

Algebra skills and traps and diagnostic teaching for the future

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Introduction

The purpose of this article is to explore some research into teaching and learning algebra, and to consider related classroom issues. This leads to the development of diagnostic instruments that may be used by senior secondary teachers as students leave the middle years and embark on rigorous mathematical study. It also outlines an effective strategy for remediating algebraic difficulties and misconceptions. This discussion, in part, parallels recent work by Ball, Stacey and Pierce (2001) and Pierce and Stacey (2001).

The challenge of computer-impacts on the algebra curriculum

In our modern era of graphics calculators and mathematics processing software (also known as computer algebra systems or CAS) we face a situation which, for secondary school algebra as a traditional curriculum territory, resembles that which was faced by the primary school arithmetic curriculum when cheap handheld numerical calculators became available in the late 1980s. Rule-bound, or algorithmic pencil-and-paper computational skills, along with mental memorising of important ‘facts’ — the traditional core of both the arithmetic and the algebra curricula — are naturally challenged by the availability of a machine that can ‘do it for us’. A similar pedagogic shift occurred in the secondary mathematics curriculum when books of logarithm and trigonometric tables, and slide-rules, were replaced by handheld scientific calculators. Earlier, the introduction of these books of tables had impacted on the pre-tables nineteenth century mathematics curriculum; as examined recently by Barling (2003).

Seven years ago Kaye Stacey (Stacey, 1997) asked what it would mean for a student to ‘understand’ algebra if the student was using a graphic calculator or CAS or, generically, ‘software’. She proposed that we could accept that a student understands what he or she is doing algebraically if the student:

- (i) knows what the procedure is for;
 - (ii) knows what sort of input for the software is required;
 - (iii) knows under what conditions the algebraic procedure can be used;
 - (iv) can make sense of the answer provided by the software; and,
- has some capacity to detect errors in the answers given by the software (i.e. proof-reading and process-checking skills related to algebraic notation and process) (p. 16)

Guiding this suggestion was Stacey's hope that forthcoming research would lead to something she called 'algebra sense', an obvious counterpart to the arithmetic-related idea of 'number sense'.

It should be stressed that the traditional pencil-and-paper approach to teaching algebra also aimed at developing what Stacey calls 'algebra sense'. But perhaps it was easier to demonstrate that a student did, or did not have 'algebra sense' when the successive algebraic steps of computation were visibly written out by the student who was, traditionally, required to 'show all working'. When the software does the inbetween steps of computation (whether numerical or algebraic), we need to look for different evidence.

The algebra curriculum itself, as well as how we teach it, and how we assess its learning, is currently changing. Senior secondary (and tertiary undergraduate) mathematics teachers, in particular, face new challenges: what should they teach? And how should they teach it? Or, more drastically, should they teach it, whatever 'it' might be? The AAMT mathematics educators' interest group listserv (archived on the EdNA website <http://www.edna.edu.au>) recently explored whether or not we should teach students how to solve, or otherwise work with, quadratic equations, functions and inequalities. The more radical view was that this level of attention to detail — which had for decades been regarded as central to the middle-secondary algebra curriculum — was no longer required. Instead general algebraic skills, understanding of notation, and understanding of underlying concepts such as 'function' were favoured.

My view is that, until definitive research proves otherwise, we may be guided by the earlier experiences of the impacts on curriculum of numerical calculators and scientific calculators. Students may not need to do large amounts of algebraic computation, whether using pencil-and-paper, or using software. But they do need to be introduced to notation, computational steps, and underlying concepts. They need to be able to talk about what algebra means and how it is used. They need to be able to draw sketches of numerical expressions and to interpret graphs and to perform step-by-step routine algebraic computations, such as expansion of bracket-expressions, simplifications and factorisations, transformation of formulas, and solution of linear, and quadratic and other generic expressions.

In the same way that learning to listen and speak, for mainstream children, precedes any introduction to written forms of language (which may be represented, perhaps simultaneously, via pencil-and-paper and ink-on-paper, and by pixels-on-screen and by typing-on-keyboards), meaningful experiences and oral consideration of variation, relationship, and numerical unknowns will

continue to precede and support both pencil-and-paper as well as keyboard and keypad software use.

It seems pedagogically simple and effective to use some pencil-and-paper experiences of algebra as a hands-on introduction to key-pressing and software use. After these initial experiences, as students move towards comfortable use of software to do their algebraic computation, pencil-and-paper skills will continue to be relevant in the necessarily related record-keeping of successive steps in algebra software computation.

