How Students Understand the Area under a Curve: A Hypothetical Learning Trajectory

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Abstract: The area under the curve is a fundamental concept for students to build their understanding of the Definite Integral. This research reveals how students comprehend the area under the curve in given contextual problems and how the Hypothetical Learning Trajectory (HLT) can help students find the concept. This research follows the development research model of Gravemeijer and Cobb, which consists of three stages: Preparing for the experiment, Experimenting, and Retrospective analysis. This research involved three students with varying abilities: low, medium, and high. Data were collected through the analysis of student work documents and in-depth interviews. The research findings indicate that students often struggle with determining the area under the curve. Students approach the area by using various polygons. With the guidance of their lecturers, students discovered that the area under the curve can be approached by partitioning the area into polygons. The more polygons used, the closer the approximation becomes. In other words, the HLT designed in this study facilitates students in understanding the concept of the area under the curve, formulated as $A = \lim_{n\to\infty} \sum_{i=1}^{n} A_i$.

Keywords: Area under the Curve, Partition, Polygon, Hypothetical Learning Trajectory, Realistic Mathematic Education.

INTRODUCTION

The definite integral is one of the important concepts in Calculus (Greefrath et al., 2021; Hashemi et al., 2020; Jones, 2015; Stevens, 2021; Ural, 2020) studied at the college level. Apart from its support for other subjects, the definite integral has broad applications, including the calculation of areas, volumes of objects, lengths of curves, work, fluid pressure, moments, the center of mass, and more (Varberg et al., 2016). This concept should be well mastered by students.

Several studies have revealed that students have difficulty understanding the concept of definite integrals. It was found that students often fail to understand the definition of a definite integral (Grundmeier et al., 2006; Rasslan & Tall, 2002). This is in line with Aniswita's research which

found that only 7% of students could answer Integral Of course questions using the definition correctly (Aniswita et al., 2023). The same thing was also found by Hidayat et al., (2021) only 10% of students were able to construct a definite integral definition. According to Purnomo et al (2022), the students' difficulties were caused by weak mathematical literacy and an incomplete mathematization process.

The concept of definite Integral is built from Riemann sums (Sealey, 2006, 2014). which begins with the concept of the area under the curve (Grundmeier et al., 2006; Orton, 1983; Yost, 2008). In other words, to understand the concept of a definite integral, students must first grasp the concept of the area under a curve.

To provide a strong understanding of concepts, learning needs to be well-designed so that it can facilitate students discovering these concepts (Clement & Sarama, 2004; Gravemeijer, 2004). Learning design or learning trajectory is a potential step to improve the learning process to achieve goals (Daro et al., 2011; Wilson et al., 2013; Clement & Sarama, 2014; Simon, 1995; Simon et al., 2018). Learning trajectory in mathematics learning was first introduced by Simon (1995) with the term Hypothetical Learning Trajectory (HLT). HLT consists of three components, namely: 1) learning objectives; 2) learning activities; and 3) the learning process hypothesis.

According to Gravemeijer (1998) and Larsen (2013), HLT in mathematics learning is connected to principles of Realistic Mathematics Education (RME), namely: 1) guided reinvention; 2) didactical phenomenology; and 3) emergent models (Gravemeijer, 1999). RME is an approach to learning mathematics based on the Freudental view that mathematics is a human activity and learning mathematics is essentially 'doing mathematics’ or 'mathematizing' (Barnes, 2005; Dickinson & Hough, 2012; Drijvers, 2018; Gravemeijer & Terwel, 2000; Kwon, 2002; Marja Van Den Heuvel-Panhuizen, 2003; Van den Heuvel-Panhuizen, M; Drijvers, 2014; Webb et al., 2011). RME was first developed in the Netherlands in 1968. This approach is inspired by the philosophy of constructivism, especially social constructivism (Gravemeijer, 2020b, 2020a) which views that knowledge is constructed by humans through their interactions with the environment (Ansyar, 2015; Schunk, 2012). So learning must start from contextual problems that are close and meaningful to students (Aziza, 2020). Students solve problems by using the knowledge they have (model of) to find concepts (model for). During the discovery process, with the help of lecturers, their knowledge develops along with the process of horizontal mathematization and vertical mathematization (Barnes, 2005; Fauzan, 2002; Gravemeijer, 2020b; Gravemeijer & Doorman, 1999; Guler, 2018; Kwon, 2002; Marja Van Den Heuvel-Panhuizen, 2003; Rasmussen, 2014; Yvain-Prébiski & Chesnais, 2019).

