Fermat's technique of finding areas under curves

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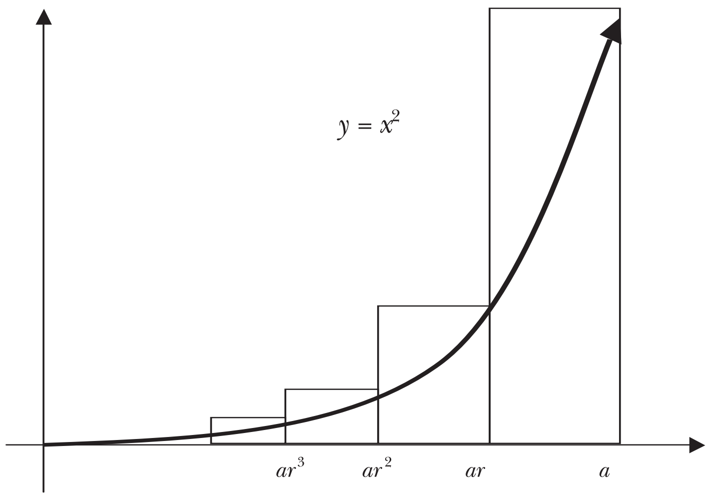
Perhaps next time you head towards the fundamental theorem of calculus in your classroom, you may wish to consider Fermat's technique of finding expressions for areas under curves, beautifully outlined in Boyer's *History of Mathematics*. Pierre de Fermat (1601–1665) developed some important results in the journey toward the discovery of the calculus. One of these concerned finding areas under simple polynomial curves. It is an amazing fact that, despite deriving these results, he failed to document any connection between tangents and areas.

The method is essentially an exercise in geometric series, and I think it is instructive and perhaps refreshing to introduce a little bit of variation from the standard introductions. The method is entirely accessible to senior students, and makes an interesting connection between the anti derivative result and the factorisation of $(1 - r^{n+1})$.

Fermat considered that the area between $x = 0$ and $x = a$ below the curve $y = x^n$ and above the x-axis, to be approximated by circumscribed rectangles where the width of each rectangle formed a geometric sequence. The divisions into which the interval is divided, from the right hand side are made at $x = a$, $x = ar$, $x = ar^2$, $x = ar^3$ etc., where a and r are constants, with $0 < r < 1$ as shown in the diagram.



If we first consider the curve $y = x^2$, the area A contained within the rectangles is given by:

$$A = (a - ar)\,a^2 + (ar - ar^2)\,(ar)^2 + (ar^2 - ar^3)\,(ar^2)^2 + \ldots$$

This is a geometric series with ratio r^3 and first term $(1 - r)\,a^3$.

Using the limiting sum formula, we have

$$\begin{aligned} A &= \frac{a^3(1-r)}{1-r^3} \\ &= \frac{a^3(1-r)}{(1-r)\,(1+r+r^2)} \\ &= \frac{a^3}{(1+r+r^2)} \end{aligned}$$

Now as $r \to 1$ the widths of the rectangles decrease and the total area approaches the area under the curve. That is to say:

$$\int_0^a x^2 dx = \frac{a^3}{3}$$

We can apply the same logic using any curve of the form $y = x^n$ for n a positive integer. The geometric series formed has the first term $a^{n+1}(1 - r)$ and the common ratio r^{n+1}. This means that the denominator in the limiting sum factorises to $(1 - r)\,(1 + r + r^2 + \ldots + r^n)$ and the general result becomes clear.

We can even apply the method to the Mid-Ordinate and the Trapezoidal rule to produce two further expressions for A. Considering the curve $y = x^2$, the expression for A using the mid-ordinate rule becomes

$$A = \frac{a^3}{4}(r+1)^2(1-r)\,(1+r^3+r^6+r^9+\ldots)$$

and for the trapezoidal rule,

$$A = \frac{a^3}{2}(r^2+1)\,(1-r)\,(1+r^3+r^6+r^9+\ldots)$$

which after determining the limiting sums in both expressions produces the desired result.