The volume of a torus using cylindrical and spherical coordinates

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Recently I was working through some problem sets on determining volumes by triple integrals in cylindrical and spherical coordinate systems. The textbook I was using included many interesting problems involving spheres, cylinders and cones and the increasingly complex solids that arose from the intersections of two or more of these. I was surprised it did not include the common torus. So naturally I tried it to see what does happen.

I am still unsure why the textbook and several others I have consulted do not include the torus in the problem sets for triple integration in cylindrical coordinates. The technique produces the solution quite neatly. However, after many hours and pages of working, I quite understand why problem sets on triple integrals in spherical coordinates avoid the torus. It is a long and arduous journey. It is however a very interesting journey with many unexpected sights along the way. I have written this brief guide for readers who need some assistance with this trek. Pack some doughnuts for visual aids and sustenance!

Of course, there are much simpler ways to determine the volume of a torus, which raises the question: why are mathematicians willing to spend hours solving a problem by a difficult method when they already know an easy method?

Rather than wasting time in lengthy argument on that issue, I will simply offer my three favourite answers:

1. because I just do not like to be beaten by a problem;
2. because that behaviour is inherent in the definition of "mathematician";
3. because it is fun.

Introducing the torus

Consider a circle in the xy-plane with centre $(R,0)$ and radius $a < R$. This is the circle

$$(x - R)^2 + y^2 = a^2$$

Rotate the circle around the y-axis. The resulting solid of revolution is a torus. It is sometimes described as the torus with inner radius $R - a$ and outer radius $R + a$. It is more common to use the pronumeral r instead of a, but later I will be using cylindrical coordinates, so I will need to save the symbol r for use there.

Before considering cylindrical and spherical coordinates, it is useful to briefly review three simple well-known methods of determining the volume of this torus.

Pappus' Centroid Theorem

The shortest method is to employ *Pappus' Centroid Theorem.*

The circle of radius a has area πa^2. When it is rotated around the y-axis its centroid, which is simply its centre, tracks a circle of radius R. Thus the centre travels a distance of $2\pi R$. Pappus' Centroid Theorem then immediately gives the volume generated as

$$V = \pi a^2 \times 2\pi R = 2\pi^2 a^2 R$$

For this method to be convincing to students they need to prove Pappus' Centroid Theorem, but the proof is within the reach of students who are studying triple integrals in cylindrical or spherical coordinates.

Two slicing techniques

There are two "volume by slicing" techniques that allow the result to be readily determined with a single integral. These will only be described very briefly since they are both well known. For example, in New South Wales, Mathematics Extension 2 students should encounter both these methods in Year 12.

In both cases, rather than rotating the full circle $(x - R)^2 + y^2 = a^2$ around the y-axis to produce the torus, it is simpler to use only the semicircle above the x-axis to generate the "top" half of the torus and double the result.

Take a thin horizontal slice of the semicircle between y and $y + \delta y$. Rotating this around the y-axis gives a prism of an annulus. It has height δy. The annulus at the base of this prism has inner radius $R - \sqrt{a^2 - y^2}$, outer radius $R + \sqrt{a^2 - y^2}$ and thus area $4\pi R\sqrt{a^2 - y^2}$. The required volume is

$$V = 2\int_0^a 4\pi R\sqrt{a^2 - y^2}\,dy = 8\pi R\int_0^a \sqrt{a^2 - y^2}\,dy$$

While the final integral can be evaluated by using a trigonometrical substitution, it is simpler to recognise it as the area of a quadrant of radius a, giving

$$V = 8\pi R \times \frac{1}{4}\pi a^2 = 2\pi^2 a^2 R$$

The other "volume by slicing" method involves taking a thin vertical slice of the semicircle between x and $x + \delta x$. Rotating this around the y-axis produces what is often called a cylindrical shell, though it is really just another prism of an annulus. It has height $y = \sqrt{a^2 - (x-R)^2}$. The annulus at the base has inner radius x, outer radius $x + \delta x$ and thus area $2\pi x\, \delta x$, ignoring the second order terms in δx. The required volume is

$$\begin{aligned} V &= 2\int_{R-a}^{R+a} 2\pi x\sqrt{a^2 - (x-R)^2}\,dx \\ &= 4\pi\int_{R-a}^{R+a} x\sqrt{a^2 - (x-R)^2}\,dx \end{aligned}$$