Research on HLT with the RME approach has been carried out quite a lot and is growing rapidly. This development inspired the Mathematics Teaching Research Journal to publish a special issue related to this research MTRJ, vol 13 no 4, winter 2021 edition (http et al., n.d.). This research was conducted at all levels, both from elementary school to university level. At the university level, multiple studies were undertaken, including investigations by Zandiah & Rasmussen (2010) on geometry, Larsen (2013) on algebra, Cárcamo et al. (2019) on algebra, Andrews-Larson et al. (2017) on linear transformation, Syafriandi et al. (2020) on statistics, and Yarman et al. (2020) on differential equations. Based on this, it is necessary to design HLT which can facilitate students to find the concept of the area under the curve. This article will reveal how students understand the

concept of the area under the curve and how the designed HLT can facilitate students in finding the formal concept of the area under the curve.

METHOD

This research is Gravemeijer and Cobb's design research model, which consists of three phases, namely: 1) Preparing for the experiment, (2) Experimenting in the classroom, and (3) Retrospective analysis (Gravemeijer & Cobb, 2013). To fulfill the research objectives of investigating students' understanding of the area under the curve and the role of HLT in facilitating their comprehension, a qualitative descriptive approach was used. In the initial phase of preparing for the experiment, the final learning goal and starting point are identified to develop an anticipated learning process and corresponding HLT. The subsequent phase of experimenting in the classroom focuses on implementing and testing the formulated learning process. Lastly, the retrospective analysis phase evaluates and potentially revises the HLT that was designed. Here is the description:

Preparing for The Experiment

The first step is to formulate the objective of studying the area under the curve. From the literature review, the learning objective of the area under the curve emphasizes the procedural ability to calculate the area under the curve using polygons. According to researchers, students need to understand that the area of a curved plane can be determined by partitioning the area into polygons. Then, the researcher interviewed ten students to determine the starting point in learning. Students were selected to represent low, medium, and high levels of academic ability. In addition, gender and communication skills were thoughtfully taken into consideration. This investigation entailed specific inquiries regarding the students' preferred learning modalities and the essential educational resources for facilitating a profound understanding of the subject matter. Finally, the author formulated an HLT for the area under the curve based on these two things.

Classroom Experiment and Retrospective Analysis

The focus of these two phases is on testing and improving the effectiveness of the HLT (Hypothetical Learning Trajectory) area under the curve that has been designed. At this stage, the HLT can transform into a localized instructional theory specifically aimed at teaching the area under the curve. Gravemeijer and Cobb (2006) emphasize that this phase involves a cyclical process where thinking and experimentation in instructional development are intertwined, as shown in Figure 1.

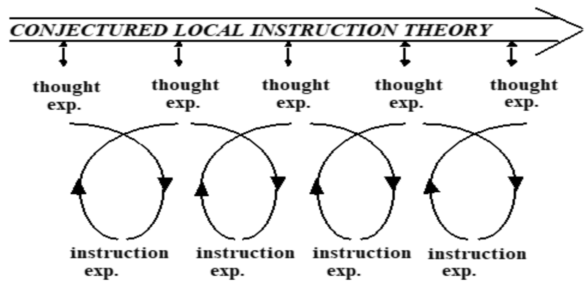


Figure 1: The reflective relation between the theory and the gap

Before conducting the experiment, the validation of HLT area under the curve was validated by five experts, including three mathematicians, one educational technology expert, one educational evaluation expert, and one language expert. The experiment involved three students who possessed high, medium, and low academic abilities, which were determined by their performance in the Differential Calculus course. These students were selected randomly. Data was collected through the analysis of student work documents and in-depth interviews conducted during the learning process. The purpose of these interviews was to understand how students approached the given activities and to determine whether their anticipation could guide them toward the desired trajectory.

RESULTS

Preparing For The Experiment

Based on the literature review and student characteristics, the HLT for the area under the curve was formulated. Iceberg from HLT is given in Figure 2.

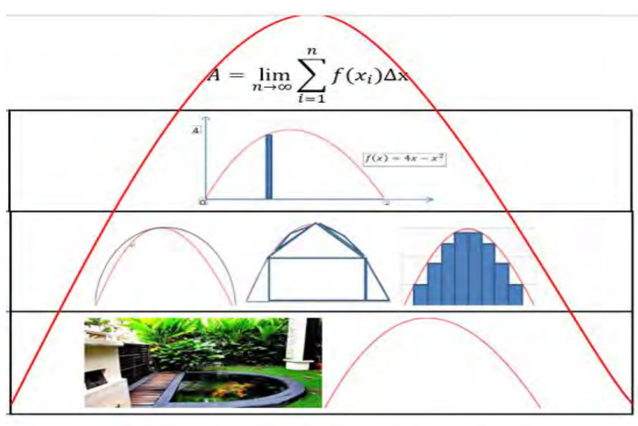


Figure 2: "Iceberg" Area under a Curve

In accordance with the "iceberg" HLT the area under the curve consists of two activities, namely:

Activity 1

The purpose of this activity in Activity 1 is for students to be able to predict/approach and determine the area using polygons. Contextual problems are given in Figure 3

Mr. Syaiful has a koi fish pond. How do you calculate the surface area of Mr. Syaiful's koi fish pond? Explain the answer!

