LANCE COAD

PAPER FOLDING in the middle school classroom *and beyond*

Introduction

Paper folding can be used in the classroom to introduce the standard results of school geometry, such as the transversal and parallel lines results along with results concerning angles in convex polygons and centres of triangles, for example. Angle bisectors, midpoints, perpendiculars are all straightforward "constructions" for the paper folder. If translucent paper is used it renders easy tasks such as duplicating an angle or a segment to any position desired. Serra (1994) indicates how these goals and others can be achieved.

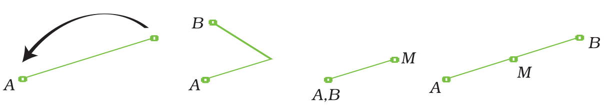
Used in this way, paper folding encourages students toward conjecture and invites the teacher to reflect upon the meaning of proof. Results are made plausible – convincingly so – by simple observations associated with the folding of paper. A significant benefit of the paper folding technique is its accessibility to students and the affective benefits this confers. The difficult question of when it is most effective to introduce deductive rigour remains.

Paper folding can also be used to solve problems that have been interesting in the context of Euclidean constructions with straightedge and compass. It is easy to trisect an angle using paper folding and also to find the cube root of two (among other numbers) – that is, to double the cube. Paper folding also enables students to visualise parabolas and other conics as folded structures.

When used together with dynamic geometry software (DGS) such as *The Geometer's Sketchpad*, paper folding becomes a powerful tool in the classroom. Its use can be extended to problems that are interesting in their own right, including folding rational angles and star polygons.

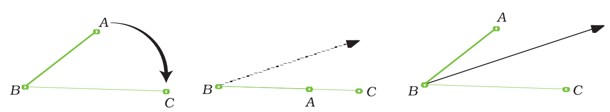
Basic folds and applications of paper folding

To begin it is good to note the fundamental folds. We immediately confront the pedagogical question of whether to elucidate axiomatically which folds underpin our paper folding technique, or whether to mediate the introduction through specific problems. A fold is a line segment l (or a line, if you wish to extend it). The segment l can be bisected by folding its ends (points A and B) together over itself.

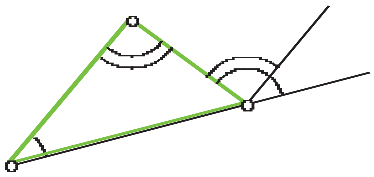


Note that this fold automatically produces a perpendicular bisector. By folding a line l onto itself, a perpendicular can be constructed through any given point P not on the line l.

Two non parallel folds make an angle. The angle is bisected quite naturally by folding the segments onto each other through the point of intersection.

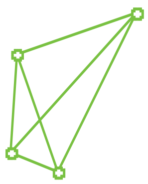


These simple folds can be posed as introductory challenges for students ("How might we construct a perpendicular? How might we bisect a segment? How can we find a perpendicular bisector? An angle bisector?"), and the success rate is high. If a straightedge is employed then it is easy to construct a line parallel to a given a line l. Without the straightedge it is still easy via the agency of two perpendicular lines.



Comparing angles is easy if two sheets of translucent paper are used. Again, this is not necessary, but it speeds the process and reduces complication, so making the desired conclusions more readily accessible to a wider range of students.

It is simple to compare the angles in a triangle and establish that they form a straight line by cutting them out and placing them with vertices together so that the three angles are seen to form a straight line. This process extends to quadrilaterals and beyond. It affords a nice opportunity to invite conjecture about the internal angle sum of polygons which can be verified in a variety of ways. The exterior triangle sum likewise yields rapidly to a simple trace and compare strategy.



A fruitful exercise is to have students consider intersection of pairs of line segments under various conditions: if the segments are same/different lengths; if they intersect at midpoints (one or both) or not; and if they intersect at right angles or not. Joining the end points of the segments produces quadrilaterals, and in this way the properties of the various quadrilaterals can be seen to be derived from the nature of their diagonals. Table 1 demonstrates four possibilities.