The substitution $u = x - R$ produces

$$\begin{aligned} V &= 4\pi\int_{-a}^{a} (u+R)\sqrt{a^2 - u^2}\,du \\ &= 4\pi\int_{-a}^{a} u\sqrt{a^2 - u^2}\,du + 4\pi R\int_{-a}^{a} \sqrt{a^2 - u^2}\,du \\ &= 4\pi\left[-\frac{1}{3}\left(a^2 - u^2\right)^{\frac{3}{2}}\right]_{-a}^{a} + 4\pi R \times \frac{1}{2}\pi a^2 \\ &= 2\pi^2 a^2 R \end{aligned}$$

where the second integral has been evaluated by recognising it as the area of a semicircle of radius a.

Reorienting the torus

Cylindrical and spherical coordinate systems often allow very neat solutions to volume problems if the solid has continuous rotational symmetry around the z-axis. While I will use the same torus as discussed above, it will be oriented differently relative to the axes to produce the required symmetry.

Readers who can be trusted with sharp knives should now take a convenient doughnut and place it on a suitable clean horizontal surface. Use a sharp knife to slice it in half with a plane parallel to the horizontal surface. Slide the bottom half out and dispose of it in the obvious manner. (You will need to keep your strength up to cope with the algebra that follows!) The newly cut annular surface of the doughnut should now be sitting on the horizontal surface. Place the origin on the horizontal surface at the centre of the hole in the middle of the doughnut. The horizontal surface is the xy-plane with the z-axis rising vertically from that plane. I will determine the volume of the half-doughnut and double the result.

Readers who do not have a doughnut to hand will have to content themselves with taking the circle $(x - R)^2 + z^2 = a^2$, $y = 0$ and rotating it around the z-axis to produce the required torus. Consider the half of the torus above the xy-plane.

Cylindrical coordinates

In cylindrical coordinates, a point is located by the triple (r, θ, z) where z is the usual rectangular z-coordinate and (r,θ) are polar coordinates in the xy-plane, θ being measured anticlockwise from the positive x-axis. For an arbitrary θ draw an r-axis in the xy-plane at an angle θ anticlockwise from the positive x-axis. The cross section of the torus in the rz-plane is as follows (Figure 1).

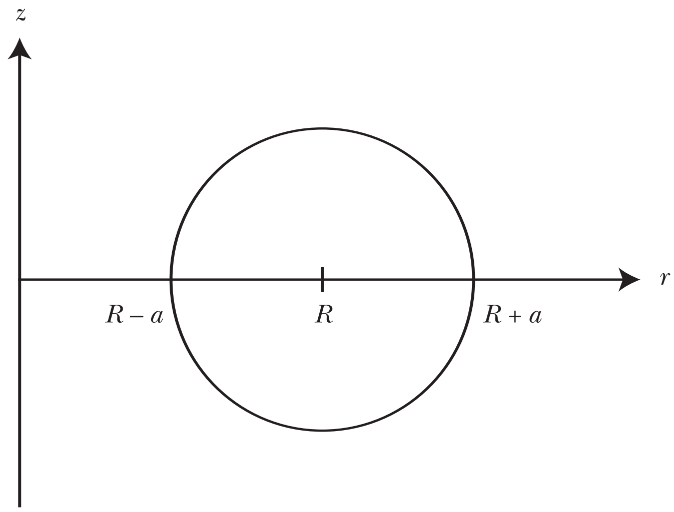

Figure 1

This cross-section is the same for all values of θ. This property hints that a solution by cylindrical coordinates is likely to be efficient. Incidentally, it also means the above figure looks identical to the cross section in the xz-plane.