Figure 3: Contextual problem Koi Fish Pond Surface Area

Hypothesis and anticipation of Activity 1 as in Table 1

The Hypothesis of Student Activities	Anticipation
Students are confused about determining the surface area of a fish pond	The lecturer asks probing questions. Take a look at the surface shape of Mr Syaiful's koi fish pond, how do you get to its area?
Students approach the surface area of a fish pond with one flat plane such as a circle, triangle, square, rectangle, trapezoid	The lecturer asks probing questions. Is the approximation you found the best approximation? What about the area that does not include the surface area of the pool?
Students approach the surface area of a fish pond by dividing the area into several polygons of different types	The lecturer asks probing questions. Is the approximation you found the best approximation? Is it easy to calculate the area of these polygons?
Students approach the surface area of a fish pond by dividing the area into several similar polygons	The lecturer asks probing questions. Is the approximation you found the best approximation? How do you make the area that does not include the area of the fish pond getting smaller and closer to the real thing?
Students approximate the surface area of a fish pond by dividing the area into more polygons	The lecturer asks probing questions. What can you conclude from the activities you did?

Students are able to find that the more the surface of the fish pond is partitioned, the area of all polygons will approach the surface area of the actual pond	The lecturer asks probing questions. When do you think the area of the entire polygon equals the actual area of the fish pond?
Students are able to find that the actual surface area of a pool can be determined by dividing the area into infinity and then adding up the areas of all the polygons.	The lecturer emphasizes the student's findings that to determine the area of a curved plane by partitioning the area into n→∞

Table 1: *Hypothetical Learning Process Koi Fish Pond Surface Area*

Activity 2

The purpose of the activity in activity 2 is for students to be able to determine the formula for the area under a curve using polygons. Contextual issues are given in Figure 4

If the curved line on the surface plan of Mr. Syaiful's koi pond is a function curve y=f(x), then determine:

1. Approximate surface area of the pool if partitioned by 8 polygons!
2. The approximate surface area of the pool if it is partitioned by n polygons!
3. What is the actual surface area of the fish pond? explain the answer!
4. Based on answer 2) construction of the formula for the area under the curve f(x) from a to b!

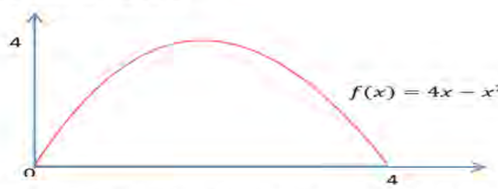


Figure 4: Contextual problem the area under the curve

Hypothesis and anticipation of Activity 2 as in Table 2

The Hypothesis of Student Activities	Anticipation
Question 1 Students partition the area into eight parts but there are still errors	The lecturer asks probing questions. Look at the partition area under the curve! Where are the polygons formed?
The student partitioned the area into eight parts but still got the size of the polygon wrong	The lecturer asks probing questions. Look at polygon 1, polygon 2, and so on, what's the width? Can it also be determined

Students are able to approximate the surface area of a fish pond by dividing the area into eight polygons	The lecturer emphasizes the students' findings, that the approximate area under the curve is the sum of the areas of eight polygons
Question 2 Students have not been able to determine the size of a polygon if it is partitioned by n	The lecturer asks probing questions. Pay attention to polygons 1, 2, …, and polygon n. Is it possible to specify the width of the polygon? Then how to determine the length of each polygon?
Students are able to determine the size of polygons but are still wrong about the surface area of a fish pond with n polygons	The lecturer asks probing questions. Notice the size of the polygons? How to determine the approximate surface area of a fish pond?
Students are able to find the approximate surface area of a fish pond with n polygons	The lecturer emphasizes the students' findings, that the approximate area under the curve is the total area of n the polygons, namely $L \approx \sum_{i=1}^{n} L_i$
Question 3 Students have not been able to find the actual surface area of the pond.	The lecturer asks probing questions. How do you determine the area under the curve (surface area of the fish pond) in the form of a curved plane?
Students are able to find the surface area of the actual fish pond	The lecturer emphasized the students' findings, that the surface area of the fish pond namely $L = \lim_{n\to\infty} \sum_{i=1}^{n} \left(4\left(\frac{4i}{n}\right) - \left(\frac{4i}{n}\right)^2\right)\frac{4}{n}$
Question 4 Students have not been able to find the area under the curve for any function f(x) from a to b	The lecturer asks probing questions. Based on the previous activity, what can you conclude to determine the area under the curve with a boundary from a to b?
Students are able to find the area under the curve for any function f(x) from a to b namely $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$	The lecturer emphasized the students' findings, that the surface area under the curve for any function f(x) from a to b namely $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ with $\Delta x = \frac{b-a}{n}$, $and\ x_i = a + i\Delta x$

