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ANGLE DEFECT and Descartes' theorem

We discuss here a delightfully simple theorem of Descartes, which will enable us to determine easily the number of vertices of almost every polyhedron.

Angle defect

We define the angle defect at a vertex of a polyhedron to be the amount by which the sum of the face angles at that vertex falls short of 2π. For example, for the regular tetrahedron, the angle defect is (in degrees) is:

$$360 - 60 - 60 - 60 = 180° = \pi.$$

The total angle defect of the polyhedron is defined to be what one gets by adding up the angle defects at all the vertices of the polyhedron. We call the total defect T.

Here is a simple exploratory exercise, good for class use!

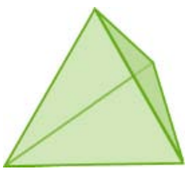
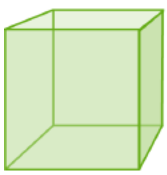
Consider some simple polyhedra, and determine the angle defects and the total angle defect T. Can you make a conjecture about T? Use Table 1.

A reasonable conjecture would seem to be that the total angle defect T is always 720° or 4π. How might we prove this?

Descartes' theorem

Rene Descartes lived from 1596 to 1650. His contributions to geometry are still remembered today in the terminology "Descartes' plane". Descartes discovered that there is a connection between the total defect, T, and the *Euler number* $V - E + F$, where V, E, F denote the number of vertices,

Table 1

Polyhedron						
Number of vertices	4					
Angle defect	180					
Total angle defect, T	720					

edges and faces of the given polyhedron. We might point out that strictly the Euler number is a relationship between the number of edges, vertices and regions ("faces") of a planar graph — a finite figure in the plane having no intersecting sides or loops. However, the number is easily applied to most polyhedra, for by looking 'through a face' of the polyhedron we can obtain a planar graph. For example, in the case of the cube, we obtain the diagram given in Figure 1.

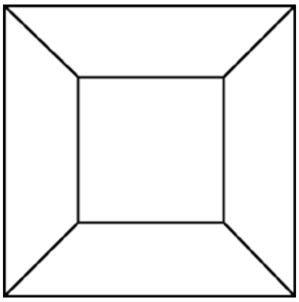


Figure 1

For the cube we have $V = 8$, $E = 12$ and $F = 6$. In the above planar representation, we have $V = 8$, $E = 12$, and the number of regions (F) is 6. The (large) square face we are looking through is ignored, and replaced by the exterior region of the figure, thus retaining the F value. This representation of the cube (and similarly of other polyhedra) is called a *Schlegel diagram*.

We can now state Descartes' Theorem.

Descartes Theorem

The total angle defect of a polyhedron and the Euler number of that polyhedron are related by

$$T = 2\pi\,(V - E + F) \qquad (*)$$

This means that for any polyhedron having Euler number 2, $T = 4\pi$, or 720°. (There are, in fact, occasional non-convex polyhedra which do not have Euler number 2.)

Proof of the theorem

Looking at the right hand side of (*), we can think of associating the quantity 2π with each vertex, edge and face of the given polyhedron. Let us relate this idea to a particular face of the polyhedron (see Figure 2).

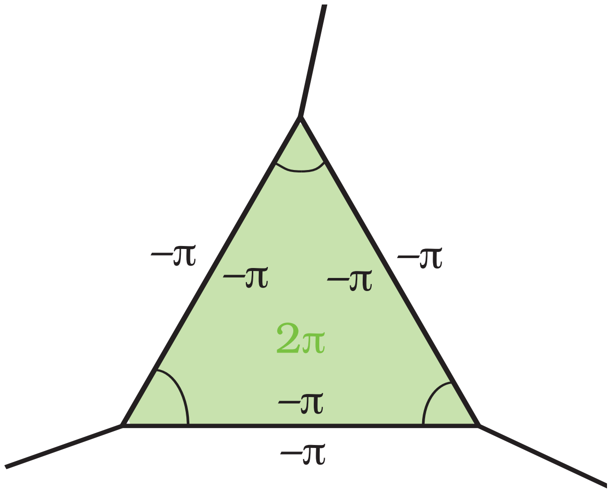


Figure 2

This face contributes 1 towards the number F, so we associate 2π with the face.

