FOSTERING MULTIP
using array-b

Multiplicative thinking has increasingly been the focus of research and writing in recent times (Clark & Kamii, 1996; Fuson, 2003; Mulligan & Mitchelmore, 1997). Several academics have written about the importance of distinguishing between additive and multiplicative reasoning (e.g., Clark & Kamii, 1996; Jacob & Willis, 2001). Sowder (2002) provides a simple but powerful example of the crucial difference between additive and multiplicative thinking using an investment scenario: one person invests $2 and gets back $8, while the other person invests $6 and gets back $12. Typically additive thinkers simply calculate the difference between the investment and the profit, and conclude that the deals are the same because both people make a $6 profit. Multiplicative thinkers, on the other hand, can appreciate that the first investment quadrupled, whereas the second investment only doubled, so the first investment is a "better deal."

0	Emergent (cannot yet count)
1	One-to-one counting (can count a single collection only)
2	Count from one with materials (counts all — two collections of materials)
3	Count from one using imaging (counts all — two screened collections)
4	Advanced Counting (counts on from one of two collections)
5	Early Additive Part-Whole (uses simple partitioning & recombining)
6	Advanced Additive Part-Whole (uses a range of additive part-whole strategies)
7	Advanced Multiplicative Part-Whole (uses a range of multiplicative p-w strategies)
8	Advanced Proportional Part-Whole (uses a range of proportional p-w strategies)

Figure 1. Overview of New Zealand Number Framework

The New Zealand (NZ) Number Framework recognises the way that multiplicative thinking builds on additive thinking, and in turn provides the foundation on which proportional reasoning can be built (see Figure 1 and Ministry of Education, 2005a, n.d.). Research evidence supports the idea that students have difficulty reasoning proportionally unless they can use multiplicative part-whole strategies, and find it difficult to reason multiplicatively unless they have a good grasp of additive part-whole strategies (Young-Loveridge & Wright, 2002). The NZ Numeracy Project credits students with Stage 7, Advanced Multiplicative Thinking, if they can use at least two difference multiplicative part-whole strategies to solve problems such as finding the total muffins in six baskets each with 24 muffins, and/or finding how many four-wheeled cars could be made from 72 wheels. Examples of multiplicative strategies for the muffins problem include:

- $6 \times 20 = 120$, $6 \times 4 = 24$, $120 + 24 = 144$ (standard place-value partitioning);
- $6 \times 25 = 150$, $150 - 6 = 144$ (compensation);

LICATIVE THINKING

ased materials

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- 6 × 24 = 12 × 12 = 144 (doubling and halving).

Possible multiplicative part-whole strategies for the wheels problem include:

- 80 ÷ 4 = 20 so 72 ÷ 4 = 20 – (8 ÷ 4) = 18 (compensation);
- 10 × 4 = 40, 72 – 40 = 32, 8 × 4 = 32, 10 + 8 = 18, so 18 × 4 = 72 (reversibility and place-value partitioning);
- 9 × 8 = 72 so 18 × 4 = 72 (doubling and halving) (Ministry of Education, 2005a).

From concrete materials to abstract entities via imaging

It is widely accepted that so-called "concrete materials" are helpful in introducing mathematics to children in the early childhood and junior primary years. However, anecdotal evidence suggests that teachers of senior primary and secondary students make little or no use of concrete materials, assuming their students can deal with numbers as abstract entities. According to many writers, students' thinking moves from a dependence on materials, through a transitional stage where imaging is used, to a stage where understanding of number properties can be used to solve problems in mathematics (Fuson, 2003; Hughes, 2002; Ministry of Education, 2005c; Pirie & Kieran, 1994).

The importance of imaging can be seen in the work of mathematics education researchers who emphasise the importance of visualisation and mental imagery (e.g., Anghileri, 2001; Yackel, 2001). According to Yackel, "carefully selected scenarios have the potential of forming an image basis for students' mathematical activity" (p. 27). Anghileri argues that apparatus and/or manipulatives (e.g., bead frames or bead strings grouped in tens by colour) are useful calculating aids for developing imagery.