Table 2: *Hypothetical Learning Process Area Under the Curve*

Experimenting in The Classroom

Activity 1

Students are asked to determine a method or strategy for calculating the surface area of Pak Syaiful's koi fish pond. Before starting Activity 1, the lecturer conveys the aim of the activity and reminds the term polygon and the area of several flat planes such as rectangles, triangles, trapeziums, and others. The three students with low (MR), medium (MS), and high (MT) abilities had difficulty determining the surface area of the pool. The lecturer, using the HLT that has been designed, guides students to find ways or strategies to calculate the surface area of the fish pond. The following is the description.

Students with Low Ability (MR)

Students with low ability (MR) are not able to understand the questions well. This can be seen from the empty student answer sheets and from interviews. Students need quite a lot of time to find a way to determine the surface area of Mr. Syaiful's fish pond. The following are the results of interviews and student worksheets during the discovery process.

Lecturer: *Do you understand what is meant by the question?*

Student (MR): *Understand Ma'am, but I cannot determine the surface area of the fish pond because the size of the pond is unknown.*

Lecturer: *Take a look at the surface shape of Mr Syaiful's koi fish pond, how do you get to its area?*

Student (MR): *hmmm (student thinks for a long time). I approximated the surface area of the fish pond using triangles, rectangles, and circles Ma'am.*

Lecturer: *What about areas that are outside the flat surface but include the surface area of the fish pond?*

Students return to the surface area of the fish pond using a trapezium. The answers as in Figure 5

Figure 5: Students MR Approach Using Circles, Triangles, Rectangles And Trapezoids

Lecturer: *Is this the best approximation?*

Student (MR): *Yes Ma'am, when compared with approaches using rectangles and triangles, the trapezoidal approach is better*

Lecturer: *OK. What about the area outside the trapezoid including the surface area of the fish pond? In your opinion, when is this approximation the best approximation?*

Student (MR): *The best approximation is when this area (pointing) gets smaller*

Lecturer: *What is the strategy to make the area even smaller?*

Students re-approximate the surface area of the pool by dividing the surface of the pool into several dissimilar polygons as in Figure 6

The more flat areas, the closer to the actual area

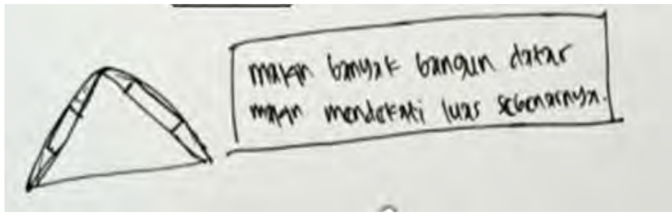


Figure 6: Students MR Approach By Dividing The Area Into Several Different Polygons

Lecturer: *Which is easier to calculate area by dividing into similar or different polygon?*
Student (MR): *It's easier if the polygon is similar*
Then, the student approximates the surface area of a pool with similar polygons as in Figure 7

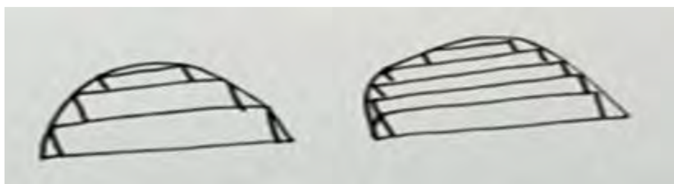


Figure 7: Students MR Approach By Dividing The Area Into Several Similar Polygons

Lecturer: *When do you think the area of all polygons is the same as the surface area of the pool?*
Student (MR): *I can't do it, Ma'am, because there will always be some leftovers.*
Lecturer: *Just imagine, what if the surface of the fish pond was divided into 100 flat areas, 1 million flat areas, and so on.*
Student (MR): *The remaining areas are getting smaller, Ma'am, almost approaching 0*
Lecturer*: When do you think the remainder of the area is equal to 0?*
Student (MR): *hmm (thinks for a long time, then hesitantly says), when divided by infinity*
Lecturer: *OK, so what can you conclude about how to determine the surface area of Mr. Syaiful's koi fish pond?*
Student (MR): *Divide the surface of the fish pond into infinite polygons.*

The answers of students with low abilities (MR) with the help of HLT can be seen from the development of models used by students to solve problems until they find formal concepts as in Figure 8.