When drawing the cross-section, only consider positive values of r. Allowing θ to run from 0 to 2π generates the whole torus, so there is no need to consider negative values of r.

The circle shown above is $(r - R)^2 + z^2 = a^2$. The top half of the circle can be generated by allowing r to run from $R - a$ to $R + a$ and allowing z to run from 0 to $\sqrt{a^2-(r-R)^2}$. Thus, using a triple integral in cylindrical coordinates the volume of the torus is

$$V = 2\int_{\theta=0}^{2\pi}\int_{r=R-a}^{R+a}\int_{z=0}^{\sqrt{a^2-(r-R)^2}} dz\; r\, dr\, d\theta$$

It was noted above that the cross section was independent of θ. As a result of this the inner two integrals are constant with respect to θ, and so they can be taken outside the outer integral as a common factor, giving

$$\begin{aligned} V &= 2\int_{\theta=0}^{2\pi} d\theta \int_{r=R-a}^{R+a}\int_{z=0}^{\sqrt{a^2-(r-R)^2}} dz\; r\; dr \\ &= 2\times 2\pi \int_{r=R-a}^{R+a} r\sqrt{a^2-(r-R)^2}\,dr \end{aligned}$$

This integral, with the dummy variable r replaced by x, has already been evaluated in the last of the simpler methods given above, the result again being

$$V = 2\pi^2 a^2 R$$

Spherical coordinates

In spherical coordinates a point is described by the triple (ρ, θ, ϕ) where ρ is the distance from the origin, ϕ is the angle of declination from the positive z-axis and θ is the second polar coordinate of the projection of the point onto the xy-plane.

Allow θ to run from 0 to 2π. For any θ, the cross-section in the resulting rz-plane is as shown in Figure 2. Only consider the half of the torus above the xy-plane. Let OB be a tangent to the semicircle.

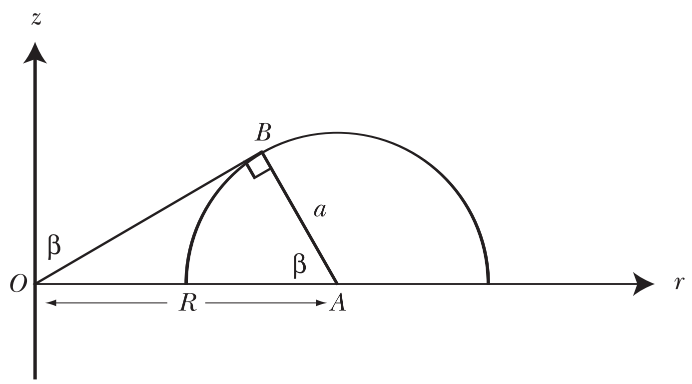

Figure 2

For the arbitrary θ, determine the integration limits for ϕ. Imagine ϕ as controlling a ray in the rz-plane, the ray being able to pivot around the origin. Start with $\phi = 0$, meaning the ray overlaps with the positive z-axis. Allow ϕ to increase, so the ray rotates clockwise around the origin. The ray first intersects the semicircle when it overlaps the tangent OB, where $\phi = \beta$. Triangle OAB is right-angled, so

$$\cos\beta = \frac{a}{R}$$

The ray continues to intersect the semicircle until the ray overlaps the r-axis where

$$\phi = \frac{\pi}{2}$$

Hence allow ϕ to run from $\beta = \cos^{-1}\left(\frac{a}{R}\right)$ to $\frac{\pi}{2}$.

Now consider an arbitrary ϕ within this range and determine the integration limits for ρ (Figure 3).