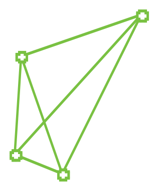
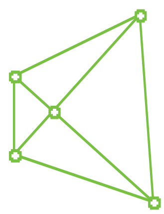
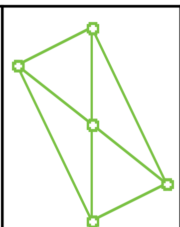
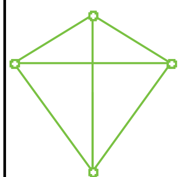
Such a categorisation reveals the relationships between the various quadrilaterals and encourages the view that some are special cases of others. For example, "same length, both at midpoints, not 90°" is a special case of "not same length, both at midpoints, not 90°", particularly so if "not" means "not necessarily". Thus students are encouraged to move through the van Hiele levels from analysis to informal deduction, or ordering. (See Senk (1985; 1989) for a discussion of van Hiele levels and evidence that students' success at writing geometric proofs can be predicted from knowledge of their van Hiele levels at the beginning of a course. Pegg (1995) also introduces the van Hiele theory).

These activities can be accomplished readily enough using dynamic geometry software (DGS). Indeed, DGS has several advantages, especially in being able to provide multiple examples of a phenomenon as a point or segment is dragged, capitalising on the inductive character of our thought. So why would we choose to fold paper?

Why choose paper folding?

Paper folding is accessible to students in a way that DGS might not be. While evidence for the use of manipulatives is mixed (see for example Sowell, 1989, but also Raphael and Wahlstrom, 1989), we might ask whether students introduced to geometric ideas via paper folding will generate better cognitive models than those who commence work on a computer. The engagement of the hands in the process of completing folds (and of the mind in the process of deciding what folds to pursue) possibly raises the cognitive models above those that might have been developed had pen and paper only been employed. This is, however, speculative. The act of selecting

Table 1

			
different lengths	same lengths	same lengths	same lengths
not at midpoints	not at midpoints	both at midpoints	one at midpoints
not 90°	90°	not 90°	90°

appropriate folds might be seen as an aid to developing the kinds of heuristics that are useful in establishing more formal proofs (what is relevant, how do I move from here to there, etc.).

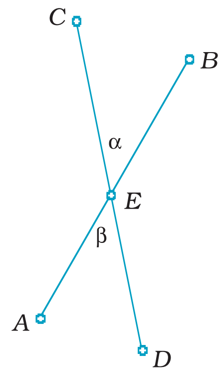
The affective aspects of paper folding are significant. It has been my experience that students enjoy paper folding. Paramount among the positive emotions appears to be a sense of pleasure brought on simply by understanding what is being done. Schlöglmann (2002) refers to the implicit emotional memory system that can be activated by the problem solving process, leading in some cases to negative reactions beyond cognitive control that effectively block learning. The value of positive experiences in the mathematics classroom should not be understated. We know that a sense of belief and worth is important for success in mathematics, so any mathematically sound process that encourages self belief is worthy of serious consideration.

Paper folding is certainly cheaper than equipping students with access to DGS. It relies less on knowledge of special procedures (how, for example, to construct line segments with DGS software) and is, in that sense, relatively transparent. Healy and Hoyles (2001) discuss issues related to tool selection. They report that in work with "less successful students, learners can... find themselves in a position where they are unable to use the tools they have in mind, even if they are convinced that their use would make sense mathematically, and they are familiar with how the tools should work" (p. 252). They note that, "the mediation of students' activities by the software is not necessarily positive for their engagement and for their learning" (ibid).

In using a paper folding approach, nevertheless, the time will come when it is determined that students will benefit from progressing to DGS in order to make conjectures more clear or to amplify the signals that paper folding is providing. This timing is likely to vary between students and so it is advantageous to have a structure in place that enables students to operate in a differentiated fashion. Indeed, I would suggest that the nature of the teacher mediated interplay between paper folding, pencil and paper and DGS will influence student achievement, and this interplay is likely to vary between students. In general, paper folding provides a useful exploratory introduction to geometry and proof after which DGS can be utilised to extend investigations and foster a deeper understanding of proof. That said, paper folding has hidden layers of depth that invite further investigation.