We are NOT ready yet to move to a paper-less classroom, or to unthinkingly discard handwriting of computations as valuable formative experiences, linking oral, visual, and physical representations of abstract concepts and processes.

Emerging research into computer-assisted algebra learning

Interestingly, Patricia Forster (2003) highlights, for advanced Year 12 students, the usefulness of ‘copying’ of algebraic expressions while using CAS software in a calculus class. Here the idea of ‘copying’ includes:

- assigning a temporary variable name or identifier to a partial calculation, to be used later in a larger computation (similar to using a memory key on a numerical or scientific calculator); and
- using an on-screen edit function to ‘click-and-drag’ to select, and then copy an expression, or part of an expression, and then pasting the copied expression somewhere else, and possibly making further changes to the copied and pasted material.

Forster notes that on-screen copying preserves the material being copied (its syntax), and avoids rounding errors that occur when a partial computation is noted to a few decimal places, and later re-entered at the same level of brevity and approximation. (This parallels the argument in lower level computation for working with exact fractions and surds, and irrationals, such as π and e , and $\sin 45^\circ$, rather than with decimal approximations.)

In the act of copying, students brought to bear order of operations, used letter symbols in the place of parameter and variable values, linked representations and formulated generalisations. Therefore, in the view that action can lead to understanding, copying warrants consideration as a means of developing understanding in these domains. (Forster, 2003, p.338)

At lower levels of the algebra curriculum, and with less ready access to mathematics enabling software, handwriting ‘copying’ serves a similar role in bringing students’ attention to order of operations, notational conventions such as superscripts and subscripts, and the relationships between specific numbers, general coefficients, and pronominal variables. Such issues arise even before the algebra curriculum is overtly begun.

Elizabeth Warren (Warren, 2003) uses open-ended arithmetic tasks to

explore students' ability to think algebraically in a pre-algebraic context of numerical computation.

Warren's Task 1

(a) Find the missing number

$$23 = 2 + 5 + \dots + 4 + 6$$

(b) Find the missing numbers

$$23 = 1 + 5 + 3 + \dots + 2 + \Delta$$

(c) Write some other sums that add to 23

Warren's Task 2

Sara shares \$15.40 among some of her friends. She gives the same amount to each person.

(a) How many people might there be; and how much would each receive?

(Give at least 3 answers.)

(b) Explain in writing how to work out more answers.

This parallels the much earlier work of Kevin Collis (Collis, 1975), and the development of the ACER Operations Test (1977), discussed by Ken Clements and John Gough (Clements & Gough, 1978).

Collis (1975) developed mathematics questions that use the same basic structure, but with different levels of cognitive demand, focussing on:

- small numbers and simple arithmetical operations;
- simple algebraic operations;
- large numbers and simple arithmetical operations; and
- more advanced algebraic operations.

Collis' diagnostic experiment used two kinds of 'elements' for the arithmetical operations: positive whole numerals and pronumerals. He further distinguished small numerals and large numerals. He also used two kinds of thinking: a 'concrete' level, using single explicit operations, and a 'formal level', using combinations of operations, and implicit operations. This gives a four-way split in the following table.

		Complexity of Operation &/or Problem	
		Concrete	Formal
Complexity of Elements	Concrete	$8 \times 3 = 3 \times [?]$	$7 - 4 = [?] - 7$
	Formal	$4283 + 517 - 517 = [?]$	$a - b = 2a \div [?]$

Clearly, many aspects of algebraic thinking are actually general forms of mathematical thinking, which may be mirrored in simpler or more complex ways by number-expressed counterparts. Warren (2003) notes that:

The majority of students are leaving primary school with limited awareness of the notion of mathematical structure and of arithmetic operations as general processes: from the instances they are experiencing in arithmetic, they have failed to abstract the relationships and principles needed for algebra. (p. 133)

This weakness confounds the secondary teachers' attempts to begin teaching algebra, and dogs students through later years of the secondary algebra curriculum, and beyond (for those who continue to need to study or use algebra).

The need to diagnose algebra skills, weaknesses and misconceptions

Right now, and in the foreseeable future, senior secondary mathematics teachers need to be able to see, quickly, what their students have already learned, or possibly mis-learned or not learned, about algebra, whether it is by means of pencil-and-paper or software. After the initial diagnostic screening of middle school exit-skills, senior secondary teachers then need guidance in effective methods of remediation.