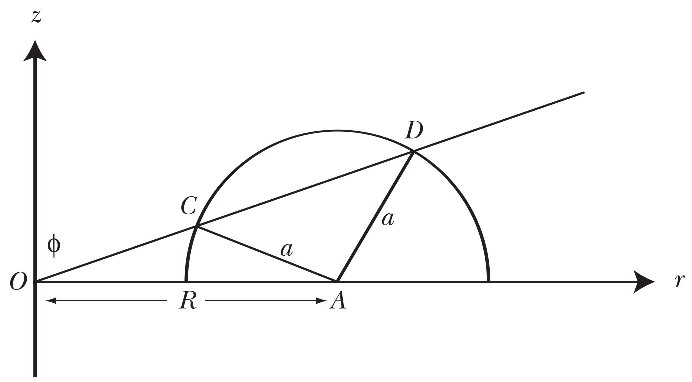

Figure 3

The points to be included in the integration are those making up the interval CD. That is, the minimum acceptable value of ρ is the length of OC and the maximum is the length of OD. The lengths of OC and OD can be determined by applying the cosine rule to triangles OAC and OAD respectively. Let j denote the length of OC in the first case and OD in the second. Curiously, both triangles give the same equation,

$$\begin{aligned} a^2 &= j^2 + R^2 - 2jR\cos\left(\frac{\pi}{2} - \phi\right) \\ &= j^2 + R^2 - 2jR\sin\phi \end{aligned}$$

This is a quadratic equation in j.

$$j^2 - (2R\sin\phi)j + R^2 - a^2 = 0$$

Solving gives

$$\begin{aligned} j &= \frac{2R\sin\phi \pm \sqrt{4R^2\sin^2\phi - 4\left(R^2 - a^2\right)}}{2} \\ &= R\sin\phi \pm \sqrt{R^2\sin^2\phi - \left(R^2 - a^2\right)} \end{aligned}$$

or

$$j = R\sin\phi \pm \sqrt{a^2 - R^2\cos^2\phi}$$

That last rearrangement is probably not an obvious thing to do and it is not clear whether it improves things or not. If the rearrangement is not done now, then at a later point in the working it will become the obvious thing to do. I will do it now since it does marginally simplify the next few steps. If this were being set as a problem for students, this is a good point to provide guidance by asking them to "Show $j = R\sin\phi \pm \sqrt{a^2 - R^2\cos^2\phi}$."

The two solutions to this quadratic are the lower and upper integration limits for ρ, corresponding to the cases where j is the length of OC and OD respectively. It now becomes important to distinguish between the two solutions. Let the lengths of OC and OD be j_1 and j_2 respectively. That is,

$$j_1 = R\sin\phi - \sqrt{a^2 - R^2\cos^2\phi}$$
$$j_2 = R\sin\phi + \sqrt{a^2 - R^2\cos^2\phi}$$

After lengthy algebra such as the above some reasonableness checks are in order.

Consider the case $\phi = \beta$. This makes the points C and D coincide at the point B of Figure 2. That is, the minimum and maximum acceptable values of ρ should coincide at the length of OB, which is $\sqrt{R^2 - a^2}$. Substituting $\phi = \beta$ into the expressions for j_1 and j_2 gives

$$j_1 = R\sin\beta - \sqrt{a^2 - R^2\cos^2\beta}$$
$$j_2 = R\sin\beta + \sqrt{a^2 - R^2\cos^2\beta}$$

where $\cos\beta = \dfrac{a}{R}$. Hence

$$\sqrt{a^2 - R^2\cos^2\beta} = \sqrt{a^2 - R^2\left(\frac{a}{R}\right)^2} = 0$$

Also, since $0 < \beta < \dfrac{\pi}{2}$,

$$\sin\beta = \sqrt{1 - \cos^2\beta} = \frac{\sqrt{R^2 - a^2}}{R}$$

Hence

$$\begin{aligned} j_1 &= R\frac{\sqrt{R^2 - a^2}}{R} - 0 \\ &= \sqrt{R^2 - a^2} \\ j_2 &= R\frac{\sqrt{R^2 - a^2}}{R} + 0 \\ &= \sqrt{R^2 - a^2} \end{aligned}$$

as expected.

The other easy check is $\phi = \dfrac{\pi}{2}$.

