Japanese temple geometry

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Sangaku

Between the 17th and 19th centuries, the Japanese government closed its borders to the outside world in an attempt to become more powerful. Foreign books were banned, people could not travel, and foreigners were not allowed to enter the country. One result of this isolation was the flourishing of sangaku — wooden tablets inscribed with intricately decorated geometry problems that were hung beneath the eaves of Shinto shrines and Buddhist temples all over Japan (Fukagawa & Pedoe, 1989). The hanging of tablets in Japanese shrines is a centuries-old custom, but earlier tablets generally depicted animals. It is believed that the sangaku arose in the second half of the 17th century, with the oldest surviving sangaku tablet dating from 1683. About 900 tablets have survived, as well as several collections of sangaku problems in early 19th century hand-written books or books produced from wooden blocks. Although their exact purpose is unclear, it would appear that they are simply a joyful expression of the beauty of geometric forms, designed perhaps to please the gods.

Many of the problems were three-dimensional, involving spheres within spheres. Only in a few cases were proofs included. Earlier tablets were generally about 50 cm by 30 cm but later tablets were sometimes as large as 180 cm by 90 cm, each displaying several geometry problems. The 1875 tablet at the Kaizu Tenman shrine (see Figure 1) has twenty-three problems.

Figure 1. The Kaizu Tenman shrine in Siga Prefecture.

The tablets were usually beautifully painted and included the name and social rank of the presenter. Although many of the problems are believed to have been created by highly educated members of the samurai class, some bear the names of women and children, village teachers and farmers. Figure 2 shows a tablet from a temple in the Osaka Prefecture.

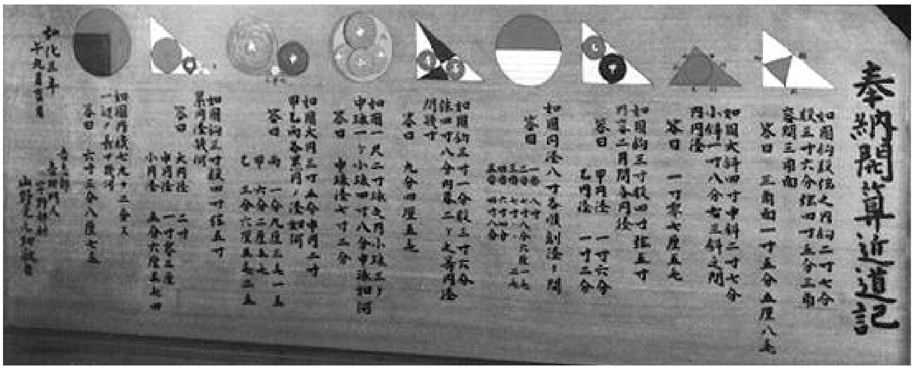

Figure 2. Temple geometry tablet from Osaka.

Sangaku problems have been made known through the work of Hidetosi Fukagawa — a Japanese high school teacher who is recognised as a sangaku expert. In 1989 Fukagawa and Dan Pedoe — a Professor Emeritus of Mathematics from the University of Minnesota — published *Japanese Temple Geometry Problems* (now out of print), dedicating it to 'those who love Geometry everywhere'. Japanese temple geometry was also described in a Scientific American article by Fukagawa and Rothman (1998).

In his preface to Japanese Temple Geometry Problems (Fukagawa & Pedoe, 1989, p. xii), Pedoe comments that Japanese temple geometry appears to have pre-dated the discovery of several well-known theorems of Western geometry, such as the Malfatti Theorem and the Soddy Hexlet Theorem, as well as including many theorems involving circles and ellipses which have never appeared in the West. Pedoe suggests that the Japanese method of always regarding an ellipse as a plane section of a cylinder, rather than of a cone, or as generated by the focus-directrix method, has given rise to theorems that would be very difficult to prove by Western methods.

In some of the examples given by Fukagawa and Pedoe original solutions are included while in other cases their solutions are modern suggestions. *Japanese Temple Geometry Problems* also includes many problems for which answers, but not solutions, are provided. Each of the following examples has been taken from *Japanese Temple Geometry Problems.* Solutions for Examples 1, 5, 7 and 8 are based on those given by Fukagawa and Pedoe, with the remaining solutions provided by the authors of this article. All diagrams have been constructed using *Cabri Geometry II.*

1. Problems involving tangent circles

The solutions to Examples 1 to 3 rely on the Pythagorean relationship. Example 4 requires calculation of the area of a segment of a circle.