June Marquis (Marquis, 1988) provided a list of twenty-two so called 'algebra demons'. These may still be used effectively as a diagnostic tool for senior secondary and undergraduate tertiary students (Gough, 2001a). Interestingly, these are presented as a list of incorrect mathematical statements, e.g., $x + y - 3(z + w) = x + y - 3z + w$, and students are asked to correct them. That is, Marquis's 'demons test' is essentially a test of proof-reading and self-correction, which implicitly identifies overt algebraic skill, and understanding.

Nearly a decade later, amongst other diagnostic probes of algebraic competence, Frank Barrington (1997) offered a one-item discrimination 'litmus test' of algebraic competence:

In the following equation, a and b are positive real numbers:
solve for x in the equation $3/x + 4/a = 5/b$ (p. 20).

Similarly, Philip Swedosh (Swedosh, 1996, 1997a, 1997b) provided a short diagnostic test of algebraic skill, along with a method of remedial intervention that Swedosh reported was highly effective in eliminating hitherto persistent misconceptions about algebraic processes.

Swedosh's (1997) 'easy and effective method' (p. 193) consists of showing simple numerical examples, by substitution, in which the misconception (which is essentially a falsely generalised process of computation, or a misuse of pronumerals and algebraic operations) leads to ridiculous conclusions. This establishes a cognitive conflict for the student: the student's reliance on (faulty) computation clashes with obviously incorrect results of substituting numbers for pronumerals and doing simple arithmetic. This then makes it possible to (re-) teach the correct method or concept. During this remedial teaching stage, different examples (different numbers, pronumerals, and values) are used, thus emphasising the general concepts, not the particular instances. For example: $4!/2! = 24/2 = 12$; but $(4 - 2)! = 2! = 2$, and hence $4!/2! \neq (4 - 2)!$

Such a direct remedial intervention approach was suggested by Robert

Davis (Davis, 1966) when he outlined a misconception remediation process called ‘torpedoing’, that is, presenting a counter-example that will refute a student’s naive conjecture or misconception (p. 2; cited in Dawson, 1969, pp. 174, 220, 221–222).

Both Swedosh (1997) and Barrington (1997) are concerned with the algebraic skills brought from secondary schools into tertiary study of mathematics. This is, implicitly, shared by John Pegg and Robyn Hadfield (Pegg & Hadfield 1999). They conclude that:

This research [into the algebraic performance of Year 12 students] does highlight the need to go beyond reporting student performance at HSC in terms of marks which allow norm-referencing. It is also clear that procedures [of instruction and assessment] which are currently in place actually mask the real extent or weaknesses of student understandings ... if teachers were made aware of their students’ performance in terms of what they actually know and understand, then a serious commencement could be made in helping students reach their true potential. (p. 423)

I believe this to be absolutely right. It is, and has always been (at least in Victoria) part of the assessment process that exam markers report to classroom teachers the persistent difficulties observed under examination, with the aim of ‘helping [later] students reach their true potential’. What is needed, as this article is steadily arguing, is an earlier diagnostic guide to student difficulties.

Elsewhere, Robert Davis (Davis, 1984) has also highlighted another persistent ‘algebra demon’, the so called ‘Professors and Students’ problem, discussed by Rosnick and Clement (1980):

At Hale university there are six times as many students as professors: Using S to represent the number of students, and P to represent the number of Professors, write an algebraic equation that summarises this statement. (p. 116)

Importantly, as reported by Davis, simple explanations for students’ mishandling of this problem do not stand up to scrutiny. Change the context, or alter the word ordering, and the error persists. Moreover, often the essential error persists when an equivalent form of this problem is presented as though it is a new problem, even after careful remedial intervention. That is, despite immediate appearances of effective remediation, students are often unable to transfer any remedial learning to modified, seemingly novel forms of the problem. In this case, the necessary remedial intervention involves alerting the student to the possibility of alternative interpretations and algebraic expressions, with the consequent need to carefully check any proposed solution, usually by simple substitution of actual numerical values to establish a match with the meaning of the problem and its proposed mathematical handling.

Apart from the challenges that arise from notorious, widely experienced

difficulties such as the ‘algebra demons’, a more general problem is a lack of agreement on, and a lack of wide-ranging research about what the algebra curriculum IS, or OUGHT to be, as well as HOW it is best taught and/or learned. Marj Horne (Horne, 1999) has announced a potentially groundbreaking research project that would develop a framework of growth points to monitor students’ comprehension of algebra in Grades 7 to 9. Within her preliminary sketch of the territory Horne includes the following structural aspects:

- generalised arithmetic laws: basic operations, commutativity, order of operations, and distributivity [but not, apparently, associativity];
- the concept of ‘=’ or ‘equals’ or ‘equivalence’;
- identity, or the value, possibly unknown, of a pronumeral or an algebraic sub-expression;
- equations, and solving equations: substituting numerical values and simplifying, checking by successive trial and error and correction, back-tracking, inverse operations;
- working with tabulated values: using tables to develop sequences of terms, translating tabular relationships into algebraic expressions; and
- graphing functions: developing the algebraic function or relation for a given graph;
- inequalities.

