A new elementary function for our curricula?

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Introduction

The concept of function is central to the teaching and learning of mathematics. Indeed it has been variously said that the single most important concept in modern mathematics is that of the function (National Council of Teachers of Mathematics (NCTM), 1989; Ferrini-Mundy & Graham, 1991; Tall, 1996). As a mathematical notion, the concept of function is fundamental, yet powerful, and is a unifying theme that is found running throughout most branches of mathematics. Kleiner (1989), in his historical account of the evolution of the function concept was quick to point out that "in fact, the concept of function is one of the distinguishing features of "modern" as against "classical" mathematics" (p. 282).

One particular area where the concept of function finds its *raison d'être* is in the so-called *special functions* (e.g., Lebedev, 1972). The special functions are those functions that arise most frequently in applications and have been studied and used for centuries. Prominent examples include the exponential, logarithmic and trigonometric functions. Because of their remarkable properties and seemingly limitless applicability, Andrews, Askey and Roy (1999) suggest that special functions could be more appropriately labelled as "useful functions". Not surprisingly therefore, the function concept is found to be central to senior secondary and beginning tertiary mathematics curricula (Australian Education Council (AEC), 1991; Ponte, 1992; Tall, 1992; Ryan, 1994; NCTM, 1989, 2000). The idea of function forms a unifying thread that begins with describing basic functional relationships between two quantities, thereby underlying much of secondary school algebra; and culminates in the study of real-valued functions of a real variable, thereby underlying most of senior secondary and introductory tertiary level calculus. As a case in point, the current introductory calculus-based senior mathematics syllabus in New South Wales goes as far as saying that, "much of this course is devoted to the study of properties of real-valued functions of a real variable" (Board of Studies NSW, 1982, p. 35). In his overview of this particular course, Pender (1999) noted that "one of the great changes from Year 10 to Year 11 is that functions and their graphs become the centre of attention" (p. 13).

With the study of real-valued functions of a real variable assuming such a central role in existing mathematics curricula, it is the so-called elementary functions (e.g., Edwards & Penney, 1990, p. 271) that figure most prominently. While many special functions are known to exist (e.g., Abramowitz & Stegun, 1972), at the senior secondary and introductory tertiary levels only the elementary functions are typically encountered. Here the elementary functions are real-valued algebraic functions (such as polynomials, rational or power functions), transcendental functions (traditionally thought of as the exponential, logarithmic, the trigonometric and hyperbolic functions together with their associated inverses (e.g., Finney, Weir & Giordano, 2001, p. 499)) or combinations of these under the operations of arithmetic and function composition. The elementary transcendental functions are the simplest of the special functions and have the widest applicability; their many applications lend considerably to their importance.

If senior secondary and introductory tertiary level mathematics curricula are to reflect mathematics as a constantly evolving and dynamic enterprise, and provide students with a glimpse of what actually goes on in contemporary research mathematics, then teachers of mathematics and mathematics educators need to seek out those areas of mathematics that are currently being developed and determine how best to appropriate aspects of this "new" mathematics into our curricula (Grimison, 1995; AEC, 1991). Certain topic areas within our existing curricula already have the ability to throw up further questions and problems that suggest further realms of mathematics. Appropriating recent developments from those areas where questions and problems naturally arise in our existing curricula, would challenge students and extend their existing conceptions by offering the view that mathematics is a constantly developing and unfolding discipline. For too long mathematics curricula at these levels have often been presented in ways that suggest mathematics is closed and complete. Stasis underlies much of school and introductory tertiary mathematics, particularly in the treatment of special functions. However, in the past thirty years, driven by discoveries of both new special functions and by the newer applications that have been found for existing special functions, mathematicians have witnessed a resurgence of interest in this area (Andrews et al., 1999).

Until quite recently, when it came to the elementary transcendental functions it was commonly perceived that everything there was to know in this domain had been discovered long ago. Recent research efforts in the field of special functions (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996) have however turned up a seemingly new elementary transcendental function whose importance had not previously been recognised but which nonetheless demanded attention. Now going by the name of the "Lambert W function" (Weisstein, 2003), this function is a notable example of what one can appropriate into our curricula with relative ease despite the fact that it is remains a topic of contemporary research interest.

In this paper I wish to introduce the Lambert W function. In doing so I hope to raise the profile of the function to a wider audience of teachers and educators of mathematics, and will argue for the case of its inclusion into our curricula. By presenting properties of this particular function and highlighting some of its applications, it will be shown how our existing curricula at the senior secondary and introductory tertiary levels stand to benefit by its introduction. Acceptance of the Lambert W function as a bona fide elementary transcendental function equal in importance to that of the well-established, traditional class of such functions, however, is expected to be no easy task. The standard set of elementary functions is so deeply and profoundly ingrained in the minds of most teachers and educators of mathematics that to even begin to suggest such a set is somehow incomplete is surely to be met with incredulity.

A question of motivation

As a way of introducing the Lambert W function, I use the historical approach from where it initially arose; namely as the solution to the transcendental equation $ye^y = x$ (Lambert, cited in Corless et al., 1996). Any further historical references surrounding its development will be deferred to a later section towards the end of this paper.