Each edge of the face is associated with the value -2π. However, each edge is shared by two faces, so with respect to the given face, we agree to associate just $-\pi$ with each edge. If the edge has n faces, then the total quantity associated with the given face so far is

$$(2 - n)\pi.$$

We note that the sum of the angles of the polygon is in fact

$$(n - 2)\pi. \qquad (**)$$

Finally, for each vertex, we wish to allocate 2π. Since each vertex is shared, we need to make a partial contribution from our given face. We agree first to give the value of the face angle at each vertex. This will be insufficient, as the angles at any vertex add to less than 2π. So let us say for the given face F, the amount contributed to the total vertex count is the sum of the face angles, plus a certain composite angle defect, δF, to compensate for the shortfall. (We could try to define this more precisely, but we are really only interested in the total angle defect.)

So the contribution of this face to the right hand side of (*) is now

$$\delta F + \text{the sum of the angles of } F + (2 - n)\pi$$
$$= \delta F \qquad \text{by } (**).$$

Summing over all the faces we obtain

$$T = 2\pi\ (V - E + F) = S\ \delta F$$

as required, T giving the total angle defect.

As we have noted, when the Euler number is 2, we obtain $T = 4\pi = 720°$.

This number is sometimes called the *total curvature of the sphere*, and is related to the surface area formula for the sphere, $A = 4\pi r^2$.

A bonus result

Descartes theorem is an unexpected and pretty result — and it is useful as well.

For any regular or semi-regular polyhedron, Euler's formula holds, and the vertices are all alike. Hence the angle defect is the same at every vertex. This means that we now have a quick way of determining the number of vertices, even for quite complicated polyhedra. Thus, working in degrees, the number of vertices is given by

$$V = \frac{720}{\text{angle defect}}$$

For example, the truncated icosahedron, which is the polyhedron giving the structure of the soccer ball, has two hexagons and a pentagon meeting at each vertex. Working in degrees, the angle defect is therefore

360 – 120 – 120 – 108 = 12°.

We deduce immediately that this polyhedron has 720 / 12 = 60 vertices. My reference book tells me that this is the right answer!

References

For a general discussion on the significance of the number 4π, see http://www.neubert.net/DESCarte.html.

For the numbers associated with various polyhedra, see Cundy, H. M. & Rollett, A. P., *Mathematical Models*. Oxford.

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editorial

Welcome to another (busy) new year! A special welcome also to new readers of *AMT*. This year we hope to bring you articles that provide much food for thought — both pedagogically and mathematically. Perhaps some articles may provoke some controversy — of the civilised, scholarly kind, of course.

At times, reading articles in journals such as *AMT* and others published by AAMT and other similar organisations, one can get the impression that mathematics educators all agree, more or less, on the broad principles that might be gathered under a "constructivist" umbrella, and implement these. Yet, experiences "in the field" can suggest otherwise.

Over the last couple of years I have worked as a teacher and researcher in schools, observed classes, listened to teachers talk, and heard stories from pre-service students and parents. There seems to persist a culture of teaching mathematics that preserves the traditions familiar to many of those educated in the middle decades of last century. The struggle by teachers to meet adequately the educational needs of students as well as the demands of educational institutions seems often to be decided in favour of meeting systemic demands. The two needs — those of the students and those of the institution of education — seem to act in opposition to each other, rather than provide a stereoscopic picture of mathematics education as it should be realised in an Australian culture. Thelma Perso, in her reflective article in this edition, contemplates this debate in the context of numeracy and mathematics.

Despite conservative, or perhaps, preservationist views on mathematics education, changes do creep in. Educational practice evolves — perhaps not quickly enough in this time of rapid change brought about by technological innovations. Many students, particularly those now in school, take for