Until the necessary research is completed, senior secondary teachers who want to start the more advanced parts of the algebra (and mathematics) curriculum, may find it helpful to conduct a survey of algebra skills. This is precisely what Brian Doig (1991) elaborated in his *Group Review of Algebra Topics*, with a set of whole-group pencil-and-paper multiple-choice worksheet-style subtests, each subtest containing thirty graded items. The subtests addressed:

- introductory concepts;
- inequalities;
- linear equations;
- manipulation;
- factorisation;
- quadratic equations;
- sequences and series; and
- word problems.

Such objective worksheet-test reviews can be an invaluable pre-test beginning (and post-test conclusion) for any one of the eight topics through middle secondary years. However, senior secondary teachers, like all teachers, are pressed for time and need rapid diagnostic guidance about the earlier success of middle secondary instruction.

To optimise effective student learning, senior teachers need to identify, as quickly and easily as possible, the existing skills and weaknesses that students are bringing into senior classes from their earlier years of learning algebra. Weaknesses, in particular, need to be exposed and remediated to avoid them impacting on learning in final years of secondary school and persisting into tertiary study (as discussed by Swedosh, 1996, 1997a, 1997b; and Barrington, 1997).

The Appendix provides a modified sample of test items from all eight of Doig's sub-tests (Gough, 2001b). It is offered here in the hope that it may fill a demonstrated gap in available screening instruments, and may also help teachers intervene early in remediating any student weaknesses that are exposed by such screening.

From each of Doig's 'Algebra Topic' tests, the first, the last, and a middle question have been taken. (Other samples could be used but this method is chosen for its simplicity.) Where possible, the multiple-choice options have been omitted. This decreases some of the diagnostic richness of the full tests and also makes this survey less objective, and harder to 'mark'. However, the open-ended scope for answering may provide different kinds of information. It also prevents the possibility of choosing one of the given answers on the basis of substituting a number for a pronumeral and simplifying to check, or checking by expanding brackets.

Parallel forms of this test could be easily constructed by selecting the second, next-to-middle, and second-last items from each of Doig's tests, or by altering the pronumerals and numerical coefficients used in the items included in the Appendix.

Although it is now possible to survey students' ability to handle algebra using a graphics calculator, or CAS-type mathematics enabling software, I recommend that the survey items in the Appendix be administered as a pencil-and-paper test for senior secondary students, precisely because I believe only handwritten algebraic computation, step by step, can reveal skills, understanding, and possible misconceptions. This would be concealed or avoided, using algebra-handling software, when a student successfully types in an algebraic expression to be simplified, solved or otherwise manipulated, and the software does the step by step working for the student. This is the same argument that can be mounted for examining a younger student's skill with numerical computation, and understanding of place value: pencil-and-paper computation reveals what reliance on a handheld calculator conceals, whatever the student's mechanical skill with a calculator.

Only where test performance is particularly weak should a follow-up test be allowed using mathematics software, to assess a student's ability to cope with schoolwork if algebra software is available.

If senior secondary students' experience of algebra has been effective, with or without access to mathematics software, the growth points or components of Horne's (1999) curriculum framework will have been covered. Equally, Stacey's (1997) 'algebra sense' will have developed. The point is that these twenty-four algebra items survey fundamental concepts and skills: have they been learned, or not?

Conclusion

As I have argued elsewhere (Gough, 2003), exactly how any algebra curriculum is to be taught, as well as what actual curriculum ought to be taught, is the subject of every teacher's groundbreaking action research into the effective use of graphics calculators, CAS and other mathematics enabling software. The anecdotal evidence of practising teachers (such as Roberts, 1997) ought to be more widely shared. Researchers should survey teachers and their students, and report the emergent experiences, reflections, difficulties, and critically analysed insights; drawing on current algebra curriculum structure, instructional materials, teaching practices, student learning, progress and pitfalls; exploring what is already familiar and looking for what is genuinely new.