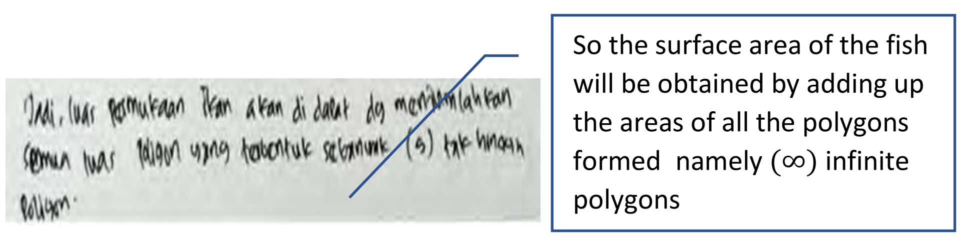


Figure 8: Students MR Can Conclude How To Calculate A Curved Plane By Partitioning n $\rightarrow \infty$

Students with Medium Ability (MS)

Students with medium abilities (MS) can understand the questions well, students immediately approach the surface area of the fish pond with various flat areas (polygons). Students need quite a long time to try out various polygon shapes.

Lecturer: *How do you determine the surface area of a fish pond?*
MS Student: *I tried to approach with triangles, rectangles, and trapezoids*
Lecturer: *Is the approximation that you found the best approximation?*
MS Student: *Not yet Ma'am, because there are still many remaining areas that are not included in the triangle, rectangle, and trapezium.*
Lecturer: *What is the strategy to make the area smaller?*
MS Student: *I divided it into several different polygons, and this is better Ma'am than just approaching it with one polygon.*
The answer as in Figure 9

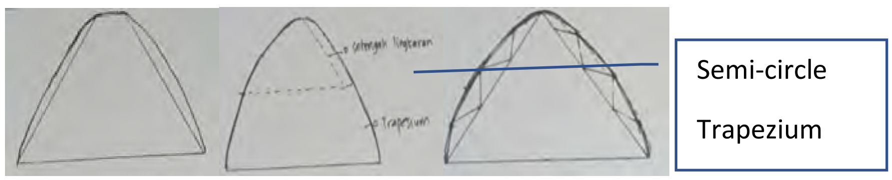


Figure 9: Students MS Approach By Dividing The Area Into Several Different Polygons

Lecturer: *In your opinion, which is easier to determine the area of similar or dissimilar polygons?*
MS Student: *It's easier to use a similar flat surface*
Lecturer: *In your opinion, when is the area of all polygons the same as the actual area? Try to imagine, what if the surface of the pool was divided into 100 polygons, 1 million polygons, and so on. What about the remaining areas?*
MS student: *the less it is, the more it goes to 0*
Lecturer: *So what can you conclude?*
MS Student: *The surface area of a pool can be determined by dividing the area into an infinite number of similar polygons.*

The answer as in Figure 10 and Figure 11

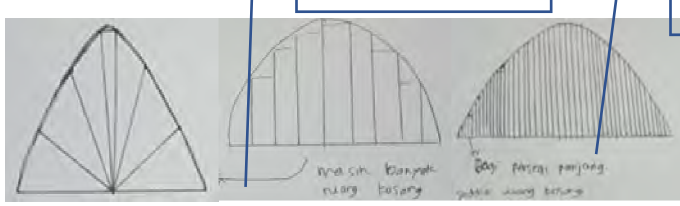


Figure 10: Students MS Approach By Dividing The Area Into Several Similar Polygons

So the surface area of the fish pond will be obtained by adding up the polygons that form infinite polygons

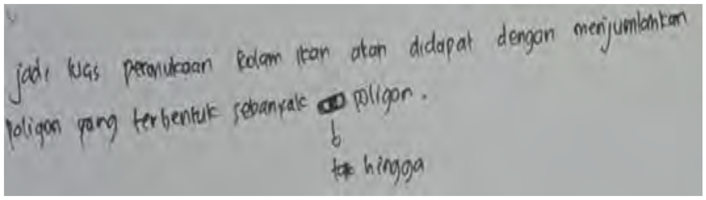


Figure 11: Students MS Conclude How To Calculate A Curved Plane By Partitioning N →∞

The answers of students with medium abilities (MS) with the help of HLT can be seen from the development of models used by students to solve problems until they find formal concepts as in Figure 9- Figure 11.

Students with High Ability (MT)

Students with high abilities (MT) can understand the questions well, students immediately approach the surface area of the fish pond with various polygons. Students need a relatively short time to try out various polygon shapes.