Seeking a closed-form solution[1] to the transcendental equation $ye^y = x$, in terms of y, we are led to the definition for the Lambert W function; it being defined as the inverse of the function $f(x) = xe^x$. Denoting the Lambert W function by $\mathrm{W}(x)$, we see that it is a solution to the equation

$$\mathrm{W}(x)\, e^{\mathrm{W}(x)} = x.$$

The above equation is known as the *defining equation* for the Lambert W function and it is central to the study of this function.

Before sketching W to assist in establishing some of the properties for this function, it is instructive to sketch the function $f(x) = xe^x$. Its curve is given in Figure 1. A little calculus reveals a turning point in this function at (–1,–1/e).

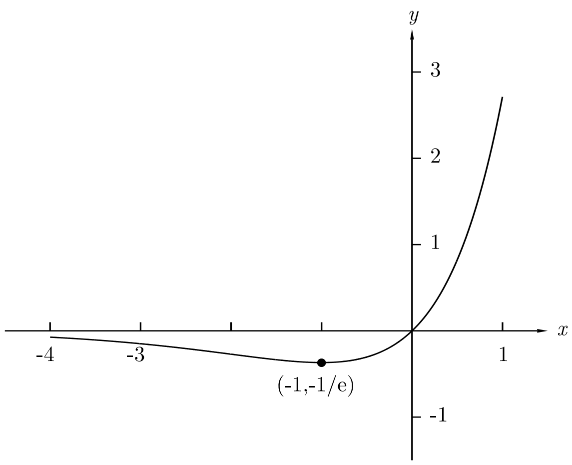

Figure 1. Plot of the function $f(x) = xe^x$.

1. A closed form solution is a solution in terms of known constants (π, *e*, √2, etc.) and known functions (log, exp, sin, etc.).

Reflecting about the line $y = x$ is the graphical link between a function and its inverse and enables W to be readily sketched (see Figure 2). Here the curve is broken up into two separate curves; it is actually two separate functions; since we recall from the definition of a function that for each x there can be at most only one value for y. These two separate functions are depicted by the solid and dashed lines in Figure 2.

The number of solutions to the defining equation for the Lambert W function varies depending on the value for x. From Figure 2 one can readily see that:

1. if $x < -1/e$ the defining equation has no (real) solutions,
2. if $x \geq 0$ the defining equation has one (real) solution, and
3. if $-1/e \leq x < 0$ the defining equation has two (real) solutions and is therefore multi-valued on this domain.

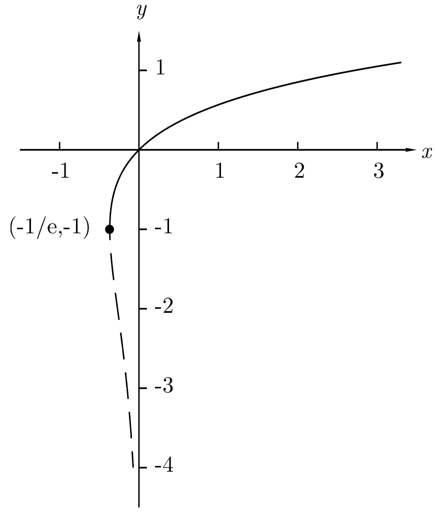

Figure 2. Plot of the Lambert W function f(x) = W(x).

The Lambert W function is therefore similar to the inverse trigonometric functions, in that it is a multi-valued function on a given domain, and a *principal branch* needs to be defined. When x is real it has two branches. In accordance with the practice now in place for naming the branches, the branch satisfying $W(x) \geq -1$ is denoted by $W_0(x)$ and is defined to be the principal branch while the secondary real branch satisfying $W(x) \leq -1$ is denoted by $W_{-1}(x)$.

Reflecting on the process taken in arriving at the Lambert W function we notice that this newly named function is defined to be the inverse of a function that has no special name. Compare this to the case of the exponential which creates a name for its inverse that has no connection with the original function's name, namely the logarithm, or the trigonometric functions whose inverses either use the general notation of f^{-1}, thus producing $\sin^{-1}$, $\cos^{-1}$ and so on, or build their new name for the inverse by modifying the original function's name, thus producing arcsin, arccos and so on.

From a pedagogical point of view the Lambert W function presents an opportunity to work with and further explore inverse functions. Moreover, it is the first and only elementary transcendental function that provides a non-trivial example of branching behaviour on the real domain. The inverse trigonometric functions, of course, also exhibit branching behaviour for real arguments, but their branches are trivial in the sense that they are essentially a shift and possible change in sign of the principal branch that one need not consider them separately. The Lambert W function therefore acts as an invaluable link in familiarising the student with a non-trivial example of a multi-branched function on the real domain, thereby foreshadowing much of what lies ahead, particularly in the higher-level study of functions of a complex argument where multi-branching behaviour is the norm.

Simple properties and applications

The Lambert W function turns out to have a surprisingly rich mathematical structure, which enables students to engage in more insightful mathematical thinking as a progressively more sophisticated exploration of functions and their inverses can be entered into. Usually the introduction of a new special function into mathematics is warranted by its importance and usefulness; and, in helping to establish the "importance" and "usefulness" of this function, some of its associated properties and simple applications will be explored in the following sections.