Imaging can also be seen in the work on so-called "diagram literacy," thought to be an essential component of students' mathematical development (Diezmann & English, 2001; NCTM, 2000). A diagram can be used as a visual representation (or image) to display information in a spatial layout. The advantage of a diagram is that it can be used to analyse and reveal the structure of a problem (e.g., the relationship between the parts and the whole), and hence provide a basis for its solution. According to Diezmann and English (2001), the ability to use diagrams effectively is critical for mathematical thinking and learning, and teachers have an important role in facilitating the development of students' diagram literacy. However, Yackel (2001) warns that the meaning of diagrams may be lost on children who do not "see" the critical features of a diagram in the same way as an adult, because of their limited experience. For example, some children do not understand that the sum of the parts of a number must always equal that number (the whole). Checking children's interpretation of diagrams is important to ensure that they understand the relationship between the diagram and the problem it represents.

Conceptual and procedural knowledge

Several authors have written about the importance of distinguishing between conceptual knowledge (knowledge of mathematical relationships that is constructed internally and connected to existing ideas) and procedural knowledge (knowledge of the rules and procedures used to carry out routine mathematical tasks) (Kamii & Dominick, 1998; Skemp, 2002; Van de Walle, 1994). Many writers have argued that students must be mentally active, reflect on the ideas presented to them, and construct for themselves relationships among the ideas (both new and old). Procedural knowledge is seen as valuable in that it allows routine tasks to be done easily, but a strong connection with conceptual knowledge is seen as a desirable goal. Related to the conceptual/procedural distinction is Skemp's differentiation of relational understanding (knowing what to do and why), from instrumental understanding (knowing the rules without reasons). According to Skemp, mathematical mismatches can occur when a student's goal is to understand instrumentally, but the student is taught by a teacher whose goal is relational understanding, or the reverse. Skemp argues that relational mathematics has several advantages, including that: it is more adaptable to new tasks; it is easier to remember; and it can be effective as a goal in itself.

Tension between conceptual and procedural knowledge is evident in the debate about when (or if) to teach formal written algorithms. Some writers warn that teaching paper and pencil algorithms before part-whole thinking is established is dangerous, and may damage students' developing number sense (Bass, 2003; Gehrke & Biddulph, 2001). The NZ Numeracy Project materials state explicitly that "students should not be exposed to standard vertical algorithms until they [can] use part-whole mental strategies" (Ministry of Education, 2005a, p. 8, bold in the original). According to Bass (2003), although traditional algorithms are "cleverly efficient" because minimal space and writing are used, they are "opaque" in that the mathematical meaning of the steps is not clearly evident. Bass asserts that "if these algorithms are learned mechanically and by rote, the opaque knowledge, unsupported by sense making and understanding, often is fragile and error-prone" (p. 326).

Counting-based and collections-based models

Yackel (2001) has identified two different conceptions of number underpinning children's solution strategies for problems involving addition, subtraction, multiplication, or division. Counting-based (or sequence-based) solutions are based on the number line. They begin with one of the numbers in the problem and involve jumping along the number line, either forwards (in the case of addition or multiplication) or backwards (in the case of subtraction or division). Even when there is no evidence of counting per se, it is assumed that abbreviated counting (e.g., counting on or skip counting) is the basis for the solution (Yackel, 2001). The empty number line developed by the Dutch is a good example of a counting-based model (Beishuizen, 1999; Carr, 1998; Klein, Beishuizen & Treffers, 1998).

Collections-based solutions involve the partitioning of numbers into component parts and the subsequent joining (in the case of addition or multiplication) or separating (in subtraction or division) of the parts to get the answer (Yackel, 2001). Standard place-value partitioning (breaking up numbers according to the value of the units, such as hundreds, tens & ones) is just one way of partitioning numbers. A different partitioning strategy might be based on doubling or halving (Young-Loveridge, 2002).

The number frameworks used in various numeracy initiatives usually begin with counting at lower levels and progress to the use of derived number facts at higher levels, implying that collections-based ways of thinking about numbers are more sophisticated than counting-based approaches (e.g., Ministry of Education, 2005a; NSW Department of Education and Training, 2001). Yackel (2001, p. 25) argues that it is "important for children to have both a collections-based and a counting-based conception of number," because of the flexibility that it gives them in terms of possible solution strategies.

New Zealand's numeracy projects

The initial Numeracy Project materials for teachers presented multiplicative thinking in terms of array models (Ministry of Education, 2001; see Appendix A). The advantage of array models is that they show two-dimensional multiplicative processes, and nicely illustrate various number properties, such as the associative property (e.g., 3 × 27 transformed into 9 × 9), and the distributive property (e.g., 24 × 6 as 20 × 6 and 4 × 6). By repositioning parts of an array, the relationships among various alternative partitionings can be made clear, an obvious advantage of using a collections-based model.