In Figure 3 this places points C and D at the intersections of the semicircle with the r-axis. The resulting lengths of OC and OD should be $R \pm a$. Substituting $\phi = \dfrac{\pi}{2}$ into the expressions for j_1 and j_2 gives

$$\begin{aligned} j_1 &= R\sin\frac{\pi}{2} - \sqrt{a^2 - R^2\cos^2\frac{\pi}{2}} \\ &= R - a \\ j_2 &= R\sin\frac{\pi}{2} + \sqrt{a^2 - R^2\cos^2\frac{\pi}{2}} \\ &= R + a \end{aligned}$$

as required; that is, both checks produced reasonable results.

Recall that j_1 and j_2 are the integration limits for ρ. Thus, using a triple integral in spherical coordinates the volume of the torus is

$$V = 2\int_{\theta=0}^{2\pi}\int_{\phi=\beta}^{\frac{\pi}{2}}\int_{\rho=j_1}^{j_2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

Since the cross section is independent of θ the inner two integrals are constant with respect to θ, and so they can be taken outside the outer integral as a common factor, giving

$$\begin{aligned}V &= 2\int_{\theta=0}^{2\pi}d\theta\int_{\phi=\beta}^{\frac{\pi}{2}}\int_{\rho=j_1}^{j_2}\rho^2\sin\phi\, d\rho\, d\phi\\ &= 2\times 2\pi\int_{\phi=\beta}^{\frac{\pi}{2}}\frac{1}{3}\left(j_2^3-j_1^3\right)\sin\phi\, d\phi\\ &= \frac{4}{3}\pi\int_{\phi=\beta}^{\frac{\pi}{2}}\left(j_2^3-j_1^3\right)\sin\phi\, d\phi\end{aligned}$$

The next step is to evaluate $j_2^3 - j_1^3$ where

$$\begin{aligned}j_1 &= R\sin\phi-\sqrt{a^2-R^2\cos^2\phi}\\ j_2 &= R\sin\phi+\sqrt{a^2-R^2\cos^2\phi}\end{aligned}$$

and to hope that the product of this result and sin ϕ is integrable.

This step is ugly however it is attempted. Perhaps the least ugly method is to set $u = R\sin\phi$ and $v = \sqrt{a^2-R^2\cos^2\phi}$, so that $j_1 = u - v$ and $j_2 = u + v$. Then wade into the algebra and find

$$\begin{aligned}j_2^3-j_1^3 &= \left(j_2-j_1\right)\left\{j_2^2+j_1j_2+j_1^2\right\}\\ &= 2v\left\{\left(u^2+2uv+v^2\right)+\left(u^2-v^2\right)+\left(u^2-2uv+v^2\right)\right\}\\ &= 2v\left\{3u^2+v^2\right\}\\ &= 2\sqrt{a^2-R^2\cos^2\phi}\left\{3R^2\sin^2\phi+a^2-R^2\cos^2\phi\right\}\\ &= 2\sqrt{a^2-R^2\cos^2\phi}\left\{a^2+3R^2-4R^2\cos^2\phi\right\}\end{aligned}$$

By this point any sane person would have concluded that spherical coordinates are just not the bright way to solve this problem, but people who use mathematics for recreational purposes are made of sterner stuff and will press on.

Substituting this result into the expression for the volume gives

$$V = \frac{8}{3}\pi\int_{\phi=\beta}^{\frac{\pi}{2}}\sqrt{a^2-R^2\cos^2\phi}\left\{a^2+3R^2-4R^2\cos^2\phi\right\}\sin\phi\, d\phi$$

This integral does not look very promising. However, the volume of a torus *does* have a nice solution so further effort is warranted. The obvious line of attack is to use a substitution to tidy up the mess within the square root sign. Hopefully some unexpected side effect of this will deal with the troublesome sin ϕ factor and allow the integral to be evaluated before I run out of different letters to use as pronumerals.

Try the substitution $R\cos\phi = a\cos w$.