Example 1. Miyagai Prefecture, 1892

A circle with centre O_1 and radius r_1 touches another circle, centre O_2 and radius r_2, externally. The two circles touch the line l at points A and B respectively.

Show that $AB^2 = 4r_1r_2$

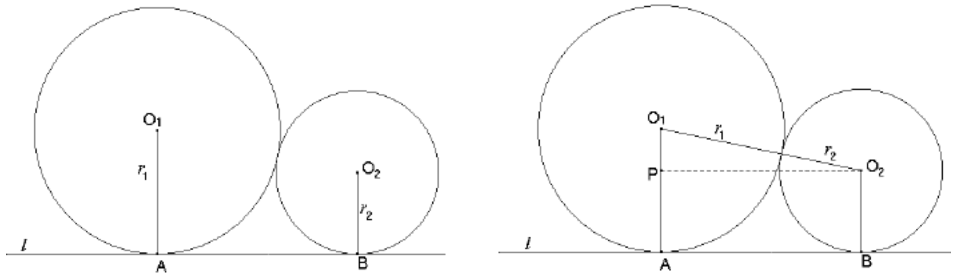

Figure 3. Two tangent circles.

$$O_1P^2 + PO_2^{\ 2} = O_1O_2^{\ 2}$$
$$(r_1 - r_2)^2 + AB^2 = (r_1 + r_2)^2$$
$$r_1^{\ 2} - 2r_1r_2 + r_2^{\ 2} + AB^2 = r_1^{\ 2} + 2r_1r_2 + r_2^{\ 2}$$
$$AB^2 = 4r_1r_2$$

Example 2. Gumma Prefecture, 1824

A circle with centre O_3 touches the line l and the two tangent circles, with centre O_1, radius r_1, and centre O_2, radius r_2.

Show that $\dfrac{1}{\sqrt{r_3}} = \dfrac{1}{\sqrt{r_1}} + \dfrac{1}{\sqrt{r_2}}$

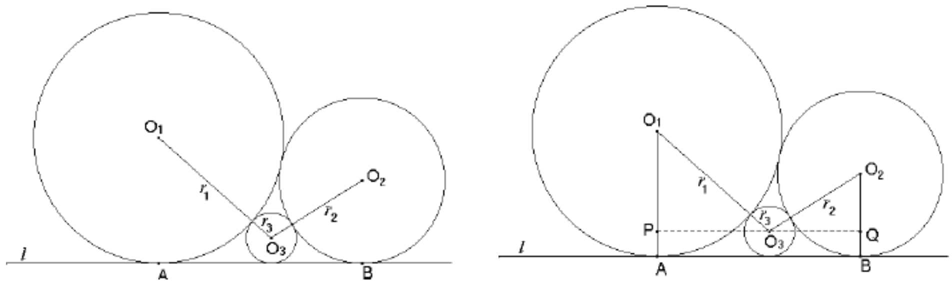

Figure 4. Three tangent circles.

$$PO_3 + O_3Q = AB$$
$$\sqrt{(r_1 + r_3)^2 - (r_1 - r_3)^2} + \sqrt{(r_2 + r_3)^2 - (r_2 - r_3)^2} = \sqrt{4r_1r_2} \quad \text{(from Example 1)}$$
$$\Rightarrow \sqrt{4r_1r_3} + \sqrt{4r_2r_3} = \sqrt{4r_1r_2}$$
$$\Rightarrow \sqrt{r_1r_3} + \sqrt{r_2r_3} = \sqrt{r_1r_2}$$
$$\Rightarrow \frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

Example 3. Aichi Prefecture, 1877

ABCD is a square of side a, and M is the centre of a circle with diameter AB; CA and DB are arcs of circles of radius a and centres D and C respectively. Find the radius, r_1, of the circle, centre O, inscribed in the area common to these circles. Find also the radius, r_2, of the circle, centre O', which touches AB and touches CA and DB externally as shown.