Since 2002, all the diagrams in the teachers' materials have been changed to number line models, reflecting a greater emphasis on counting-based processes (Ministry of Education, 2005a). These diagrams essentially show a repeated addition model of multiplication, with a linear (unidirectional) process of jumps along a line, from left to right, or right to left, depending on the operation involved. Yet the literature on multiplication strategies argues that repeated addition is a more primitive strategy than an array-based approach (Fuson, 2003; Mulligan & Mitchelmore, 1997). It is unfortunate, in my view, that the array diagrams from 2001 were not retained and presented alongside the number line models to help teachers appreciate both counting-based and collections-based conceptions of number.

Multi-coloured/shaded grids

Recently I became intrigued with a piece of computer software (Maddy the Multiplier), that allows students to partition a grid showing a multi-digit multiplication problem by starting with their existing knowledge of multiplication (the known), and building onto this the remaining part/s of the array (the unknown) (for details, see The Learning Federation, n.d.). A computer mouse is used to partition the array into parts, by sliding a horizontal dividing line up and down and a vertical dividing line from side to side (by clicking & dragging), to stretch or shrink the known array (shown in one colour). The remainder of the array can then be further partitioned into two or three smaller arrays (each shown by a different colour/shade). I developed a "paper version" of the software using a structured grid and different-coloured highlighter pens to show the smaller arrays within the overall larger array (see Appendix B). The grid is structured to show sections of 10 by 10 (heavy black lines), and the 5 by 5 sub-sections within these (fine double lines). A multi-digit multiplication problem up to 35 × 25 can be shown using the grid. The base-ten structure of the grid makes it easy to see the six blocks of 100, the five blocks of 50 (5 x 10) that together make another 250, and a block of 5 x 5 that completes the array. Summing the partial products: 600 + 250 + 25, gives a total of 875, the product of 35 by 25.

Students start by colouring the array for a multiplication fact they do know. They then use different colours for the remaining arrays. Adding the partial products of each smaller array yields the overall product. We can see how the system works with a simple multiplication problem like 3 × 12 (see Figure 2 i). For example, colouring/shading a 3 by 10 array in one colour/shade leaves an array of 3 by 2, which accounts for the remaining 6 (see Figure 2 ii). The array can also be made from two identical blocks of 3 by 6 that are side by side (see Figure 2 iii). Moving one of those blocks directly below the other reveals the equivalence of a 6 by 6 array to the original 3 by 12 array (see Figure 2 iv). An alternative way of partitioning the 3 by 12 array is to distinguish three identical blocks of 3 by 4 that are side by side (see Figure 2 v). Moving the second and third blocks directly below the first block reveals the equivalence of a 9 by 4 array to the original 3 by 12 array (see Figure 2 vi). Three strips of 12, each a different colour/shade, shows the repeated addition strategy of 12 + 12 + 12 (see Figure 2 vii). This final model differs from the other six in being a linear (one-dimensional) additive model, composed of three single rows of each colour. The other six examples are two-dimensional multiplicative models, each composed of