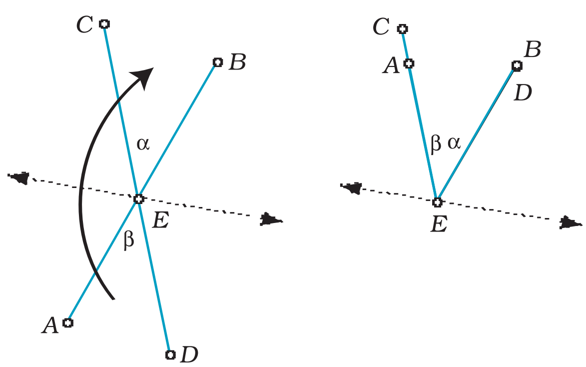
Paper folding, DGS and proof

If we adopt the view that it is not unreasonable to introduce proof gradually, then we can use students' natural facility with paper folding to enable construction of simple class approved proofs. For example, in proving that opposite angles are equal, one might follow a path like this:



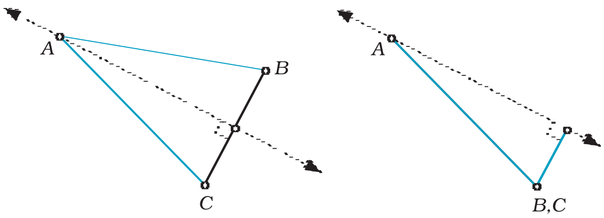
Let us call acute angles *CEB* and *DEA* α and β respectively. Then, as *AEB* is a straight line, we have $CEA = 180° - \alpha$; but we also have a straight line *CED*, and so $CEA = 180° - \beta$. Hence $180° - \beta = 180° - \alpha$ and so $\alpha = \beta$.

As an alternative, we could simply produce a fold so that the angles α and β were superimposed upon each other. The fold line is the bisector of angles *CEA* and *BED*.



From this fold, it is "obvious" that $\alpha = \beta$. Of course, this places a certain weight on the observation that might be greater than some are prepared to accept. It nevertheless establishes a result (opposite angles are equal) quickly and believably so — all the more believable, I might suggest, to the students concerned because there has been no attempt to shroud in rigour what is obvious. Better, perhaps, to reserve more rigorous proofs for circumstances in which rigorous proofs are necessary.

To prove that base angles of an isosceles triangle are equal becomes the work of a moment. One merely has to fold a perpendicular bisector:



It is worth noting that prior to or in the course of making this proof, students can fold perpendicular bisectors to the base in non isosceles triangles to "establish" that only in isosceles triangles does the perpendicular bisector pass through the opposite vertex. Such an activity is an investigation in its own right. The property that isosceles base angles are equal will fall out as a corollary. As a consequence, the act of creating an isosceles triangle can then be managed simply by folding a perpendicular bisector to a base segment (see later). The vertex can be any point on the perpendicular bisector. In this sense, then, the "proof", whilst lacking in rigour, carries the power of explanation and definition (of properties).

In cases such as this, proving becomes an action, a process. Tall (1995) might describe it as enactive, whilst acknowledging that moving between categories of proof, from enactive to visual to manipulative (often meaning algebraic) can involve a "huge cognitive struggle" (p. 36). The use of axioms is embodied in the sense that one physically enacts what are in effect axioms of the geometry. Students can be encouraged to consider what fundamental folds are utilised in making explorations. That is, a search for axioms can be undertaken. The Euclidean axioms are well established, and have been for some time, but it is only relatively recently that efforts have been made to axiomatise paper folding.

Students can be required to determine the folds that can be used to produce given states — that is, to "prove" or derive given "theorems" (folded states or constructions) by reference to "axioms". This is fundamentally a proof process. Proofs that detail the sequence of steps are proofs that convince. At the next level, students can be presented with sequences of folds to produce a given state, such as the trisection of a segment, and asked to explain why, possibly in algebraic terms, the process works (see Hanna & Jahnke, 1996) for a discussion of types and purposes of proof). For example, why do the sequence of folds presented later in this paper trisect a line, or an angle, or produce an equilateral triangle?

DGS can be used to encourage students to further explore geometrical situations and to make conjectures Traditional geometrical results can be cast as open explorations with scaffolding and constraints applied to guide the search process so that results become personal discoveries. Christou et al. (2004), forward the concern raised by others that the power of DGS to facilitate students "seeing" results (conjectures) might militate against them feeling a need to explain why the results hold. This is the "gap between deduction and experimentation" (p. 340). They conclude, in concert with others (e.g., Jones, 2002) that care and skill in task construction and teacher guidance to encourage in students a desire to validate results is important:

> In the DGS environment students acquire understanding through verifying their conjectures and in turn this understanding solicits further curiosity to explain "why" a particular result is true. However, students working in the DGS environment are able to produce numerous configurations easily and rapidly, and thereby they may have no need for further conviction/verification (Hölzl, 2001). Although students may exhibit no further need for conviction in such situations, it is important for teachers to challenge them by asking why they think a particular result is true (De Villiers, 1996, 2003). Students quickly admit that inductive verification merely confirms but the why questions urge them to view deductive arguments as an attempt for explanation, rather than verification (Hölzl, 2001). Thus, the challenge of educators is to convey clearly to the students the interplay of deduction and

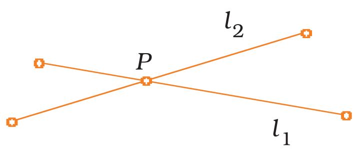
experimentation. (Hanna, 2000, pp. 342–343).

