PENTIMALS

A recent exercise with primary pre-service teachers at my institution asked them for as many facts as possible about the number 36. Apart from the usual factor details, someone mentioned that it is three dozen. A discussion of 'dozens' then ensued. Examples of where dozens are still used is readily available. Of more mathematical interest was a discussion of how 12 forms a base other than 10. Some points to arise were:

- dozen was a base used in England from medieval times;
- dozen is a good commercial base because it has many factors;
- the Babylonian base 60 may have been chosen for the same reason;
- if numbers were written in base 'dozen' using digits as is the case with base 10, two more digits would be required;
- since dozen was used, why is there no record of two such digits;
- numbers were not written in the base 10 manner but used words – one dozen, two and a half dozen, quarter of a dozen.

In base 10, hundreds (10^2), thousands (10^3) etc. are used. Base 12 uses gross (12^2). Hence, 200 would be one gross, four dozen and eight. Gross is still part of the English vocabulary, but now rarely used in that form, but is the origin of another more common word. Rutherford (1998, pp. 521–522) explains how in fourteenth century England, groups of general wholesalers sold only by the gross and so were the 'grossers', which is the root of the modern day 'grocer'.

How did dozens give way to the metric system? The metric system was brought into law by the French Government in 1792 after work by a Government committee which included two mathematicians, Lagrange and Laplace. The mathematicians wanted the base ten system. Initially, when the committee presented their recommendations, the Government rejected the base ten system because they wanted the British base twelve system. In fact, a compromise of base eleven was suggested. Fortunately, wisdom prevailed and everyone saw the folly of a base of eleven. The committee was established by Louis XVI and survived the French Revolution (unlike Louis).

Base 10 has now been universally accepted and the connection between 10 and the number of digits on our hands is obvious However, would not base 5 (the system of the Gumatj tribe in north eastern Arnhemland) have been equally as obvious? Are two hands better than one? What are the benefits in using base 10 instead of base 5?

Base 10 versus Base 5

There are five main characteristics of our number system:

1. it uses only 10 symbols;
2. it has place value so that 43 and 34 are different;
3. it is additive so that 432 = 400 + 30 + 2;
4. numbers are in powers of ten so that $432 = 4 \times 10^2 + 3 \times 10^1 + 2 \times 10^0$; and
5. it has a zero so that 12 and 102 are not confused.

For base 5, the characteristics would be identical except for replacing any '10' with '5'. Also, the algorithms for the four operations would match identically. In favour of 10 is that shorter strings are required for writing numbers (e.g. $214_{10} = 1324_5$). However, a short string argument would favour bases

or why 10 is a better base than 5

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higher than 10 (e.g. 12), and so does not alone provide a solid case in favour of base 10. On the other hand, base 5 may require longer strings, but has the advantage of requiring fewer digits. Scores seem equal. Are there any other significant advantages for choosing base 10 over base 5?

Pentimals

So far only whole numbers have been considered. What about numbers smaller than one?

In base ten, the decimal 0.243 reads as 2 tenths, 4 hundredths, 3 thousandths. What would be the interpretation in base 5? In this case, the word decimal is clearly inappropriate and the word *pentimal* has been chosen as a natural equivalent. With a pentimal, tenths are replaced with fifths. Hence,

$$0.1_5 = \frac{1}{5} = 0.2_{10}$$

$$0.2_5 = \frac{2}{5} = 0.4_{10}$$

$$0.3_5 = \frac{3}{5} = 0.6_{10}$$

$$0.4_5 = \frac{4}{5} = 0.8_{10}$$

For pentimals with a higher number of digits, hundredths are replaced by twenty fifths, thousandths by one hundred and twenty fifths, etc. For example:

$$0.01_5 = \frac{1}{25} = 0.04_{10}$$

$$\begin{aligned} 243_5 &= \frac{2}{5} + \frac{4}{25} + \frac{3}{125} = \frac{73}{125} \\ &\quad \left(\begin{array}{l} \text{not } \dfrac{243}{125} \text{ as often first} \\ \text{thought by students} \end{array} \right) \\ &= 0.584_{10} \end{aligned}$$

Hence, terminating pentimals convert to nice terminating decimals. This is not surprising given the multiplicative relationship between 5 and 10. On the other hand, the lack of divisibility makes base 10 and base 12 not similarly compatible. For example:

$$0.1_{12} = \frac{1}{12} = 0.08\overline{3}_{10}$$

Also, the structure of pentimals being the same as decimals provides other common results. To multiply a decimal by 10, the decimal point is moved one place to the right. With pentimals, multiplying by 5 gives (for example)

$$0.01_5 = 5 \times \frac{1}{25} = \frac{1}{5} = 0.1_5$$

Exactly the same result for the 'pentimal' point.