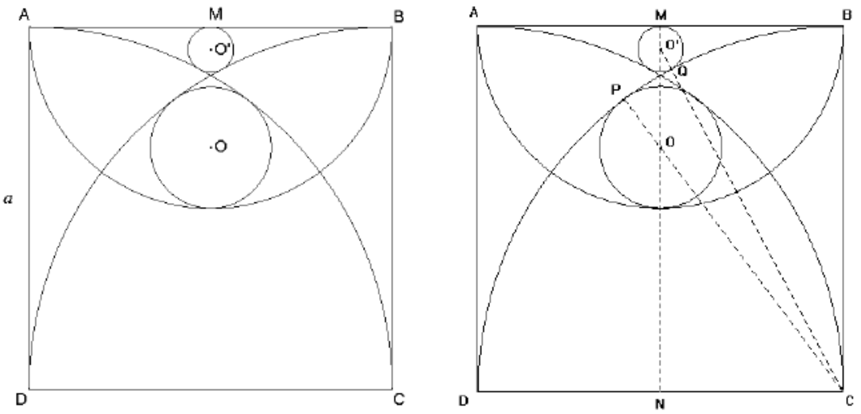

Figure 5. Inscribed circles.

$AB = a,\ OP = r_1,\ O'Q = r_2$

In ΔONC,

$$ON^2 + NC^2 = OC^2$$

$$\left(\frac{a}{2} + r_1\right)^2 + \left(\frac{a}{2}\right)^2 = (a - r_1)^2$$

$$\Rightarrow \frac{a^2}{4} + ar_1 + r_1^{\,2} + \frac{a^2}{4} = a^2 - 2ar_1 + r_1^{\,2}$$

$$\Rightarrow 3ar_1 = \frac{a^2}{2}$$

$$\Rightarrow r_1 = \frac{a}{6}$$

In $\Delta O'NC$,

$$O'N^2 + NC^2 = O'C^2$$

$$(a - r_2)^2 + \left(\frac{a}{2}\right)^2 = (a + r_2)^2$$

$$\Rightarrow a^2 - 2ar_2 + r_2^{\,2} + \frac{a^2}{4} = a^2 + 2ar_2 + r_2^{\,2}$$

$$\Rightarrow 4ar_2 = \frac{a^2}{4}$$

$$\Rightarrow r_2 = \frac{a}{16}$$

Example 4. Fukusima Prefecture, 1883

ABCD is a rectangle with $AB = \sqrt{2}BC$. Semicircles are drawn with AB and CD as diameters. The circle that touches AB and CD passes through the intersections of these semicircles. Find the area S of each lune in terms of AB.

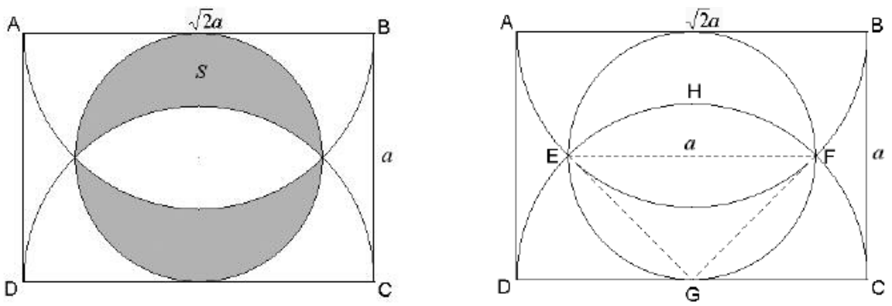

Figure 6. Lunes problem.

First of all we will show that the circle does pass through the intersection of the two semicircles. Consider semicircle *DHC*. Let $BC = a$. So $AB = \sqrt{2}a$.

$$EF = 2\sqrt{EG^2 - \left(\frac{a}{2}\right)^2} = 2\sqrt{\left(\frac{\sqrt{2}a}{2}\right)^2 - \left(\frac{a}{2}\right)^2} = a$$

Now we will find the area of each lune.

$$\angle EGH = 90^\circ \text{ (angle in a semicircle)}$$

$$\text{Area of segment } EHF = \text{ Area of sector } GEHF - \text{ Area of } \Delta EFG$$

$$= \frac{1}{4}\left(\pi\left(\frac{\sqrt{2}a}{2}\right)^2\right) - \frac{1}{2}(a)\left(\frac{a}{2}\right)$$

$$= \frac{\pi a^2}{8} - \frac{a^2}{4}$$

$$\text{Area of each lune } (S) = \text{Area of semicircle on } EF - \text{ Area of segment } EHF$$

$$\Rightarrow S = \frac{1}{2}\left(\pi\left(\frac{a}{2}\right)^2\right) - \left(\frac{\pi a^2}{8} - \frac{a^2}{4}\right)$$

$$\Rightarrow S = \frac{a^2}{4} = \frac{AB^2}{8}$$

The area of the lune is therefore independent of π. Hippocrates is also known to have investigated lunes and shown that it was possible to square the region between two circles (Cooke, 1997).