Fundamental identities

A few fundamental identities for the Lambert W function follow immediately from its definition:

$$\mathrm{W}(0) = 0$$

$$\mathrm{W}\left(\frac{-1}{e}\right) = -1$$

$$\mathrm{W}(x) = \frac{x}{e^{\mathrm{W}(x)}}$$

$$\mathrm{W}(x) = \ln\left(\frac{x}{\mathrm{W}(x)}\right)$$

These identities will prove useful later on, particularly the last two, which enable, for example, a golden ratio like connection for the Lambert W function to be made and allow iterative processes to be explored.

Special values

We know from previous experience in working with the more familiar elementary transcendental functions that for given arguments, special values are known to exist. The Lambert W function should therefore be no different. From inspection of the graph of W (see Figure 2), it is immediately obvious that

$$\mathrm{W}_0(0) = 0 \quad \text{and} \quad \mathrm{W}_0\left(\frac{-1}{e}\right) = \mathrm{W}_{-1}\left(\frac{-1}{e}\right) = -1.$$

Such values were already noted in the previous section.

A question that naturally arises is can we find others? Consider $W_0(e)$. From the defining equation for the Lambert W function, setting $x = e$ yields

$$W_0(e)e^{W_0(e)} = e$$

which on inspection gives $W_0(e) = 1$. In fact, it is not too difficult to recognise that since the Lambert W function is the inverse of the function $f(x) = xe^x$, in general the following simplification rule must hold

$$x = \begin{cases} W_0(xe^x) & \text{for } x \geq -1 \\ W_{-1}(xe^x) & \text{for } x \leq -1 \end{cases}$$

Infinitely many exact values for W can now be generated from this simplification rule. For example,

$$W_0\left(2e^2\right) = 2$$
$$W_0\left(\frac{9}{2}e^{\frac{9}{2}}\right) = \frac{9}{2}$$
$$W_{-1}\left(-2e^{-2}\right) = -2$$

and so on.

Numerical values

The Lambert W function can take on irrational values, $W_0(1)$ being one notable example (Weisstein, 2003). The computation of such values needs to be performed numerically. One way in which this can be readily achieved is by the use of Newton's method. As an excellent application of this method, an approximate value for $W_0(1)$ can be found by finding an approximate solution to the equation $te^t = 1$, where $t = W_0(1)$. If a beginning approximation of 0.5 is used, one application of Newton's method gives a next best approximation of $W_0(1) \approx 0.5710$. Repeated application of Newton's method yields $W_0(1) = 0.567143290\ldots$, the first nine decimal places for this number.

Alternatively, as an illustrative application of the use of technology in the curriculum, a graphics calculator can be used. On a Texas Instruments TI-89 numerical values for the Lambert W function can be obtained using a user-defined function in the following manner:

1. For the principal branch of the Lambert W function, $W_0(x)$ is created by entering:

   Define w(x) = nSolve(te^t=x,t)|t>–1

2. For the secondary real branch of the Lambert W function, $W_{-1}(x)$ is created by entering:

   Define wm1(x) = nSolve(te^t=x,t)|t<–1

In both instances, values for W are calculated by solving the defining transcendental equation for the Lambert W function numerically. A screen shot, taken off a TI-89 showing some numerical values calculated for the Lambert W function using the above user-defined functions is given in Figure 3. Additional values for W are given in Table 1.

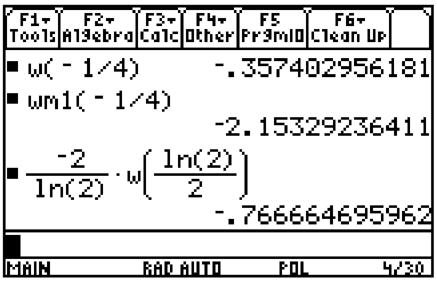

Figure 3. Some values for the Lambert W function obtained using a user-defined function on a graphics calculator.

Table 1. Some values for the Lambert W function.

x	$W_0(x)$	$W_{-1}(x)$
$2e^2$	2	–
e	1	–
2	0.8526…	–
1	0.5671…	–
$1/e$	0.2784…	–
0	0	–
$-1/4$	–0.3574…	–2.1532
$-2/e^2$	–0.4064…	–2
$-1/e$	–1	–1

As the Lambert W function is presently not found on any scientific/graphics calculators, the evaluation of arbitrary values is somewhat more involved than the now trivial procedure of pushing the appropriate buttons on the calculator for the familiar elementary transcendental functions. Recall, however, that it was not that long ago when finding arbitrary values for the more familiar elementary transcendental functions required the use of tables! Once the ubiquitous nature of the Lambert W function is duly recognised, the author believes it will only be a matter of time before we see a designated button for this function appearing on most scientific/graphics calculators.

A special number

Interestingly, since $W_0(1)$ is the solution to the following equations

$$e^{-\Omega} = \Omega$$

$$\ln\left(\frac{1}{\Omega}\right) = \Omega$$

it can be considered a sort of "golden ratio" for exponentials (Weisstein, 2003). From the likeness in this exponential connection to the golden ratio, the number $W_0(1) = 0.567143290\ldots$ is singled out as being a special number associated with the Lambert W function. It is usual for this special number to be referred to as the *omega constant*, Ω (Weisstein, 2003) and is not unlike the practice of special irrational numbers being associated with the other elementary transcendental functions, such as e with the exponential and logarithmic functions and π with the trigonometric functions.