When students use paper folding and DGS to explore and make discoveries a teacher is able to foster the development of explanatory proof.

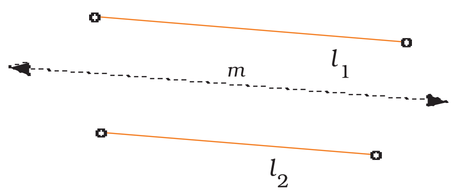
Axioms

Paper folding (origami) axioms have been developed by various mathematicians. Geretschlager (1995) presents the following set:

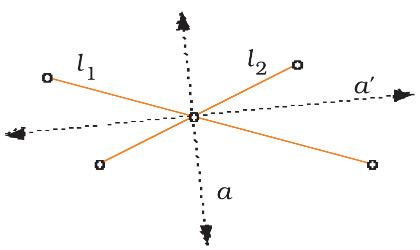
1. Given two non parallel straight lines l_1 and l_2, one can determine their unique point of intersection $P = l_1 \cap l_2$.



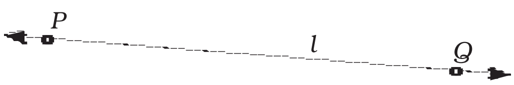
2. Given two parallel straight lines l_1 and l_2, one can fold the line m parallel to and equidistant from them ("mid-parallel").



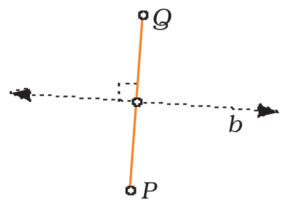
3. Given two intersecting straight lines l_1 and l_2, one can fold their angle bisectors a and a'.



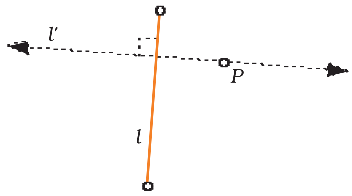
4. Given two non-identical points P and Q, one can fold the unique straight line connecting both points.



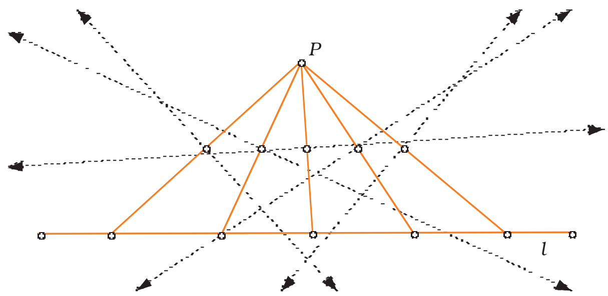
5. Given two non-identical points P and Q, one can fold the unique perpendicular bisector b of the line segment PQ.



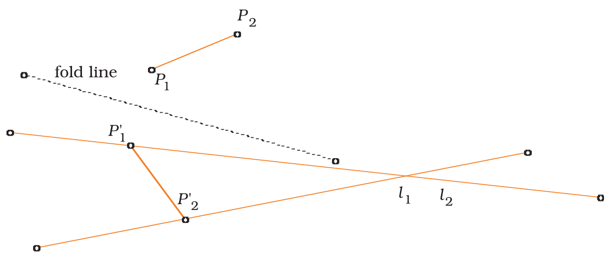
6 Given a point P and a straight line l, one can fold the unique line l' perpendicular to l and containing P.



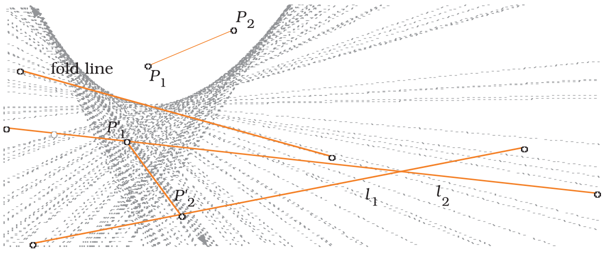
7 Given a point P and a straight line l, one can fold any tangent of the parabola with focus P and directrix l. Specifically, given a farther point Q, one can fold to the parabola tangents that contain Q.



