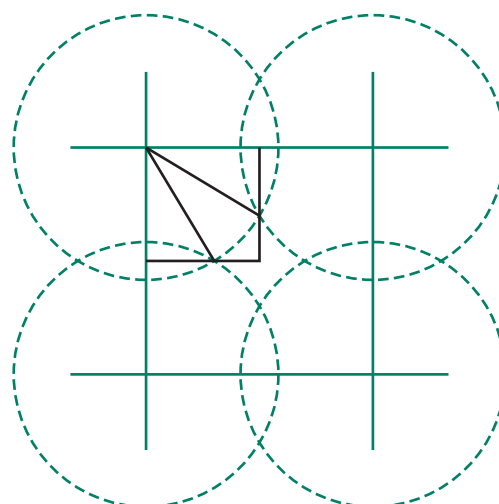


Figure 4

In this case,

$$P(C) = \frac{2 \left( s \sqrt{r^2 - \left(\frac{s}{2}\right)^2} + r^2 \left( \frac{\pi}{2} - 2 \cos^{-1} \left( \frac{s}{2r} \right) \right) \right)}{s^2}$$

## More history

A century after Buffon posed the needle problem, the Swiss mathematician and astronomer Johann Rudolf Wolf (1816–1893) had the idea that by comparing the theoretical probability in Buffon's experiment with a value obtained experimentally, one could obtain an estimate for the value of  $\pi$ . This surprising result is the essence of what have come to be called Monte Carlo methods (after a 1949 paper by Stanislaw Ulam and Nicholas Metropolis).

Buffon's formula, when rearranged, gives the relation

$$\pi = \frac{2l}{Pd}$$

where  $P$  is the probability. Wolf obtained a probability estimate by performing the needle experiment a predetermined number of times with fixed values of  $l$  and  $d$ , counting the number of favourable outcomes. He then found  $\pi$  (approximately) by substitution.

The coin-and-chessboard experiment can be used in a similar way. Fortunately, it is easily simulated with a spreadsheet program

and can be quickly iterated millions of times — for many more times than Buffon or Wolf would have had the patience.

In this case, since

$$P = \frac{\pi r^2}{s^2}$$

we can use

$$\pi = \frac{Ps^2}{r^2}$$

with  $P$  estimated by the proportion of successes in a large number of trials of the experiment.

## Simulation

The essential things a simulated coin-and-chessboard experiment must do are:

- randomly place the centre of the coin in a two dimensional coordinate system;
- check whether each trial has been a success; and
- keep a tally while iterating the experiment some specified number of times.

For simplicity, we can let the chessboard squares have side length 1. To locate the centre of the coin, two random numbers are needed, one each for the  $x$ - and  $y$ -coordinates. In Excel, we can use the `RAND()` function: it generates random numbers between 0 and 1 with an approximately uniform distribution.

Instead of testing whether the centre of the coin lands in one of the corner quadrants, we may as well combine the quadrants into one circle at the centre of the square. The coin is just as likely to land in this central circle as in the corner quadrants. We can put the origin of the coordinate axes in the centre of the square.

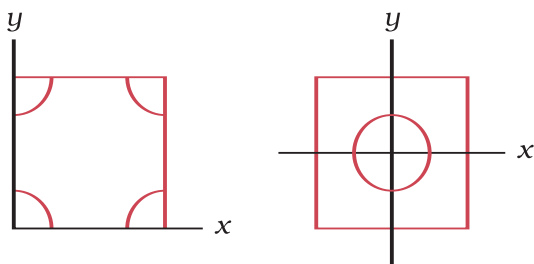


Figure 5

With these arrangements, both the  $x$ - and  $y$ -coordinates of the coin centre are obtained using the formula `=RAND()-0.5`. Testing for a success is now a matter of checking how far the coin centre is from the origin.

Excel works faster if several calculations are combined into one cell. So, a formula like `=SUMSQ(RAND()-0.5,RAND()-0.5)` can be used to get the squared distance of the centre from the origin, and this can then be compared with the square of the coin's radius. In my implementation this calculation is embedded in an even bigger formula (see Figure 6).

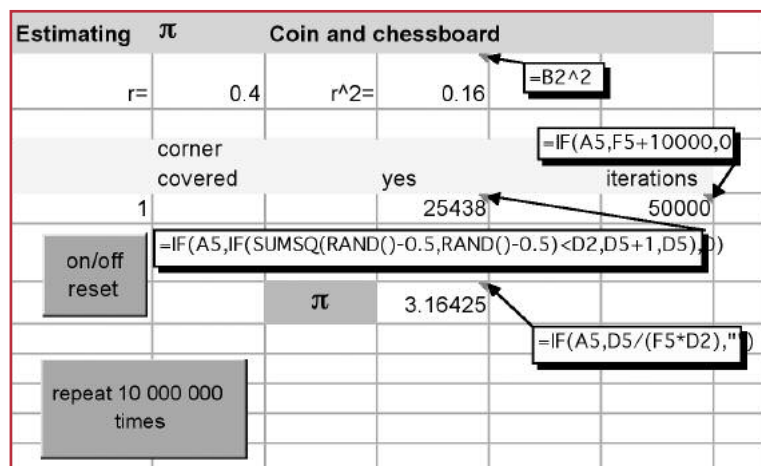


Figure 6

I found it convenient to automate the coin tossing process with a Visual Basic for Applications procedure.