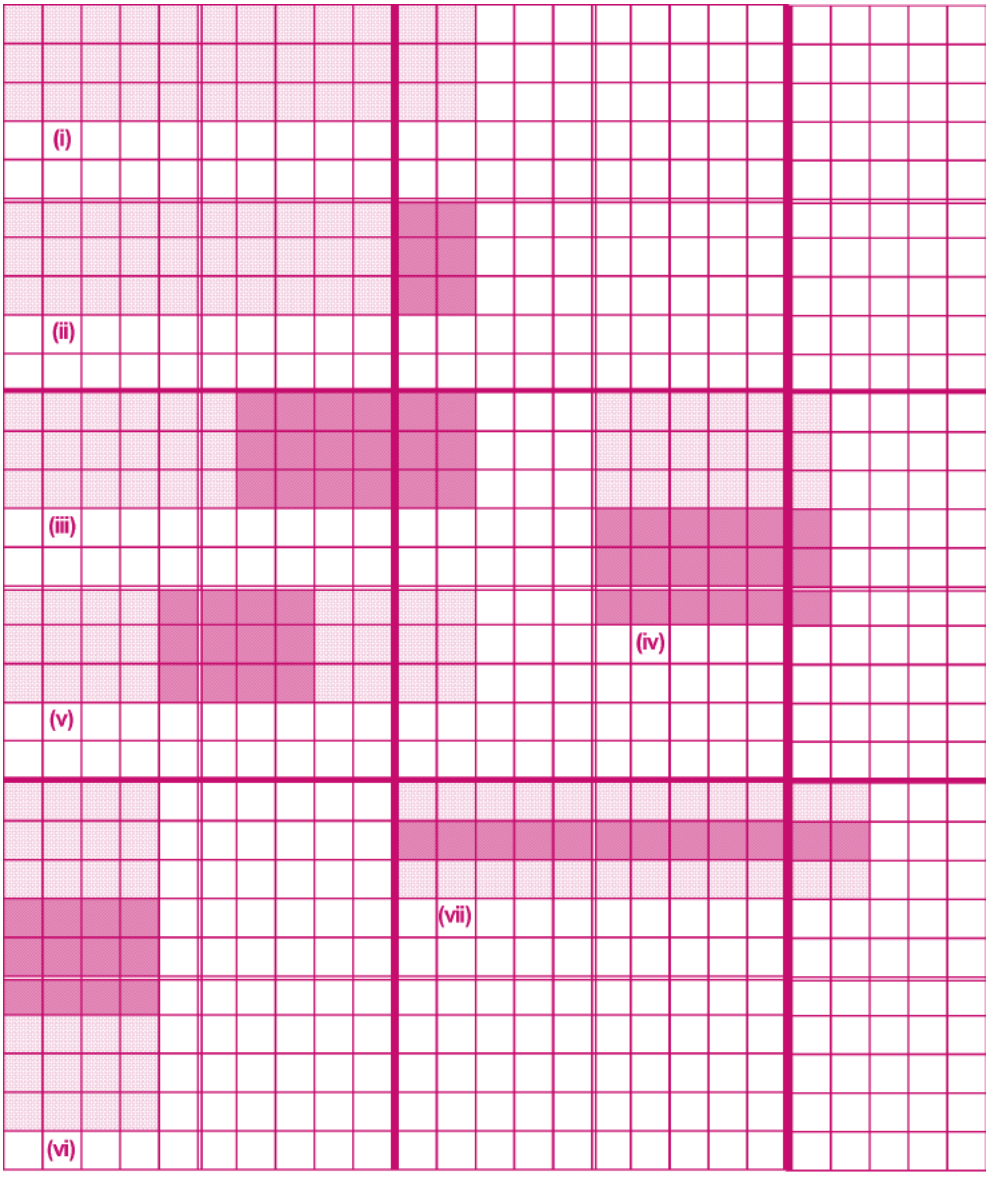

Figure 2. A range of ways to partition 3 × 12 = 36 using multiplicative (array-based) partitioning (i to vi) and additive (linear) partitioning (vii), shown with different colouring/shading.

different combinations of smaller arrays within the larger array. One of the advantages of this paper version with different colouring/shading is that a permanent record can be made of the alternative partitionings, and this can be shared with other students and with the teacher.

The need to strengthen students' multiplicative thinking

The importance of strengthening students' multiplicative thinking has been demonstrated repeatedly by the NZ Numeracy Project evaluation reports (Irwin & Niederer, 2002, 2004; Irwin, 2003; Young-Loveridge, 2004). Typically, only about 10 percent of students at the Year 7–8 level (11- to 13-year-olds) had reached Stage 7, Advanced Multiplicative Part-Whole thinking, at the beginning of the Numeracy Project. After their teachers participated in a professional development programme that provided them with tools to help strengthen students' use of advanced part-whole strategies, the proportion of students in this age group increased to about 25%. However, this is still a fairly small proportion of students at a stage in their schooling where algebra has been or is about to be introduced, particularly in view of the idea that algebraic thinking may depend on students being able to use multiplicative part-whole thinking. The relatively small proportion of students shown to be at stage 7 (Advanced Multiplicative Part-Whole) indicates that much more needs to be done to help students develop advanced multiplicative part-whole strategies.