Lecturer: *How to determine the surface area of a fish pond?*

Student (MT): *Partition it into several polygons, Ma'am, then calculate the area of the area of all polygons.*

Lecturer: *Is the area of the entire polygon the surface area of the fish pond?*

Student (MT): *No Ma'am, because there are areas that are not included in the surface area of the pool.*

Lecturer: *So what can you conclude?*

Student (MT): *If the surface of the pool is divided more and more, the area of the entire area will be closer to the actual area.*

The answer as in Figure 12

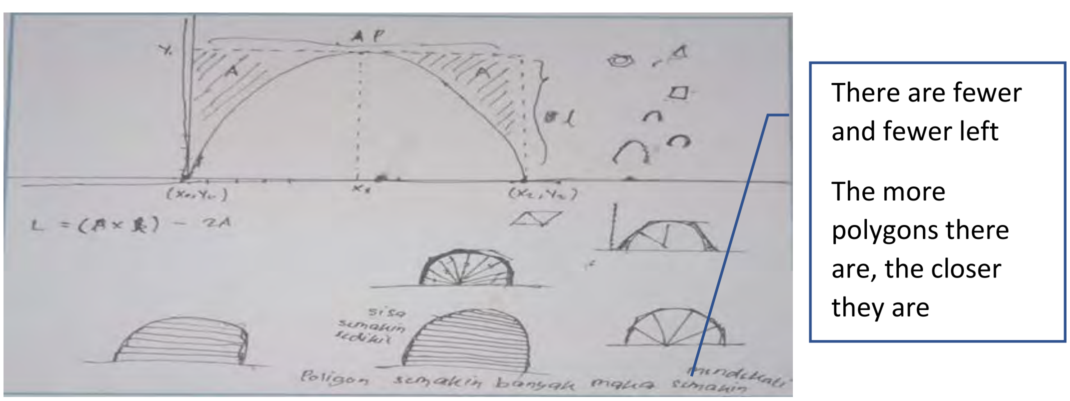
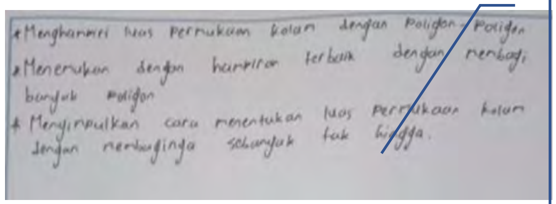


Figure 12: Students MT Approach By Dividing The Area Into Several Polygons

Lecturer: *When do you think the area of a polygon is the same as the actual area?*

Student (MT): *hmm (thinks for a moment) When divided by n it goes to infinity.*

The answers of students with high abilities (MT) with the help of HLT can be seen from the development of models used by students to solve problems until they find formal concepts as in Figure 13


*Menghampiri luas permukaan kolam dengan poligon-poligon
*Menentukan dengan hampiran terbaik dengan membagi banyak poligon
*Menyimpulkan cara menentukan luas permukaan kolam dengan membaginya sebanyak tak hingga.

* Approximate pool surface area with polygons
* Finds the best approximation by dividing many polygons
* Summarize how to determine the surface area of a pool by dividing it infinitely

Figure 13: Students MT Conclude How To Calculate A Curved Plane By Partitioning N $\rightarrow\infty$

Activity 2

Students are asked to find a formula for determining the area under the curve from the problem of the surface area of a fish pond whose curved side is a graph of the function f(x)=4x-x^2. Before starting the activity, the lecturer conveys the purpose of the activity and reminds you about the sigma function and notation. The three students with low (MR), medium (MS) and high (MT) abilities still have difficulty determining this. The lecturer, using the HLT that has been designed, guides students to find the formula for determining the area under the function curve f(x) from a to b. Here's the description:

Students with low ability (MR)

Students with low ability (MR) need quite a long time to determine the surface area of Mr. Syaiful's fish pond, especially when the area is divided into n polygons. The following are the results of interviews and student worksheets during the discovery process. Students (MR) divide the area into 8 parts but it is not correct.

Lecturer: *Look at the partition, is it divided into eight?*

Student (MR): *hmm (thinking and paying attention to the picture), not yet ma'am (smiling)*

Lecturer*: Can you determine the area of each polygon?*

Student (MR): *No*, *I can't, Ma'am, because the length of the polygon is unknown.*

Lecturer: *Look at the curved line (while pointing at the graph), if the x value is known, can this value (while pointing at the graph) be determined?*

Student (MR): *(smiling) Yes, Ma'am, by substituting the value of x into the function f(x)*

Then students (MR) determine the size of the polygon. Calculate the area of the polygons and add up their areas as an approximation of the surface area of a fish pond with partitions of eight as in Figure 14

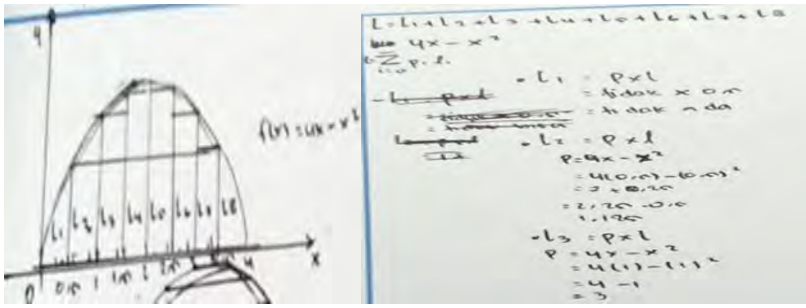


Figure 14: Students Determine The Approximate Surface Area Of A Fish Pond With Eight Partitions

To answer the second question, students (MR) are confused about how to divide it into n parts, this can be seen from the students' answer sheets which are still empty.