Solution of equations

One area where the Lambert W function would find ready use and applicability in a senior secondary or introductory tertiary level mathematics course would be in the solution of equations. Many equations that involve exponentials (or logarithms) can be solved in terms of the Lambert W function. The general strategy to solving such equations is to move all instances of the unknown to one side of the equation, make it look like the form of the defining equation, namely $f(x)e^{f(x)}$, at which point the Lambert W function provides the solution to the equation. Such an exercise reinforces the importance of form recognition in problem solving while at the same time introduces the method of implicit solution. Let us look at two examples.

As a first example, consider the solution to the equation $x + e^x = 0$. This rather innocuous looking equation can not be solved in closed-form in terms of any of the known elementary (or higher) transcendental functions one is traditionally familiar with. In the past numerical methods have been required in order to find an approximate solution for x. If, however, we rewrite this equation as

$$x = -e^x$$

and move all instances of the unknown to the left hand side we have

$$xe^{-x} = -1.$$

Next, writing the left hand side of the above equation in the form of the defining equation, namely

$$-xe^{-x} = 1$$

enables this equation to be solved in terms of the Lambert W function as

$$-x = W_0(1)$$

or

$$x = -W_0(1) = -0.567143290\ldots$$

Here the principal branch is chosen since the argument is greater than zero. Substituting the numerical value for x into the initial equation can be used to confirm the validity of this solution.

One issue is whether or not the number $-W_0(1)$ is a closed form solution? Is it closed in the way $\sin(2)$, $\ln(2)$, etc., are? Certainly the latter are expressible in terms of familiar functions, and after all, it is familiarity that is important since the solver must regard the function as the final answer and not simply as another question. Just as the student is unlikely to regard the exponential function as providing the solution to the equation $\ln(x) = 2$ until the moment they formally encounter it, the same could therefore be said for the Lambert W function.

A second interesting example that makes use of the Lambert W function in its solution comes from solving the equation $x^2 = 2^x$. By inspection, two solutions to this equation are $x = 2$ and $x = 4$, but are there others? Plotting the two curves $y = x^2$ and $y = 2^x$ on the same diagram reveals that a third solution exists to this equation as the two curves intersect not only at the points $x = 2$ and $x = 4$ but at a third point close to $x = -1$.

Taking the square root of both sides of the above equation, leaves one with

$$x = \pm\sqrt{2^x} = \pm 2^{\frac{x}{2}}$$

Two cases therefore need to be considered. Considering the positive case first, taking the logarithm of both sides and simplifying gives

$$\ln x = \frac{x \ln 2}{2}$$

Exponentiating and moving all instances of the unknown to one side gives

$$xe^{\left(\frac{-x\ln 2}{2}\right)} = 1$$

or

$$-\frac{x\ln 2}{2} e^{\left(-\frac{x\ln 2}{2}\right)} = -\frac{\ln 2}{2}$$

Solving for x results in two solutions since the argument $\frac{-\ln 2}{2}$ (0.346…) lies between $\frac{-1}{e}$ (0.367…) and zero and therefore both branches of the Lambert W function need to be considered on this domain. The two solutions are

$$x = -\frac{2}{\ln 2} W_0\left(-\frac{\ln 2}{2}\right) = 2 \qquad (1)$$

and

$$x = -\frac{2}{\ln 2} W_{-1}\left(-\frac{\ln 2}{2}\right) = 4 \qquad (2)$$

While the solutions to the positive case of this equation can be written in terms of the Lambert W function, on evaluation they reduce to the two trivial solutions, namely that of 2 and 4, which were already known in advance. More importantly, however, these trivial solutions make the suggestion that a simplification rule for W exists. This simplification rule will be established in the next section.

The third solution to this equation must come from a consideration of the negative case. Solving, then, for the negative case in an analogous manner to that used for the positive case, we arrive at the third solution of

$$x = -\frac{2}{\ln 2} \mathrm{W}_0\left(\frac{\ln 2}{2}\right) = -0.76666469\ldots \quad (3)$$

Admitting the Lambert W function to the standard set of known elementary transcendental functions permits a larger class of equations to be solved in closed-form. In fact, one could argue that the examples given above illustrate that the Lambert W function is the simplest example of a root (or roots) to what can be thought of as an *exponential polynomial*, which is a "polynomial" of sorts in that it contains an exponential term. Exponential polynomials have been said to give rise to the next simplest class of functions after the polynomials (Corless & Jeffrey, 2002). So, for example, we have seen that the exponential polynomial $E(x) = x + e^x$ has as its root $x = -\mathrm{W}_0(1)$ while the exponential polynomial $E(x) = x^2 - 2^x$ has as its roots $x = 2$, 4 and

$$-\frac{2}{\ln 2} \mathrm{W}_0\left(\frac{\ln 2}{2}\right)$$

A simplification rule for W

The two rational solutions to the equation $x^2 = 2^x$ for $x > 0$ (see solutions (1) and (2) in the previous section) suggest the solution to the more general equation of