2. Problems based on triangles

Examples 5 and 6 rely on knowledge of similar triangles. Example 7 requires visualisation of a circle and an ellipse as plane sections of a cylinder.

Example 5. From an early 19th century book of temple geometry problems

ABC is a triangle with a right angle at *C*, and *C′* is the point on *AB* such that *BC′* = *BC*. P is the point on *BC* such that the line *C′P* divides the triangle into two parts of equal area. Show that 2*PC′* = *AB*.

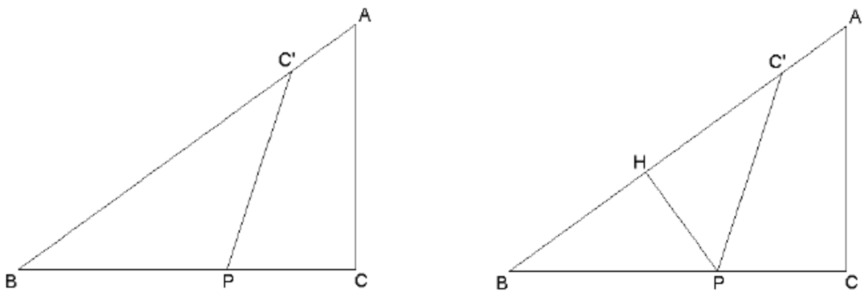

Figure 7. Equal areas problem.

Draw a perpendicular PH from P to AB.

Area of $\Delta BPC' = \frac{1}{2}(BC')(PH)$

$= \frac{1}{2}(BC)(PH)$

Area of $\Delta ABC = \frac{1}{2}(BC)(AC)$

Since Area of $\Delta BPC' = \frac{1}{2}(\text{Area of } \Delta ABC)$,

$\frac{1}{2}(BC)(PH) = \frac{1}{2}\left(\frac{1}{2}(BC)(AC)\right)$

Hence $PH = \frac{1}{2}(AC)$

$\angle BHP = \angle BCA = 90°$

$\angle HBP$ is common to ΔABC and ΔPBH

$\Rightarrow$ ΔABC is similar to ΔPBH

So $BH = \frac{1}{2}BC = \frac{1}{2}BC'$ (since $PH = \frac{1}{2}AC$)

Hence $\Delta BPC'$ is isosceles, and $PB = PC'$

But $PB = \frac{1}{2}AB$ (ΔABC is similar to ΔPBH)

So $PC' = \frac{1}{2}AB$, that is, $2PC' = AB$

Example 6. Miyagi Prefecture, 1877

$ABCD$ is a square of side a, and M is the midpoint of AB. AC meets MD at F. Find the radius of the incircle in triangle DCF.

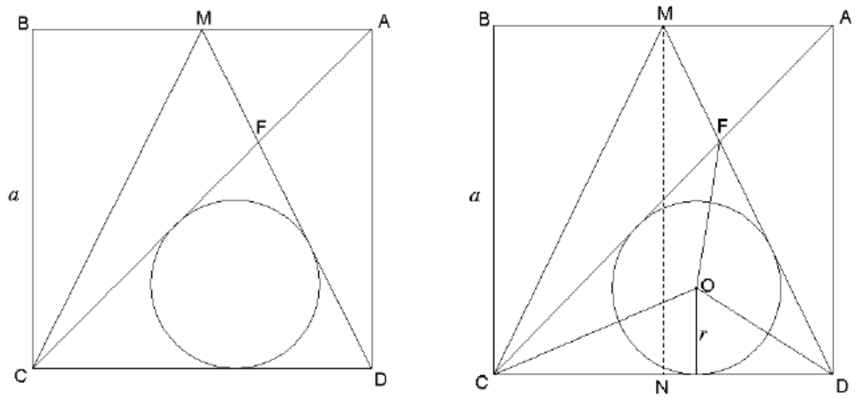

Figure 8. Inscribed circle problem.