Lecturer: *Can you calculate the area of all polygons?*
Student (MR): *(while thinking and hesitating to answer) I can't, Ma'am, because the value of n is unknown.*
Lecturer: *Haven't you studied sigma notation? Try to pay attention to the area of polygons 1, 2, 3, and so on. Can you see the connection?*
Student (MR): *hmm (thinks for a long time then smiles) yes Ma'am*
Then students determine the surface area of the fish pond with partition n as ini Figure 15

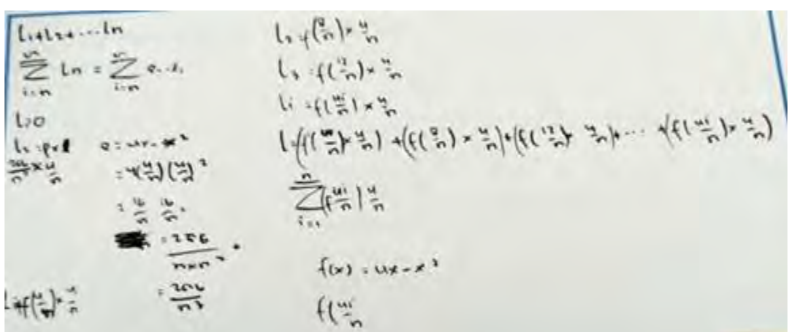


Figure 15: Students Determine The Approximate Surface Area Of The Mikan Pool With Partition n

To answer question 3 to determine the surface area of a fish pond, MR students can actually answer in words but are confused about how to formulate it.

To answer question 4, determine the formula for the area under the curve of the function f from a to b, students (MR) have difficulty. This can be seen from the empty Student (MR) answer sheets.
Lecturer: *Pay attention to your answer to question number 3, if you substitute for any function f(x) and the limit is not from 0 to 4 but from a to b, can you find the formula?*
Student (MR): *(hmm), yes Ma'am*

The answers of students with low abilities (MR) with the help of HLT can be seen from the development of models used by students to solve problems until they find formal concepts as in Figure 16:

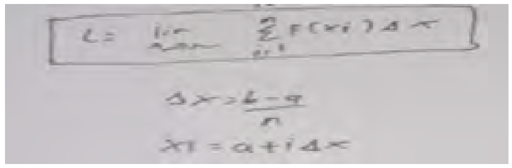


Figure 16: Students Generalize The Area Formula For The Area Under The Curve

Students with Medium Ability (MS)

Students with a medium ability (MS) are almost the same as students with low ability for questions number 1, 3, and 4. However, students (MS) tend to solve them more quickly. For question number 2, MS students were able to determine the size and area of the polygon but there were still errors. The following are interviews and student work results:

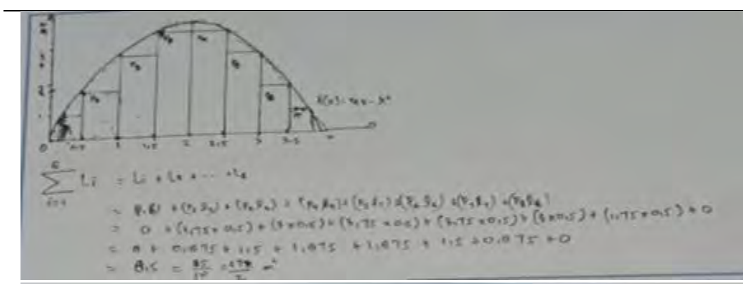
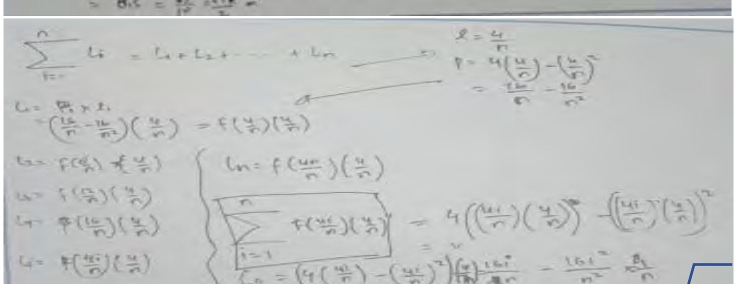
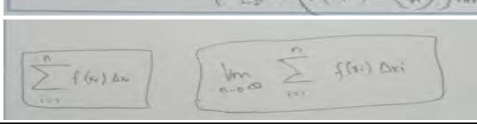
Lecturer: *How to determine the surface area of a pool if divided by n?*