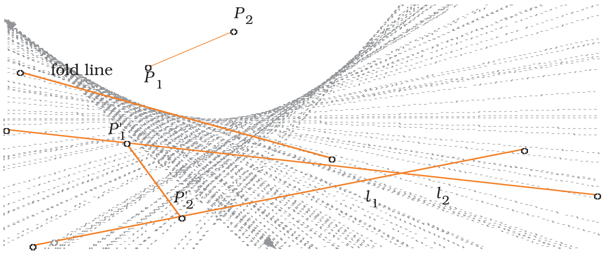
7.* Given (possibly identical) points P_1 and P_2 and (possibly identical) lines l_1 and l_2, one can fold the common tangents of the parabolas p_1 and p_2 with foci P_1 and P_2 and directrices l_1 and l_2, respectively.



The parabola p_1 with focus P_1 and directrix l_1, showing that the fold line is a tangent.



The parabola p_2 with focus P_2 and directrix l_2, showing that the fold line is a tangent.



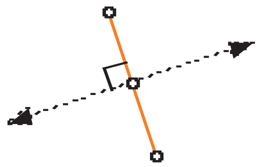
This last axiom states that given two points P_1 and P_2 and two lines l_1 and l_2, it is possible with a single fold to fold P_1 onto l_1 and P_2 onto l_2. Note that this is to be interpreted to mean that it can be done under certain conditions. If the two lines l_1 and l_2 are too far apart, for example, then it will not be possible to fold P_1 onto l_1 and P_2 onto l_2. Geretschlager notes that it is this last procedure (7*) that makes origami (paper folding) different from Euclidean geometry. He shows that Euclidean constructions are equivalent to origami built from 1 to 7, but that 7* amounts to the solution of a cubic problem, which is not achievable using Euclidean methods. It is this axiom, in fact, that allows paper folding methods to solve the classic problems of doubling the cube and trisecting the angle.

Note that this axiom list is not presented as though it were the "correct" set: Alperin (2000) produces a reduced set of six origami axioms and discusses the associated field theory. Hull (2003) likewise lists six axioms developed by Humiaki Huzita (see also Hull, 1996). For our purposes, we can note that origami is at least as rich, and in fact richer, than traditional Euclidean geometry, and that it is subject to rigorous treatment.

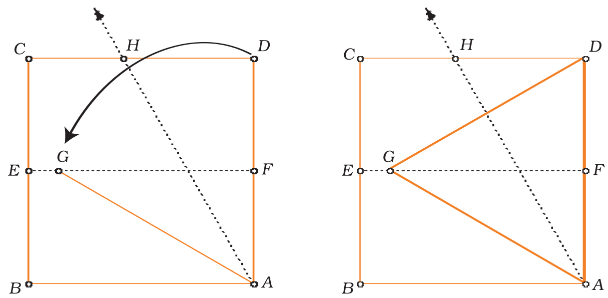
Folded constructions that beg proofs that explain

As indicated above, students can be asked to explain why certain constructions "work". A variety is presented here, ranging from simple to more complex, to indicate the breadth that is available within the context of school mathematics. The historically noteworthy cases of the trisection of the angle and the doubling of the cube are described.

1. One might begin with simple results, such as folding a perpendicular bisector enables the construction of an isosceles triangle. Why?

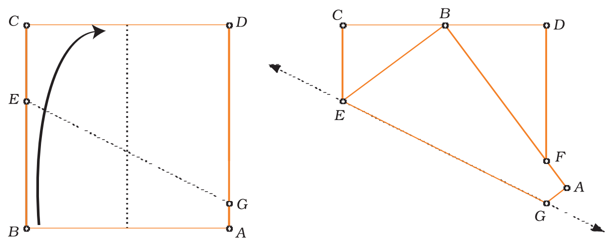


2. Simple folds can also be used to construct equilateral triangles.
Fold a sheet of paper in half and then fold so that the segment AD meets EF at G.



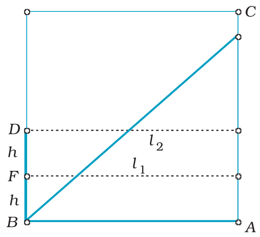
Then it is easy enough to see that GD is equal to AG and hence to AD. Thus triangle AGD is equilateral. While at it, one can note that angle DAH is 30°.