In ΔMND,

$$MD^2 = a^2 + \left(\frac{a}{2}\right)^2$$

$$\Rightarrow MD = \frac{\sqrt{5}a}{2}$$

$\Delta MAF \sim \Delta DCF$ and $AM = \dfrac{1}{2}CD$,

So $MF = \dfrac{1}{2}FD$

$$FD = \frac{2}{3}MD = \frac{\sqrt{5}a}{3}$$

$$AC = \sqrt{2}a$$

$$CF = \frac{2}{3}AC = \frac{2\sqrt{2}a}{3}$$

$$\text{Area of } \Delta DCF = \Delta COD + \Delta COF + \Delta FOD$$

$$= \frac{1}{2}r(CF + FD + CD)$$

$$= \frac{r}{2}\left(\frac{2\sqrt{2}a}{3} + \frac{\sqrt{5}a}{3} + a\right)$$

$$\text{But area of } \Delta DCF = \frac{1}{2}(CD)\,(CF)\sin 45°$$

$$= \frac{a}{2}\left(\frac{2\sqrt{2}a}{3}\right)\frac{1}{\sqrt{2}}$$

$$= \frac{a^2}{3}$$

$$\text{Hence } \frac{r}{2}\left(\frac{2\sqrt{2}a}{3} + \frac{\sqrt{5}a}{3} + a\right) = \frac{a^2}{3}$$

$$\Rightarrow \frac{r}{2}\left(2\sqrt{2} + \sqrt{5} + 3\right) = a$$

$$\Rightarrow r = \frac{2a}{2\sqrt{2} + \sqrt{5} + 3}$$

Example 7. Miyagi Prefecture, 1822

A, B and C are three points on an ellipse with centre O, semimajor axis a and semiminor axis b such that the areas S_1, S_2 and S_3 of the curvilinear triangles are equal.
Show that the area of triangle $ABC = \dfrac{3}{4}\sqrt{3}ab$.

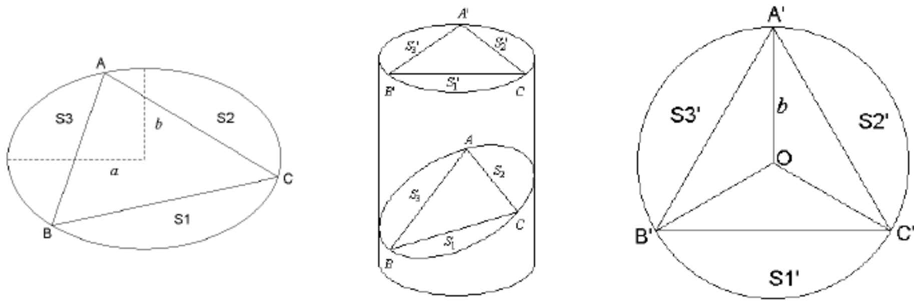

Figure 9. Ellipse problem.

Japanese temple geometers were conversant in affine transformations, whereby an ellipse may be transformed into a circle. The ellipse in this example transforms into a circle with radius b by scaling the major axis by a factor of b/a. The condition of equal areas for the curvilinear triangles in Figure 9 transforms into the condition that triangle $A'B'C'$ is equilateral.

$$\text{Area of } \Delta B'OC' = \frac{1}{2}b^2 \sin 120° = \frac{\sqrt{3}}{4}b^2$$

$$\text{Area of } \Delta A'B'C' = \frac{3\sqrt{3}}{4}b^2$$

Hence, by reverse affine transformation,

$$\text{Area of }\Delta ABC = \frac{a}{b}\left(\frac{3\sqrt{3}}{4}b^2\right) = \frac{3\sqrt{3}ab}{4}$$

3. Problems based on pentagons

These problems suggest that, like the Pythagoreans, some Japanese temple geometers found the form of the pentagon, associated with the golden ratio, a geometrically pleasing one. Fukagawa and Pedoe (1989, p. 134) note that the temple geometers knew the values of cos π/n, sin π/n and tan π/n for many values of n, probably derived from their relationship to the sides of regular n-gons inscribed in or circumscribed about a circle of unit diameter. The authors of this article have used the exact value of cos 36° in the solution to Example 9.

Example 8: From an early 19th century book of temple geometry problems

Construct a regular pentagon by tying a knot in a strip of paper of width a. Calculate the side t of the pentagon in terms of a.

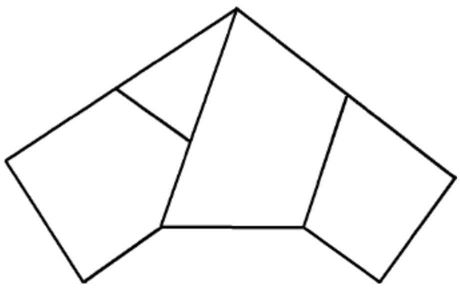

Figure 10. Pentagon formed from tying a knot in a strip of paper.