$$x^y = y^x \text{ for } x, y > 0$$

ought to be considered. This equation was first considered by Euler in 1748 (Euler, cited in Knoebel, 1981) and subsequently by many others (e.g. Sved, 1990; Churchhouse, 1995; Bennett & Reznick, 2004). In solving this equation we acknowledge the solution $y = x$. The equation can also be solved in terms of the Lambert W function. Taking the natural logarithm of both sides of this equation and rearranging gives

$$\ln y = \frac{\ln x}{x} y$$

Exponentiating both sides and rearranging yields

$$y \exp\left(-\frac{\ln x}{x} y\right) = 1$$

or

$$-\frac{\ln x}{x} y \exp\left(-\frac{\ln x}{x} y\right) = -\frac{\ln x}{x}$$

after multiplying both sides of this equation by $-\frac{\ln x}{x}$.
Upon solving for y we have

$$y = -\frac{x}{\ln x} \mathrm{W}_k\left(-\frac{\ln x}{x}\right)$$

where $k = -1,0$ denotes the two real branches for the Lambert W function.

From the $y = x$ solution the following simplification rule for the Lambert W function then follows:

$$-\ln x = \begin{cases} W_0\left(-\dfrac{\ln x}{x}\right) \text{ for } 0 < x \le e \\ W_{-1}\left(-\dfrac{\ln x}{x}\right) \text{ for } x \ge e \end{cases} \tag{4}$$

By replacing x with $\frac{1}{x}$ the negative signs in the above simplification rule can be removed resulting in the more compact result of

$$-\ln x = \begin{cases} W_0\left(x\ln x\right) \text{ for } x \ge \dfrac{1}{e} \\ W_{-1}\left(x\ln x\right) \text{ for } 0 < x \le \dfrac{1}{e} \end{cases} \tag{5}$$

Observe how solving a far simpler problem, namely $x^2 = 2^x$, leads to a general simplification rule for W being found. Arriving at the above simplification rule therefore provides the student with an excellent characterisation of the nature of mathematics. It illustrates how the solution to a simpler problem can lead to a more general result being discovered and therefore is representative of a mathematical process often exploited by mathematicians in arriving at interesting new results.

Iteration

Iteration, or the repeated application of some mathematical process (be it a computation, construction, algorithm, etc.) on some initial state, shows up across many areas of mathematics. It can be an extremely important tool in problem solving or as the subject of investigation. For example, in the senior secondary or introductory tertiary level curricula, iteration makes its appearance in Newton's method, which utilises this process in an essential way. The Lambert W function readily lends itself to the process of iteration, and it is therefore natural to explore iterative processes further within this context.

It should be apparent, from the last two fundamental identities for W, that the function is defined in an iterative way. Starting with

$$W(x) = \frac{x}{\exp W(x)}$$

and iterating indefinitely by back substituting W on the left hand side for W on the right hand side, the following continued fraction-like formula for W begins to emerge

$$W(x) = \cfrac{x}{\exp\cfrac{x}{\exp\cfrac{x}{\exp\cfrac{x}{\ddots}}}}$$

Likewise, if we start with

$$W(x) = \ln\left(\frac{x}{W(x)}\right)$$

iterating gives

$$W(x) = \ln\cfrac{x}{\ln\cfrac{x}{\ln\cfrac{x}{\ln\cfrac{x}{\ddots}}}}$$

These two curious looking formulae are sure to arouse the interest of teacher and student alike, and they highlight the inherently iterative nature of this function. The iterative formulae for W couple iterative processes to the Lambert W function, in an intriguing way, at a level that is readily accessible to the student.

One other example connected with the process of iteration and that makes use of the Lambert W function is iterated exponentiation, a perennial problem that has attracted the attention of mathematicians since the time of Euler (e.g., Knoebel, 1981; de Villiers & Robinson, 1986).

Consider the iterated exponential of Euler fame

$$h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$$

The equation consists of an infinite power tower of x-s such that the powers are read from the top down. Euler was the first to prove that this iteration converges on the interval

$$\frac{1}{e^e} < x < e^{\left(\frac{1}{e}\right)}$$

(Euler, cited in Knoebel, 1981). If we take the natural logarithm of both sides of the iterated exponential, we can write

$$\begin{aligned} \ln h(x) &= \ln x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \\ &= x^{x^{\cdot^{\cdot^{\cdot}}}} \ln x \\ &= h(x) \ln x \end{aligned}$$

Upon exponentiating both sides of the above equation we have

$$h(x) = e^{h(x)\ln x}$$

which upon rearranging gives

$$h(x)e^{-h(x)\ln x} = 1$$

On multiplying both sides of the above equation by $-\ln x$ we have

$$-h(x)\ln x e^{-h(x)\ln x} = -\ln x$$

which can be solved for $h(x)$ in terms of the Lambert W function to give

$$-h(x)\ln x = \mathrm{W}_0(-\ln x)$$

so that

$$h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = \frac{\mathrm{W}_0(-\ln x)}{-\ln x} \qquad (6)$$

The Lambert W function therefore provides a neat, closed-form expression to the problem of iterated exponentiation and I should mention that it was on seeing this result in the paper by Corless et al. (1996) that my interest in the Lambert W function was initially aroused.