3. Slightly less obvious, the following shows that a line segment can be trisected. We fold in half again, and then fold B up to the midpoint of CD.

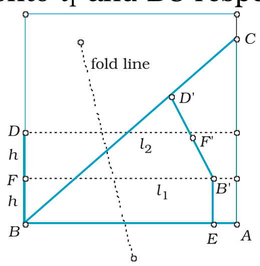


We find that $DF = \frac{2}{3}\,AD$. That is, F is a point of trisection of AD. The proof is an exercise in similar triangles.

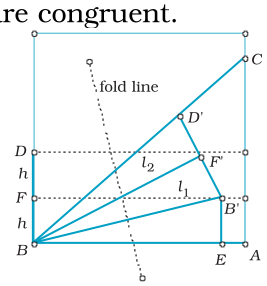
4. To trisect an angle one utilises 7*. If we begin with a square sheet of paper, then to trisect the angle *ABC* we first fold a line parallel to *AB*. Call this line l_1. Use as a fold over which to reflect *AB*, so that a new segment l_2 can be determined.



From here the trick is to use 7* to fold *B* and *D* onto l_1 and *BC* respectively.

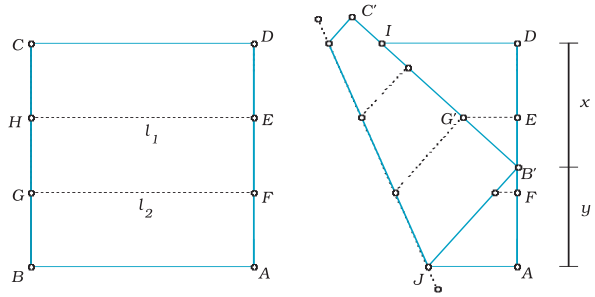


A perpendicular is constructed from the line *AB* through *B*′. The trisection can be appreciated now by recognising that the three right triangles *EBB*′, *BF*′*B*′ and *BF*′*D*′ are congruent.



Details, including useful diagrams, are available at various websites including hverrill.net/pages~helena/origami/trisect.

5. To double a cube, by which is meant to construct a segment of length , we can proceed as follows. Note that 7* is required. This solution is due to Peter Messer, from Problem 1054 in *Crux Mathematicorum*, *12*(10), 1986, pp. 284–285. It and more are available from Thomas Hull's origami pages at www.merrimack.edu/~thull/origamimath.html. Having trisected the side of a square, we can extend this and fold the square into thirds. All that is then required is to apply 7* to bring *B* and *G* to *AD* and respectively:



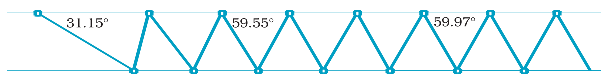
It can now be demonstrated that

$$\frac{DB'}{B'A} = \sqrt[3]{2}$$

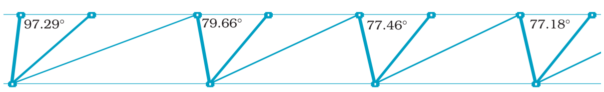
To see this, let $DB' = x$ and $B'A = y$, as shown above. Since $AJ + JB' = x + y$ we can use Pythagoras' theorem to express each of *AJ* and *JB*′ in terms of *y*. Noting that $DE = G'B' = \frac{1}{3}(x + y)$ and that right triangles *G*′*EB*′ and *JAB*′ are similar helps develop the desired result. This is a challenging result for a student in the middle school, but it is good to have challenges ready should occasion demand. It can be shown more generally that folding with 7* produces solutions to the general cubic equation (e.g., see Geretschlager, 1995), but that is beyond our present scope.

Further avenues for exploration

Anyone interested in pursuing further aspects of paper folding could investigate folding ellipses and hyperbolae by using circular paper (Yates, 1943) or turning to any of: Hilton and Pedersen (1983), (1993); Polster (2004); Froemke and Grossman (1988); to investigate a technique that allows the approximation of any rational angle with simple sequences of angle bisections. These sketches show convergence to 60° $\left(\frac{180°}{3}\right)$ from an initial estimate:



and to $\frac{3 \times 180°}{7} \approx 77.14°$:



The procedure can be carried out using cash register tape and involves some simple but useful work with fractions and also the application of an algorithmic approach. What is more, it demonstrates convergence and has the added benefit of producing the reward of constructible polygons, including star polygons at the end of the process. For the interested, the field is open to number theoretic investigation.