Fukagawa and Pedoe note that 'the method of construction is "right over left and under". The paper is then smoothed out, and the pentagon appears. *Billet*[s] *doux* [love letters] were sent thus in Japan' (p. 132). In Figure 11, BE and CD are the edges of the strip of paper, so that FC is the width a.

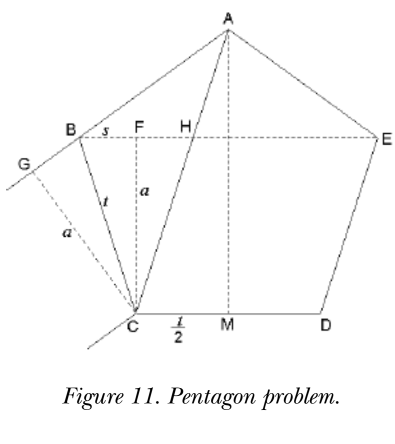

Figure 11. Pentagon problem.

Fukagawa and Pedoe provide a solution from the 1810 book *Sanpo Tenshoho Sinan* (Geometry and Algebra) by Anmei Aida (1747–1817). The solution does not indicate how the relationship $t=\left(1+\sqrt{5}\right)s$ was obtained. We assume that it was derived algebraically, rather than from knowledge of the exact value of sin 18° (it can be shown that $\angle BCF=\angle BCG=\angle CAM=18°$). If the exact values for sin 18° and cos 18° had been known, it would be expected that the temple geometers would have gone straight to the relationship $a=t\cos 18°$.

Since $\angle BCF=\angle CAM$, $BF:BC=CM:AC$

$$t\left(\frac{t}{2}\right)=sp, \text{ where } BF=s \text{ and } p=BE=2s+t, \text{ that is, } t=\left(1+\sqrt{5}\right)s$$

From $(BC)^2=(BF)^2+(FC)^2$

$$t^2=\left(\frac{t}{1+\sqrt{5}}\right)^2+a^2$$

$$\Rightarrow a^2=\left(\frac{5+\sqrt{5}}{8}\right)t^2$$

$$\Rightarrow a=\frac{\sqrt{10+2\sqrt{5}}}{4}t$$

$$\Rightarrow t=\sqrt{2-\sqrt{\frac{4}{5}}}a$$

The following solution, provided by the authors of this article, includes steps that were omitted in the original solution.

In ΔBCG and ΔCAM,

$\angle BCG=\angle CAM$, $\angle BGC=\angle CMA=90°$

Hence $\Delta BCG\sim\Delta CAM$

$$\frac{BF}{t}=\frac{CM}{CA}=\frac{CM}{BE}=\frac{\frac{t}{2}}{BE}=\frac{t}{2BE}$$

$$t^2=2BF\times BE$$

$$t^2=2BF(2BF+t)$$

Let $BF=s$, where $s=\sqrt{t^2-a^2}$

$$\Rightarrow t^2=2s(2s+t)$$

$$\Rightarrow 4s^2-2st-t^2=0$$

$$\Rightarrow s=\frac{2t\pm\sqrt{4t^2+16t^2}}{8}$$

$$\Rightarrow s=\left(\frac{2\pm2\sqrt{5}}{8}\right)t$$

Taking the positive solution $s=\left(\frac{1+\sqrt{5}}{4}\right)t$

$$\text{In } \Delta BFC,\ \ FC^2 = BC^2 - BF^2$$

$$a^2 = t^2 - s^2$$

$$a^2 = t^2 - \left(\frac{1+\sqrt{5}}{4}\right)^2 t^2$$

$$= \left(1 - \frac{6+2\sqrt{5}}{16}\right)t^2$$

$$= \left(\frac{10-2\sqrt{5}}{16}\right)t^2$$

$$t^2 = \left(\frac{8}{5-\sqrt{5}}\right)a^2$$

$$= \left(\frac{8}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}}\right)a^2$$

$$= \left(\frac{40-8\sqrt{5}}{20}\right)a^2$$

$$\Rightarrow t = \sqrt{\frac{10-2\sqrt{5}}{5}}\,a$$

$$\Rightarrow t = \sqrt{2-\sqrt{\frac{4}{5}}}\,a$$

It can also be seen from Figure 11 that $t = \frac{a}{\cos 18°}$.