Having a closed-form expression for the iterated exponential allows for some competition-type questions to be readily answered. For example, if $x = \sqrt{2}$, then

$$\begin{aligned}\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}} &= \frac{\mathrm{W}_0\left(-\ln\sqrt{2}\right)}{-\ln\sqrt{2}} \\ &= -\frac{2}{\ln 2}\mathrm{W}_0\left(-\frac{\ln 2}{2}\right) \\ &= 2\end{aligned}$$

where use of the first of the simplification rules given by equation (4) has been made. As a second example, if $x = \frac{1}{4}$, then

$$\begin{aligned}\frac{1}{4}^{\frac{1}{4}^{\frac{1}{4}^{\cdot^{\cdot^{\cdot}}}}} &= \frac{\mathrm{W}_0\left(-\ln\left(\frac{1}{4}\right)\right)}{-\ln\left(\frac{1}{4}\right)} \\ &= \frac{\mathrm{W}_0\left(2\ln 2\right)}{2\ln 2} \\ &= \frac{1}{2}\end{aligned}$$

where use of the second of the simplification rules for W as given by equation (5) has been used.

Note that in the above two examples the value for x chosen lies within the interval

$$\frac{1}{e^e} < x < e^{\frac{1}{e}} \quad (0.0659\ldots < x < 1.444\ldots)$$

for which the iterated exponential converges. Other iterated exponentials whose final solutions are no longer rational are expressible in terms of the Lambert W function and follow from the closed-form expression given by equation (6).

A little calculus

The calculus of the Lambert W function provides a useful exercise in the application of techniques that one often employs in the symbolic differentiation and integration of functions.

The derivative of the Lambert W function is found by implicitly differentiating the defining equation with respect to x (Corless et al., 1996). So

$$\frac{d}{dx}\left(\mathrm{W}(e)e^{\mathrm{W}(x)}\right)=1$$

$$\mathrm{W}'(x)e^{\mathrm{W}(x)}+\mathrm{W}(x)\mathrm{W}'(x)e^{\mathrm{W}(x)}=1$$

$$\mathrm{W}'(x)\left(1+\mathrm{W}(x)\right)e^{\mathrm{W}(x)}=1$$

Upon solving for $\mathrm{W}'(x)$, the following expression for the derivative of W is obtained

$$\begin{aligned}\mathrm{W}'(x)&=\frac{1}{\left(1+\mathrm{W}(x)\right)e^{\mathrm{W}(x)}}\\&=\frac{\mathrm{W}(x)}{x\left(1+\mathrm{W}(x)\right)}\quad \text{if } x\neq 0\end{aligned}$$

Higher-order derivatives for W then follow using the quotient rule for example (see, e.g., Corless et al., 1996).

From Figure 2 it appears as though the principal branch for the Lambert W function is an increasing function on the domain $x>\frac{-1}{e}$. Recall that a function $f(x)$ is said to be increasing if $f'(x)>0$ holds for all values of x in its domain. Geometrically the curve of an increasing function slopes upwards so that a tangent drawn to the curve has a positive gradient. Using the above expression for the derivative of the Lambert W function we can show that $\mathrm{W}_0(x)$ is a monotonically increasing function for $x>\frac{-1}{e}$.

If $\mathrm{W}_0(x)$ is to be an increasing function, it must satisfy $\mathrm{W}'_0(x)>0$ on some interval for x. Thus from the expression for the derivative of W we have

$$\frac{1}{\left(1+\mathrm{W}_0(x)\right)\exp\left(\mathrm{W}_0(x)\right)}>0$$

Since $\exp(\mathrm{W}_0(x))>0$ and $\mathrm{W}_0(x)>-1$, provided $x>\frac{-1}{e}$, this ensures the denominator of the above inequality is always positive for $x>\frac{-1}{e}$. The principal branch of the Lambert W function is therefore a monotonically increasing function on its domain except at the branch point $x=\frac{-1}{e}$.

Symbolic integration of the Lambert W function is also possible (Corless et al., 1996). A specificity of the Lambert W function is that it is defined as an inverse function, thus the problem of integrating expressions containing the Lambert W function is a special case of integrating expressions containing inverse functions. It relies on a substitution, followed by the method of integration by parts. This approach is not too dissimilar to the method used for integrating expressions that contain the logarithmic function.