As noted at the beginning, it may be hoped that researchers will publish the results of their ongoing work in teacher journals, such as this one, as well as in journals and conference proceedings published for researchers. At the University of Melbourne, in particular, considerable research has been underway since the late 1990s into the uses and impacts of graphics calculators and Computer Algebra Systems, especially in senior secondary mathematics classes. For example, the early Technology and Mathematics Education (TAME) project, investigating graphics calculators, and now the current Computer Algebra Systems in Schools: Curriculum, Assessment and Teaching (CAS-CAT project,; McCrae, Asp & Kendal, 1999; Stacey & Ball, 2001). Practical results of such research, along with diagnostic instruments and guidance for software-enhanced instruction and curriculum, need to be made more widely available in journals for teachers at state and national level.

Appendix: Algebra concepts and skills survey

Adapted from Gough (2001b). The questions in this draft survey have been adapted from Brian Doig's Group Review of Algebra Topics (ACER, 1991). Note that the prefix-abbreviations for each question identify the source sub-test and item number in Doig's earlier version. For example, Question 1 has been adapted from Item 1 in Doig's sub-test 'Introductory Concepts'. The first eight questions range across the first eight items in each of Doig's graded sub-tests, and successive sets of eight questions continue this pattern, successively increasing in graded difficulty.

1. (Intr.1) $a + a = ?$
2. (Ineq.1) For which value of x is $3 < 2x + 5 < 7$?
3. (LE.1) If $n + 3 = 9$, then $n = ?$
4. (Man.1) If $0.5x + 2 = 5$, then $x = ?$
5. (Fac.1) Simplify by factorising $2x^3 - x^2$
6. (QEq.1) If the solutions of a quadratic equation are $x = -4$ and $x = 3$, then the equation could be:
 - (a) $x^2 - x + 12 = 0$
 - (b) $x^2 + x + 12 = 0$
 - (c) $x^2 + x - 12 = 0$
 - (d) $x^2 - x - 12 = 0$
7. (Seq.1) Because $3^2 + 4^2 = 5^2$, the trio of numbers {3, 4, 5} is called a Pythagorean triad. Which of the following is a Pythagorean triad?
 - (a) {1, 2, 3}
 - (b) {2, 3, 4}
 - (c) {6, 8, 10}
 - (d) {7, 9, 11}
8. (WP.1) Kel was counting frogs, and Lan was counting tadpoles. Altogether they counted 56 frogs and tadpoles. If Lan counted n tadpoles, how many frogs did Kel count?
9. (Int.15) Simplify, by removing the brackets correctly, $p - (3q + 2r)$
10. (Ineq.15) The pronumerals a and b are ordinary numbers, and $a > b$. When $ax < bx$, what can you say about the value or size of the number x ?
11. (LE.15) If $\frac{3}{2} + \frac{x}{3} = \frac{7}{2}$ then $x = ?$
12. (Man.15) If $a = 2.04$ and $b = 4.02$, circle the correct symbol, $<$, $=$, or $>$, to make the following statements correct:

$$2a < = > b$$

$$a^2 < = > b$$
13. (Fac.15) Factorise (re-express as the product of two brackets) $4x^2 - 9$
14. (QEq.15) Given that $-10 < x < 0$, the solutions of $x^2 + 15x + 36 = 0$ are ?
15. (Seq.15) The n th term of a sequence is $(n - 2n^2)$. What is the sixth term?

16. (WP.15) Seven times an unknown number is five more than 13 decreased by twice the unknown number.
Re-express this statement, using numbers and X to represent the unknown number.
17. (Int.30) Simplify

$$\frac{a^{-1}m^{-4}n}{a^{-2}mn^{-3}}$$
18. (Ineq.30). If $(a + b)(b - a) > 0$, choose the expression which is true for all possible values of a and b : $a < b$, or $a = b$, or $b > a$.
19. (LE.30) Find the whole number closest in value to the solution of
 $24Y - 9(2Y - 8) = 97 - 7Y$
20. (Man.30) The operation $\#$ means 'square the first number and divide the result by the second number'. For example, $2 \# 3$ means '2 squared, divided by 3'.
What is 'one and one-third $\#$ two-thirds'? That is [missing?]
21. (Fac.30). Expand and simplify so no fractions are included in the result:
 $(x + 1/3)^2 = 3$
22. (QE.30) Here are the successive steps in the solution of a quadratic equation. Each step is numbered, from 1 to v.
 $(2X + 3)^2 = 25$ (i)
 $2X + 3 = 5$ (ii)
 $2X = 5 - 3$ (iii)
 $2X = 2$ (iv)
 $X = 1$ (v)
 Check the steps, and comment on whether each step is correct, or faulty.
23. (Seq.30). Which digit does 3^{17} end in?
24. $X = 5x + 2b$, $Y = a - 2b$. If $X = Y$, find possible values for a and b .

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