The exact value of cos 18° is $\frac{\sqrt{10+2\sqrt{5}}}{4}$, so that the reciprocal of cos 18° is $\frac{4}{\sqrt{10+2\sqrt{5}}}$ which can be shown to be equal to $\sqrt{\frac{10-2\sqrt{5}}{5}}$.

Example 9. Miyagi Prefecture, 1912

$A_1\ A_2\ A_3\ A_4\ A_5$ is a regular pentagon of side *a*, and, as shown in Figure 12, six congruent right angled triangles fan out along the sides of the pentagon. Find the length of the hypotenuse, *t*, of these triangles.

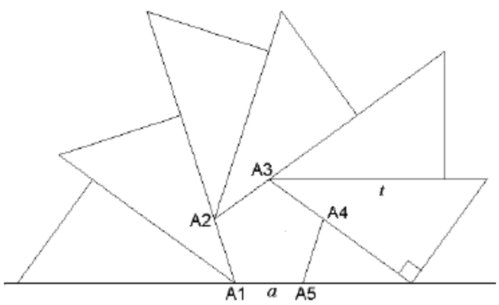

Figure 12. Pentagon with six congruent right-angled triangles.

$A_3H = t$ and let $A_3P = c$, $A_3A_5 = l$ and $A_3M = h$ (see Figure 13).

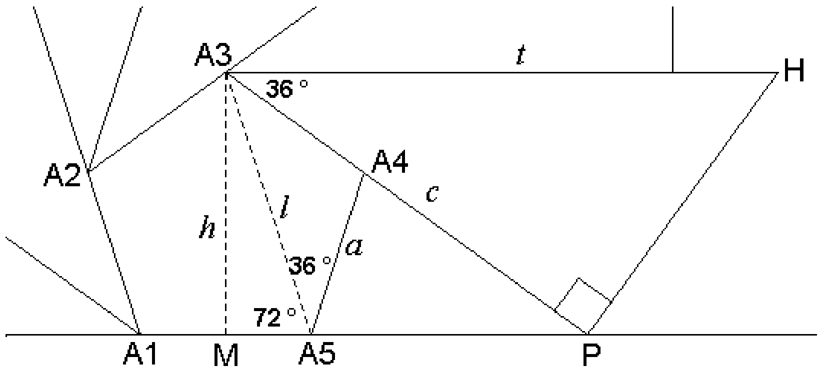

Figure 13. Construction for solution of Example 9.

In $\Delta A_3A_4A_5$,

$$l = 2a\cos 36°$$

In ΔA_3MA_5,

$$h = l\sin 72°$$

$$\Rightarrow h = 2a\cos 36°\sin 72°$$

In ΔA_3MP,

$$\cos 54° = \frac{h}{c}$$

$$\Rightarrow c = \frac{h}{\cos 54°}$$

$$\Rightarrow c = \frac{2a\cos 36°\sin 72°}{\cos 54°}$$

In ΔA_3PH,

$$\cos 36° = \frac{c}{t}$$

$$\Rightarrow t = \frac{c}{\cos 36°}$$

$$\Rightarrow t = \frac{2a\cos 36°\sin 72°}{\cos 54°\cos 36°}$$

$$\Rightarrow t = \frac{2a\sin 72°}{\cos 54°}$$

$$\Rightarrow t = \frac{4a\sin 36°\cos 36°}{\cos 54°}$$

$$\Rightarrow t = \frac{4a\cos 54°\cos 36°}{\cos 54°}$$

$$\Rightarrow t = 4a\cos 36°$$

$$\Rightarrow t = a(1+\sqrt{5}) \text{ since } \cos 36° = \frac{1+\sqrt{5}}{4}$$

Some problems to solve

Examples 10, 11 and 12 have been included for the reader to solve.

Example 10. Fukusima Prefecture, 1888

AB is a chord of a circle with radius r, and M is the midpoint of AB. The circle with centre O_1 and radius r_1, where $r_1 < r$, touches AB at M and also touches the outer circle internally. P is any point on AB distinct from A, M and B. A circle with centre O_3, which also has radius r_1, touches AB at P, on the other side of AB. The circles with centres O_2 and O_4 and radii r_2 and r_3 respectively both touch AB, and touch the outer circle internally and the circle with centre O_3 externally.