What do repeating pentimals look like?

$$\begin{aligned} 0.\bar{1}_5 &= 0.111\ldots_5 \\ &= \left(\frac{1}{5} + \frac{1}{25} + \frac{1}{125} \ldots \right) \end{aligned}$$

This is an infinite geometric progression with first term and common ratio both $\frac{1}{5}$.

$$\begin{aligned} \text{Sum} &= \frac{\frac{1}{5}}{1 - \frac{1}{5}} \\ &= \frac{1}{4} \end{aligned}$$

Similarly:

$$0.\overline{2}_5 = 0.222\ldots_5$$
$$= \left(\frac{2}{5} + \frac{2}{25} + \frac{2}{125}\ldots\right)$$
$$= \frac{\frac{2}{5}}{1 - \frac{1}{5}}$$
$$= \frac{2}{4}$$

$$0.\overline{3}_5 = 0.333\ldots_5$$
$$= \left(\frac{3}{5} + \frac{3}{25} + \frac{3}{125}\ldots\right)$$
$$= \frac{\frac{3}{5}}{1 - \frac{1}{5}}$$
$$= \frac{3}{4}$$

$$0.\overline{4}_5 = 0.444\ldots_5$$
$$= \left(\frac{4}{5} + \frac{4}{25} + \frac{4}{125}\ldots\right)$$
$$= \frac{\frac{4}{5}}{1 - \frac{1}{5}}$$
$$= \frac{4}{4}$$

This pattern follows the $\frac{1}{9}, \frac{2}{9}, \cdots \frac{9}{9}$ pattern for one digit repeating decimals. So it appears that pentimals (like base 5 whole numbers) follow the same patterns as decimals.

Decimals as pentimals

Pentimals have been shown to match the structure and user friendliness of decimals. However, the above conversion of repeating pentimals shows the first disadvantage compared to base 10. The 'nice' fractions of quarter, half and three-quarters translate into the 'nice' decimals of 0.25, 0.5 and 0.75. However, quarter, half and three-quarters are not 'nice' pentimals. The fact that 2 is a factor of 10, but not of 5 is significant to the ease of writing our most basic fractions in decimal and pentimal form. To go one step further, consider another basic fraction $(\frac{1}{10})$ as a pentimal:

$$0.1 < 0.2$$
$$\Rightarrow \frac{1}{10} < \frac{1}{5}$$
$$\Rightarrow 0.1_{10} = 0.0???_5$$

$$0.08 < 0.1 < 0.12$$
$$\Rightarrow \frac{2}{25} < \frac{1}{10} < \frac{3}{25}$$
$$\Rightarrow 0.1_{10} = 0.02???_5$$

$$\frac{2}{25} - \frac{1}{10} = \frac{1}{50}$$
$$0.016 < 0.02 < 0.024$$
$$\Rightarrow \frac{2}{125} < \frac{1}{50} < \frac{3}{125}$$
$$\Rightarrow 0.1_{10} = 0.022???_5$$

The pattern emerging suggests that

$$= 0.1_{10} = 0.0222\ldots_5 = 0.0\overline{2}_5$$

Checking shows this is the correct pattern.

$$0.0\overline{2}_5 = \left(\frac{2}{25} + \frac{2}{125} + \frac{2}{625}\ldots\right)$$

This is an infinite geometric progression with first term $\frac{2}{25}$ and common ratio $\frac{1}{5}$.

$$\text{Sum} = \frac{\left(\frac{2}{25}\right)}{\left(1 - \frac{1}{5}\right)}$$
$$= \frac{1}{10}$$

Some other pentimal versions for simple fractions are:

$0.2_{10} = 0.1_{10} \times 2 = 0.0\overline{2}_5 \times 2 = 0.0\overline{4}_5$
$0.3_{10} = 0.1_{10} + 0.2_{10} = 0.0\overline{2}_5 + 0.0\overline{4}_5 = 0.1\overline{2}_5$
$0.4_{10} = 0.1_{10} + 0.3_{10} = 0.0\overline{2}_5 + 0.1\overline{2}_5 = 0.1\overline{4}_5$
$0.7_{10} = 0.3_{10} + 0.4_{10} = 0.1\overline{2}_5 + 0.1\overline{4}_5 = 0.3\overline{2}_5$

Repeating decimals hardly bear thinking about. Trying to find

$$\frac{1}{9}$$

leads quickly to a mire of fractions with large denominators. The author quit at 0.021424... with no distinct pattern in sight. Is it possible that a rational number base 10 could be irrational in another base?

Conclusion

Other bases such as 5 and 12 provide the same structural place value benefits as base 10. However, when numbers less than one are concerned, base 10 provides friendly decimals for the most common fractions of half, quarter, three-quarters. Base 5 is not user friendly at all in this regard. Base 12 would provide nice *dozenimals*(?) for the same fractions, but not for the commonly used tenths or fifths. Of course, it may be that the reason these are the commonly used fractions is that they do match base 10 so well. However, the conclusion drawn here is that the wisdom of the mathematicians like Lagrange and Laplace, even when compelled to oppose political forces, is vindicated and we have, for practical purposes, a number system which stands up strongly to scrutiny.

Reference

Rutherford, E. (1998). *London*. London: Arrow Books Limited.

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Postscript

The other commonly used base is two (binary). Binary form provides a 'mind reading' game as follows.
The numbers 1 to 15 are distributed on 4 cards as follows.

1	3	5	7
	A		
9	11	13	15

2	3	6	7
	B		
10	11	14	15

4	5	6	7
	C		
12	13	14	15

8	9	10	11
	D		
12	13	14	15

A person then selects a number and tells the mind reader which of cards A, B, C, D the number is on. The mind reader immediately picks the correct number. This works because the top left hand corner of each card is 1, 2, 4, 8 (the binary place holders). Each card then contains the numbers which have a value in that place holder. For example, $13 = 1 + 4 + 8$ and so is on cards A, C, D; $10 = 2 + 8$ and so is on cards B and D.

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