Conclusion

Paper folding, then, is a far from trivial enterprise. It is a rich field of mathematics that has the advantage of being readily accessible to middle school students. It introduces students to geometrical ideas that can be developed further using DGS and it encourages an emergent appreciation of proofs that convince and proofs that explain. It is accessible and engaging and so carries affective benefits that traditional approaches to proof via Euclidean geometry have perhaps lacked.

References

Alperin, R. (2000). A mathematical theory of origami constructions and numbers. *New York Journal of Mathematics, 6*, 119–133.

Christou, C. et al. (2004). Proofs through exploration in dynamic geometry environments. *International Journal of Science and Mathematics Education, 2*, 339–352.

Froemke, J. & Grossman, J. (1988). An algebraic approach to some number-theoretic problems arising from paper-folding regular polygons. *The American Mathematical Monthly, 95*(4), 289–307.

Geretschlager, R. (1995). Euclidean constructions and the geometry of origami. *Mathematics Magazine, 68*(5), 357–371.

Hanna, G. & Jahnke, H. N. (1996). Proof and proving. In A. J. Bishop et al. (Eds), *International Handbook of Mathematics Education* (pp. 877–908). Dordrecht: Kluwer Academic Publishers.

Healy, L. & Hoyles, C. (2001). Software tools for geometrical problem solving: Potentials and pitfalls. *International Journal of Computers for Mathematical Learning, 6*, 235–256.

Hilton, P. & Pedersen, J. (1993). Geometry: A gateway to understanding. *The College Mathematics Journal, 24*(4) 298–317.

Hilton, P. & Pedersen, J. (1983). Approximating any regular polygon by folding paper. *Mathematics Magazine, 56*(3), 141–155.

Hull, T. (1996). A note on “impossible” paper folding. *The American Mathematical Monthly, 103*(3), 240–241.

Huzita, H. (1992). Understanding geometry through origami axioms. In J. Smith (Ed.), *Proceedings of the First International Conference on Origami in Education and Therapy* (COET91, pp. 37–70). Cited in Hull, T. (2003) at http://www.merrimack.edu/~thull/origamimath.html.

Jones, K. (2002). Implications for the classroom: Research on the use of dynamic software. *Micromath*, Autumn.

Pegg, J. (1995). Learning and teaching geometry. In J. Pegg & L. Grimison (Eds), *Teaching Secondary School Mathematics*. Sydney: Harcourt Brace.

Polster, B. (2004). Variations on a theme in paper folding. *The American Mathematical Monthly, 111*(1), 39–47.

Raphael, D. & Wahlstrom, M. (1989). The influence of instructional aids on mathematics achievement. *Journal for Research in Mathematics Education, 20*(2), 173–190.

Schlöglmann, W. (2002). Affect and mathematics learning. In A. D. Cockburn & E. Nardi (Eds), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (pp. 185–192). UK: University of Norwich.

Senk, S. (1985). How well do students write geometric proofs? *Mathematics Teacher, 78*(6), 448–456.

Senk, S. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education, 20*(3), 309–321.

Serra, M. (1994). *Patty Paper Geometry*. Emeryville: Key Curriculum Press

Sowell, E. (1989). Effects of manipulative materials in mathematics instruction. *Journal for Research in Mathematics Education, 20*(5), 498–505.

Tall, D. (1995). *Cognitive Development, Representations and Proof* (paper presented at the conference on Justifying and Proving in School Mathematics, Institute of Education, London, December 1995). Accessed at http://www.warwick.ac.uk/staff/David.Tall/pdfs/dot1995f-repns-proof.pdf

Verrill, H. (2002). Accessed at http://hverrill.net/pages~helena/origami/trisect/.

Williams, G. (1999). Now, how to amuse them today? In K. Baldwin & J. Roberts (Eds), *Mathematics: The Next Millennium* (Proceedings of the Seventeenth Biennial Conference of the Australian Association of Mathematics Teachers). Adelaide: AAMT Inc.

Yates, R. (1943). Folding the conics. *The American Mathematical Monthly, 50*(4), 228–230.

Editors’ Note

This paper is based on a presentation at the *Quality Mathematics in the Middle Years* conference (Fremantle, 2005), the proceedings of which are being published by AAMT.