The code for this is activated by a button on the sheet that causes the virtual coin to be tossed ten million times. There is also a button to reset the experiment.

Setting the radius of the coin to 0.4 will result in successes in about half the trials. To save Excel the trouble of calculating the square of the radius repeatedly, it is possible to set a value for the radius between 0 and 0.5, and have the VBA procedure calculate the square only once for each run of the experiment.

Cell A5 switches the process on or off depending on whether its value is set to '1' or '0'. Formulas in other cells test whether A5 is set to *on* or *off* by treating its contents as a boolean variable: '1' in cell A5 evaluates to *true*, '0' evaluates to *false*.

Cells D5 and F5 include circular references. These enable the contents of the cells to be incremented as the experiment proceeds. To

use the circular reference feature in Excel 2000, go to the Tools menu and select Options and the Calculation tab. Check the Iteration box and set Maximum Iterations to 1 (see Hartley Hyde's 'Cactus' article in *The Australian Mathematics Teacher* 60(4), 2004.)

While at the Calculation tab dialogue box, select the Manual button. This will make it possible for the Visual Basic for Applications code to control which cells are to be updated on each trial.

To put Command Buttons on a spreadsheet and attach code to them, first activate the Control Toolbox on the View: Toolbars menu. Work in Design Mode, selected from the Control Toolbox, making use of the Properties and View Code options.

## Visual Basic for Applications code

'This procedure resets the experiment.

```
Private Sub CommandButton1_Click()
    Dim state As Integer
    state = Range("A5")
    If state = 1 Or state = 0
    Then Range("A5") = 1 - state
    Else Range("A5") = 0
    If Range("A5") = 0
    Then Range("A1:F7").calculate
End Sub
```

This procedure repeats the simulated coin toss 10 million times.

```
Private Sub CommandButton2_Click()
    Dim counter As Long, outer_counter As Long
    outer_counter = 0
    If Range("A5") = 1 Then
    Do
        outer_counter = outer_counter + 1
        counter = 0
    Do
        counter = counter + 1
        Range("D5").calculate
    Loop Until counter = 10000
    Range("F5,D7").calculate
    Loop Until outer_counter = 1000
End If
End Sub
```

## A wave of the hand

In 10 000 000 trials of the coin-and-chess-board experiment, there could be any number of successes between 0 and 10 000 000. To each one of these outcomes, it is possible to assign a probability depending on the number of successes in the outcome and on the probability of a success in a single trial. The assignment of probabilities among the outcomes in this case is of a particular kind, called a binomial distribution.

We need to distinguish between the exact number  $\pi$  and what is called the *estimator*

$$\hat{\pi} = \frac{ms^2}{nr^2}$$

(where  $m$  is the number of successes,  $n$  is the number of trials,  $s = 1$  is the side of a chess-board square and  $r$  is the radius of the coin). Because the number  $m$  varies each time the experiment is performed, it is a random variable — with a binomial probability distribution. The estimator  $\hat{\pi}$  is also a random variable. Its probability distribution has the same shape as the distribution for  $m$ .

It can be shown that the mean or average value of  $\hat{\pi}$  is  $\pi$ , in the sense that  $\hat{\pi}$  is likely to approach  $\pi$  when  $n$  is very large. It can also be shown that the variance of  $\hat{\pi}$ , a measure of the degree of spread of its probability distribution, is given by

$$\text{Var}(\hat{\pi}) = \frac{\pi}{n} \left( \frac{1}{r^2} - \pi \right)$$

From this formula we can see that maximising the coin radius will minimise the variance. If the variance is small, most of the probability is concentrated within a narrow range around the mean. So, for a given number of trials in the experiment, the estimate for  $\pi$  is likely to be more accurate if the variance is small.

In the spreadsheet simulation then, it would be wise to set the coin radius to 0.5, its largest possible value. By trying out different coin radii between 0 and 0.5 one can test the validity of the claims made above.

According to my calculations, with 10 000 000 iterations per experiment and a coin radius of 0.5, the estimate of  $\pi$  will be accurate to two decimal places in more than 99.9% of experiments. It will be accurate to three decimal places about 53% of the time,

and accurate to four decimal places in less than 8% of experiments.

Looked at another way, the estimate will be accurate to two decimal places 95% of the time if the experiment consists of  $4.14 \times 10^3$  trials; it will be accurate to three decimal places 95% of the time if the experiment has  $4.14 \times 10^5$  trials; and it will be accurate to four decimal places 95% of the time if there are  $4.14 \times 10^7$  trials.

Clearly this is not a good way to calculate  $\pi$ , but it does provide an interesting way to get a feel for a statistical process.

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**Paul Turner**

Melrose High School, ACT  
 pturner@melrosehs.act.edu.au

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