Hence $R^2\cos^2\phi = a^2\cos^2 w$ and $R\sin\phi\, d\phi = a\sin w\, dw$. Turning to the limits:

$$\phi = \beta = \cos^{-1}\left(\frac{a}{R}\right) \Rightarrow w = 0 \ \text{ and } \ \phi = \frac{\pi}{2} \Rightarrow w = \frac{\pi}{2}$$

The substitution has simplified the lower limit, an unexpected bonus.
Applying the substitution gives

$$\begin{aligned} V &= \frac{8}{3}\pi \int_{w=0}^{\frac{\pi}{2}} \sqrt{a^2 - a^2\cos^2 w}\left\{a^2 + 3R^2 - 4a^2\cos^2 w\right\}\frac{a}{R}\sin w\, dw \\ &= \frac{8}{3}\pi \int_{w=0}^{\frac{\pi}{2}} a\sin w\left\{a^2 + 3R^2 - 4a^2\cos^2 w\right\}\frac{a}{R}\sin w\, dw \\ &= \frac{8a^2\pi}{3R}\int_{w=0}^{\frac{\pi}{2}} \sin^2 w\left\{a^2 + 3R^2 - 4a^2\cos^2 w\right\}dw \end{aligned}$$

Finally, the integral is looking manageable. Using the results

$$\begin{aligned} \sin^2 w &= \frac{1}{2}(1 - \cos 2w) \\ \text{and } \ \cos^2 w &= \frac{1}{2}(1 + \cos 2w) \end{aligned}$$

gives

$$\begin{aligned} V &= \frac{8a^2\pi}{3R}\int_{w=0}^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 2w)\left\{a^2 + 3R^2 - 2a^2(1 + \cos 2w)\right\}dw \\ &= \frac{4a^2\pi}{3R}\int_{w=0}^{\frac{\pi}{2}} (1 - \cos 2w)\left\{3R^2 - a^2 - 2a^2\cos 2w\right\}dw \\ &= \frac{4a^2\pi}{3R}\int_{w=0}^{\frac{\pi}{2}} 3R^2 - a^2 - \left(3R^2 + a^2\right)\cos 2w + 2a^2\cos^2 2w\, dw \end{aligned}$$

Then $\cos^2 2w = \frac{1}{2}(1 + \cos 4w)$ gives

$$\begin{aligned} V &= \frac{4a^2\pi}{3R}\int_{w=0}^{\frac{\pi}{2}} 3R^2 - a^2 - \left(3R^2 + a^2\right)\cos 2w + a^2(1 + \cos 4w)dw \\ &= \frac{4a^2\pi}{3R}\int_{w=0}^{\frac{\pi}{2}} 3R^2 - \left(3R^2 + a^2\right)\cos 2w + a^2\cos 4w\, dw \end{aligned}$$

Symmetry could now be used to argue that the terms in cos $2w$ and cos $4w$ will evaluate to zero on integration over 0 to $\frac{\pi}{2}$. Alternatively, integrating in the normal manner gives

$$V = \frac{4a^2\pi}{3R}\left[3R^2w - \frac{1}{2}\left(3R^2 + a^2\right)\sin 2w + \frac{1}{4}a^2 \sin 4w\right]_0^{\frac{\pi}{2}}$$
$$= \frac{4a^2\pi}{3R} \times 3R^2 \frac{\pi}{2}$$
$$= 2\pi^2 a^2 R$$

Of course, I am not suggesting that using triple integrals in spherical coordinates is an efficient way to find the volume of a torus. Given the great potential for algebraic errors, it does not even make a particularly good exam question. It is more the type of problem that could be tackled as a group exercise, where students can check their progress with each other frequently and find the errors before they carry them through too many steps.

What I find most fascinating about this problem is the number of different mathematical concepts that need to be applied to reach the result. Students sometimes seem to treat mathematics as a series of unrelated techniques each with their own clearly demarcated area of application. This problem demonstrates the need to reach for tools that would not normally be thought of as part of the area of knowledge labelled "triple integrals".