Conclusion

If our goal is to promote mathematical thinking and help children become flexible problem solvers, then it is important to show students multiple representations of a problem. Because it is important to help students develop both counting-based and collections-based conceptions of number, we should be showing students both number line (counting-based) and array (collections-based) models of multiplication. Arrays allow students to develop a deeper and more flexible understanding of multiplication/division, and to fully appreciate the two-dimensionality of the multiplicative process. Because the grids presented here are structured to show the ten-based structure of the number system, and enable various different partitioning possibilities to be shown using different colouring/shading, they provide a basis for students to image or visualise multiplicative processes. Arrays have enormous potential at the senior primary and secondary school levels to help strengthen students' multiplicative thinking. The multi-coloured/shaded grids described here provide a practical means of doing this.

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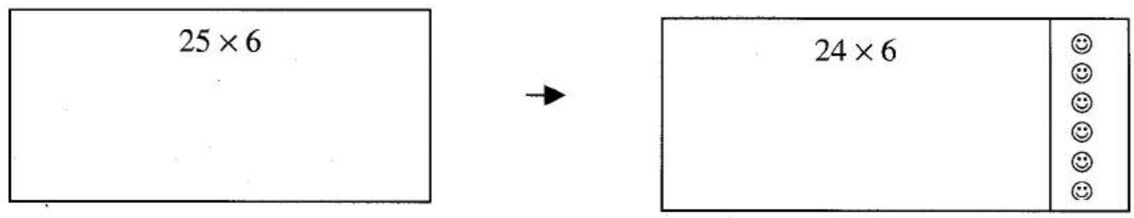
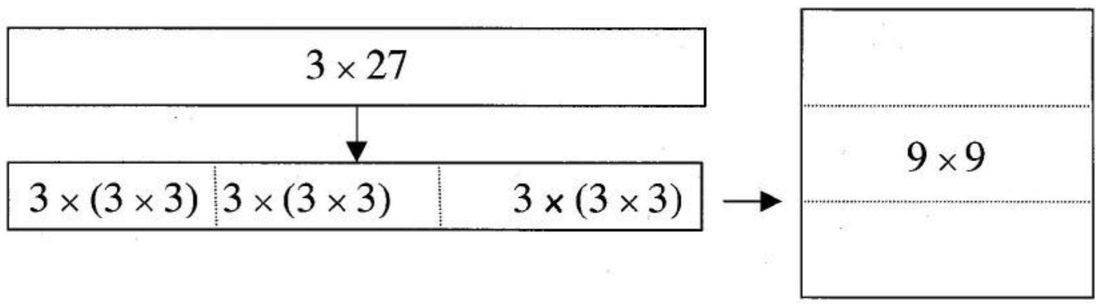
Appendix A

Strategies used by students at Stage 7 Advanced Multiplicative stage to solve multiplication and division problems
(Note: In 2001 this was Stage 6, see Ministry of Education, 2001, Section A, p. 6)

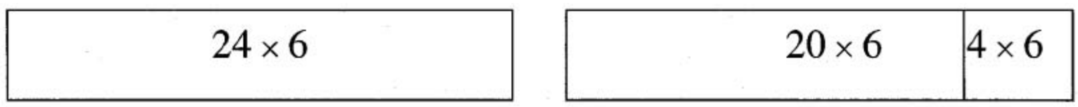
(i) Compensation with tidy numbers
Example: 24×6 as $25 \times 6 - 6$



(ii) Associative property with proportional reasoning (tripling and "thirding")
Example: 3×27 as $3 \times 3 \times 3 \times 3 = 9 \times 9$

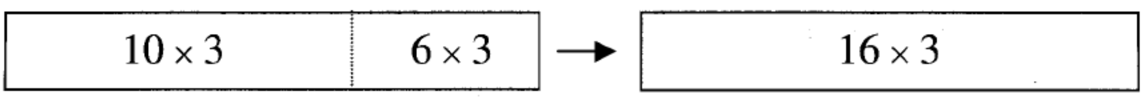


(iii) Distributive property with standard place value partitioning
Example: 24×6 as $(20 \times 6) + (4 \times 6)$



(iv) Proportional reasoning with doubling and halving
Example: $80 \div 5$ as $(80 \div 10) \times 2$

(v) Reversing
Example: $48 \div 3$ as $? \times 3 = 48$,
$10 \times 3 = 30$ and $6 \times 3 = 18$, $16 \times 3 = 48$, so $48 \div 3 = 16$



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