Starting with the defining equation for the Lambert W function and using the change of variable $w=\mathrm{W}(x)$, the defining equation becomes $we^w=x$ such that $dx=(w+1)e^w dw$. Thus

$$\int W(x)dx = \int w(w+1)e^{w}dw$$

The integral on the right is now integrated readily by parts. We obtain

$$\begin{aligned}\int W(x)dx &= \left(w^2 - w + 1\right)e^{w} + C\\ &= \left(W^2(x) - W(x) + 1\right)e^{W(x)} + C\\ &= x\left(W(x) - 1 + \frac{1}{W(x)}\right) + C\end{aligned}$$

Many other expressions that contain W can be integrated in this way. For example,

$$\begin{aligned}\int \frac{W(x)}{x}dx &= \int e^{-W(x)}dx\\ &= \int e^{-w}\left(1+w\right)e^{w}dw\\ &= \int \left(1+w\right)dw\\ &= w + \frac{w^2}{2} + C\\ &= W(x) + \frac{W^2(x)}{2} + C\end{aligned}$$

and

$$\begin{aligned}\int \sin W(x)dx &= \int \left(1+w\right)e^{w}\sin w\,dw\\ &= \frac{1}{2}\left(\left(\mathrm{w}+1\right)\sin w - w\cos w\right)e^{w} + C\\ &= \frac{1}{2}\left(\left(W(x)+1\right)\sin W(x) - W(x)\cos W(x)\right)e^{W(x)} + C\end{aligned}$$

Since expressions containing W can be symbolically integrated, we expect this to account for some of the applications of the Lambert W function. For example, a simple mathematical model for combustion (e.g., O'Malley, 1991) is

$$\frac{dy}{dt} = y^2\left(1-y\right),\quad y(0) = \varepsilon > 0$$

where ε is a positive constant. Since the equation is separable, the model problem can be solved using the method of separation of variables. The following implicit solution in y is obtained

$$\ln\left|\frac{y}{1-y}\right| - \frac{1}{y} = t + C$$

The above logarithmic equation can be solved explicitly for y in terms of the Lambert W function. We leave it as an exercise to show that when this is done

$$y(t) = \frac{1}{W_0\left(e^{-t-1-C}\right)+1}\quad \text{where } 0 < y < 1$$

Eliminating the constant of integration by substituting for the initial condition, we finally get

$$y(t) = \frac{1}{W_0\left(ue^{u-t}\right)+1} \quad \text{where } u = \frac{1}{\varepsilon - 1}$$

A brief historical account of the Lambert W function

The evolving realisation that the Lambert W function, whose origin can be traced back to the mid-eighteenth century, was an elementary transcendental function in its own right can be said to consist of anticipations and near misses. Best remembered today for his proof of the irrationality of π and his early treatment of the hyperbolic functions (Barnett, 2004), the work of Johann Heinrich Lambert in 1758 foreshadowed the modern day definition of this function (Lambert, cited in Corless et al., 1996). Lambert himself did not provide any special notation or name for this function as he did not recognise nor see the need in defining it as a separate function. Subsequent work by the great Leonard Euler in 1779 (Euler, cited in Corless et al., 1996) led to some of the mathematical properties of this function being worked out, at least indirectly, but again its importance went unrecognised. Since that time it appears as though many authors working in disparate fields discovered and rediscovered this function, but its existence as a special function in its own right was not formally established until quite recently.

In the mid-1990s, Corless and co-workers (Corless et al., 1996) became sufficiently convinced that what many others had previously been dealing with in isolation, was in fact a too often overlooked elementary transcendental function waiting to be named. Having hit upon this initial realisation, Corless et al. (1996) went ahead and named it as a special function and gave a systematic account of many of its properties. If what we now know as the Lambert W function had not been provided with a convenient and standard name of its own, its wide-ranging applicability would continue to go unnoticed. Most people, when coming across it, would think they were dealing with an isolated transcendental equation rather than a simple elementary transcendental function on par with the existing well-known elementary transcendental functions of mathematics. In a relatively short period of time after having received its name, the usefulness of the function in various fields beyond mathematics, such as in physics and engineering, was quickly established (Barry, Parlange, Li, Prommer, Cunningham, & Stagnitti, 2000). Such apparent general applicability rapidly established the Lambert W function as one of the important elementary transcendental functions of mathematics. Part of its initial popularity seems to have stemmed from its early inclusion in certain computer algebra systems such as Maple, where initially it was simply called "W" (Corless, Gonnent, Hare, & Jeffrey, 1993). Despite the apparent lack of association that the initial choice in the name "W" seemed to have to "anyone" or "anything", quite fortuitously an English mathematician by the name of Edward M. Wright had studied the complex values of this function in the late nineteen forties (Wright, cited in Corless et al., 1996) and it is probably through such work that this function acquired its "W" epithet. The function itself is named after Johann Lambert in honour of the initial work he performed in pre-empting its definition. An entry for the Lambert W function can now to be found in Eric Weisstein's weighty encyclopedic tome of mathematics (2003, pp. 1684–1685) and reflects its nascent recognition, as it is included alongside the more familiar elementary transcendental functions.

Conclusion

In this paper I have introduced the recently defined Lambert W function and have argued the case for its inclusion into our curricula. It is a simple yet accessible function which could be purposefully introduced into either the senior secondary or introductory tertiary level mathematics curricula, relying on little more than the concept of an inverse function having been properly prepared beforehand.

The Lambert W function, which is rapidly emerging as one of the important elementary transcendental functions of mathematics, has a surprisingly rich mathematical structure and arises in many applications due to its simplicity — simple functions occur often. Using examples that are readily accessible to the senior secondary or introductory tertiary level student, it has been shown how analytic solutions to a variety of equations involving exponentials, iterated exponentiation, and a simple combustion model, can be solved in terms of W. Despite its growing applicability, its presence often continues to go unrecognised. Of intrinsic mathematical interest, it was shown how the Lambert W function has properties that are akin to those of the golden ratio and how expressions containing this function can be symbolically integrated and differentiated. Pedagogically, its introduction could be a means for consolidating and reinforcing work on inverse functions, and use of it would go some way towards preparing the student for future work on multi-valued functions.