Show that $r = r_1 + r_2 + r_3$.

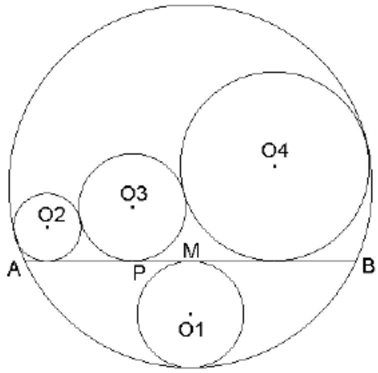

Figure 14. Five circles problem.

Example 11. Miyagi Prefecture, 1877

DPC is an equilateral triangle within the square *ABCD*, and *AP* meets *BC* in *Q*. Show that we can find *R* on *CD* so that *AQR* is equilateral. Let *AR* meet *PD* in *S* and *QR* meet *PC* in *T*.

The incircle of triangle *ADS* has centre O_1 and radius r_1, and the incircle of triangle *CRT* has centre O_2 and radius r_2. Show that $r_2 = 2r_1$.

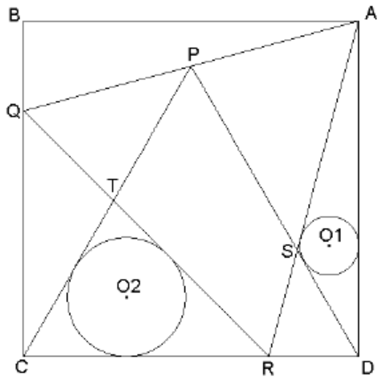

Figure 15. Two incircles problem.

Example 12. Fukusima Prefecture, 1893

The triangle *ABC* is right-angled at *C* and *DEFG* is a rectangle. The three circles are incircles of the triangles *AGD*, *FBE* and *GFC*.

Show that when the area of the rectangle is a maximum, then $r_1^2 + r_2^2 = r_3^2$, where r_1, r_2 and r_3 are the respective radii of the three circles.

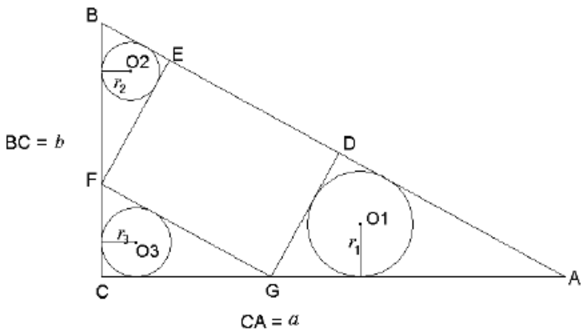

Figure 16. Rectangle area problem.

Conclusion

The examples shown here are but a small selection of over 250 problems presented by Fukagawa and Pedoe. The Sangaku leave many unanswered questions: how did the custom originate? Were the problems devised by the presenters? Where solutions were not given, how did the presenters intend the problems to be solved? To really enter the minds of the temple geometers, it would be necessary to explore in greater depth the mathematical techniques that they had at their disposal during the 17th–19th centuries.

Acknowledgements

The authors thank Tony Rothman for permission to use Hidetosi Fukagawa's photograph of the Kaizu Tenman shrine (Figure 1) and Kotera Hiroshi for permission to use the photograph of the Osaka temple geometry tablet (Figure 2). Both Kotera Hiroshi and Tony Rothman responded enthusiastically to our interest in Japanese temple geometry.

Source of examples

The following table shows the sources of the examples and indicates whether the solutions are based on those provided by Fukagawa and Pedoe or have been produced by the authors of this article.

Example	Fukagawa & Pedoe (1989)	Source of solution
1	Example 1.1, pp. 3, 79	Fukagawa & Pedoe
2	Problem 1.1.1, p. 3	Authors
3	Problem 3.3.1, p. 41	Authors
4	Problem 3.4.3, p. 43	Authors
5	Example 4.1, pp. 46, 129	Fukagawa & Pedoe
6	Problem 3.1.3, p. 38	Authors
7	Example 7.1, pp. 62, 153	Fukagawa & Pedoe
8	Example 4.3, pp. 49, 132–133	Fukagawa & Pedoe; Authors
9	Problem 4.3.2, p. 49	Authors
10	Problem 1.3.6, p. 7	
11	Problem 3.1.6, p. 38	
12	Problem 2.3.6, p. 31	

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