The teacher of mathematics is not often presented with the opportunity to introduce and discuss "new" mathematics from the current mathematical literature that is even remotely accessible to our students. I maintain that the Lambert W function is one surprising exception. Whilst I concede that any attempt to change long ingrained thinking towards the apodictic set of familiar elementary transcendental functions is to be regarded as largely a quixotic endeavour, I encourage you to think otherwise.

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References

Abramowitz, M. & Stegun, I. A. (Eds) (1972). *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.* New York: Dover Publications.

Andrews, G. E., Askey, R. & Roy, R. (1999). *Special Functions.* Cambridge: Cambridge University Press.

Australian Education Council (AEC) (1991). *A National Statement on Mathematics for Australian Schools.* Carlton South: Curriculum Corporation.

Barnett, J. H. (2004). Enter, stage center: The early drama of the hyperbolic functions. *Mathematics Magazine, 77*(1), 15–30.

Barry, D. A., Parlange, J.-Y., Li, L., Prommer, H., Cunningham, C. J. & Stagnitti, F. (2000). Analytical approximations for real values of the Lambert W function. *Mathematics and Computers in Simulation, 53*(1–2), 95–103.

Bennett, M. A. & Reznick, B. (2004). Positive rational solutions to $x^y = y^{mx}$: A number-theoretic excursion. *The American Mathematical Monthly, 111*(1), 13–21.

Board of Studies NSW (1982). *Mathematics 2/3 Unit Years 11–12 Syllabus.* Sydney: Author.

Churchhouse, R. F. (1995). Solutions of the equation $x^y = y^x$. *Bulletin of the IMA, 31*, 106.

Corless, R. M., Gonnet, G. H., Hare, D. E. G. & Jeffrey, D. J. (1993). Lambert's W function in Maple. *Maple Technical Newsletter, 9* (Spring), 12–22.

Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J. & Knuth, D. E. (1996). On the Lambert W function. *Advances in Computational Mathematics, 5*(4), 329–359.

Corless, R. M. & Jeffrey, D. J. (2002). On the Wright ω function. In J. Calmet, B. Benhamou, O. Caprotti, L. Henocque & V. Sorge (Eds), *Proceedings of the Joint International Conferences on Artificial Intelligence, Automated Reasoning, and Symbolic Computation* (pp. 76–89). London: Spring–Verlag.

de Villiers, J. M. & Robinson, P. N. (1986). The interval of convergence and limiting functions of a hyperpower sequence. *The American Mathematical Monthly, 93* (1), 13–23.

Edwards, C. H. & Penney, D. E. (1990). *Calculus and Analytic Geometry (3rd ed.).* New Jersey: Prentice-Hall.

Ferrini-Mundy, J. & Graham, K. G. (1991). An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. *The American Mathematical Monthly, 98*(7), 627–635.

Finney, R. L., Weir, M. D. & Giordano, F. R. (2001). *Thomas' Calculus: Early transcendentals (10th ed.).* Boston: Addison Wesley.

Grimison, L. (1995). What is mathematics and why teach it? In L. Grimison & J. Pegg (Eds), *Teaching Secondary School Mathematics: Theory into Practice* (pp. 1–14). Southbank: Harcourt Brace.

Kleiner, I. (1989). Evolution of the function concept: A brief survey. *The College Mathematics Journal, 20*(4), 282–300.

Knoebel, R. K. (1981). Exponentials reiterated. *The American Mathematical Monthly, 88*(4), 235–252.

Lebedev, N. N. (1972). *Special Functions and their Applications.* New York: Dover Publications.

National Council of Teachers of Mathematics (NCTM) (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM) (2000). *Principles and Standards for School Mathematics.* Reston, VA: Author.

O'Malley, R. E. (1991). *Singular Perturbation Methods for Ordinary Differential Equations* (Vol. 89 of Applied Mathematical Sciences). New York: Springer.

Pender, B. (1999). An overview of the 2 unit course. *Reflections, 24*(3), 10–21.

Ponte, J. P. (1992). The history of the concept of function and some educational implications. *The Mathematics Educator, 3*(2), 3–8.

Ryan, J. (1994). The function concept: Making connections within and between representations. In R. Killen (Ed.), *Educational Research: Innovation and Practice. Proceedings of the Annual Australian Association for Research in Education Conference.* Accessed http://www.aare.edu.au/94pap/ryan94.028

Sved, M. (1990). On the rational solutions of $x^y = y^x$. *Mathematics Magazine, 63*(1), 30–33.

Tall, D. O. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity and proof. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 495–511). New York: Macmillan.

Tall, D. O. (1996). Functions and calculus. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds), *International Handbook of Mathematics Education* (pp. 289–325). Dordrecht: Kluwer.

Weisstein, E. W. (2003). *CRC Concise Encyclopedia of Mathematics (2nd ed.)*. Boca Raton: Chapman & Hall.