Student (MS): *Calculating the area of all polygons then adding them? But I'm confused about how to determine the total size and area because n is unknown*

Lecturer: *Is there a relationship between the areas of polygon 1, 2, 3, and so on?*

Student (MS): *(smiling) yes Ma'am.*

The answers of students with medium abilities with the help of HLT can be seen from the development of models used by students to solve problems until they find formal concepts as in Table 3:

	Students determine the approximate area of the pool using 8 polygons
	Students determine the surface area of the pool using n polygons
	Students formulate the surface area of the fish pond

So to find the area under a curve using polygons, divide the area into as many polygons as possible and then add them up.

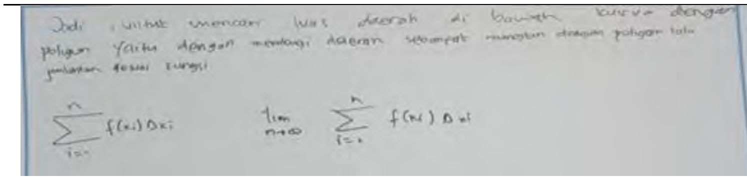
	Students generalize the formula for the area under the curve

Table 3: *Student Answers (MS) During the Discovery Process in Activity 2*

Students with High Ability (MT)

Students with high abilities (MT) were able to solve question number 1, question 3, and question 4. Students (MT) had a little difficulty solving question number 2 to determine the surface area of a fish pond if divided into n polygons.

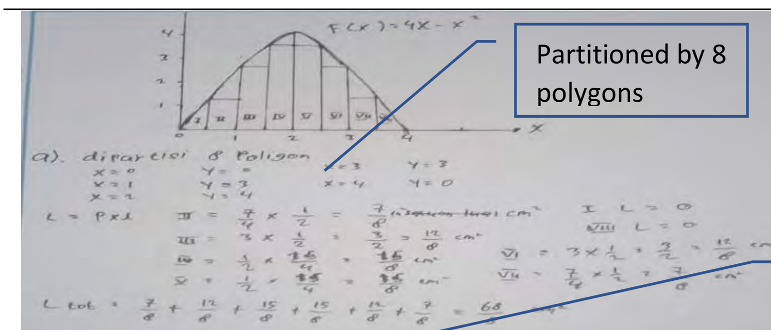
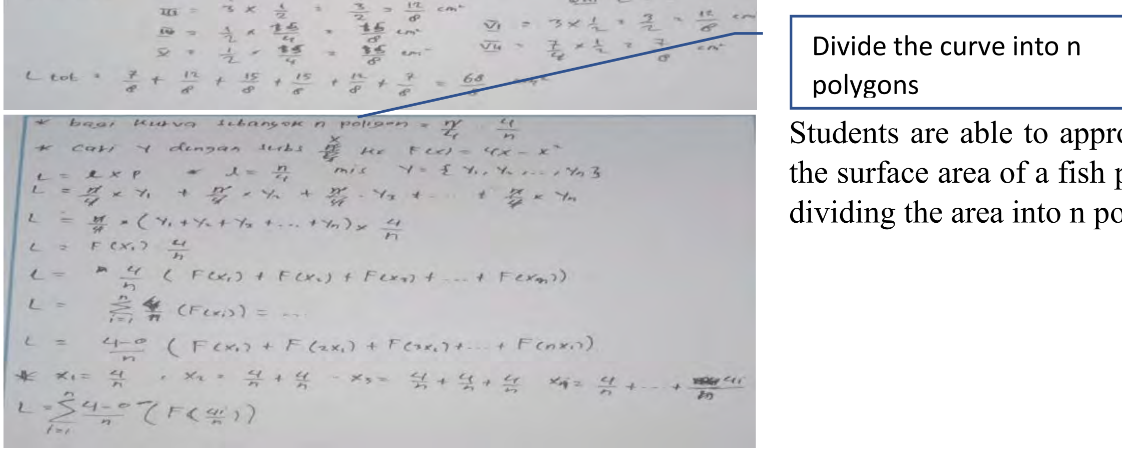
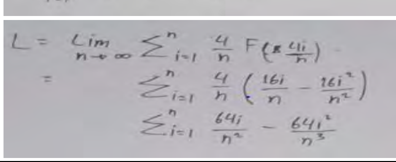
Lecturer: *How to determine the surface area of a fish pond if it is approached by n polygons?*

Student (MT): *Determine the area of each polygon then add them up, ma'am. But I am constrained because there are n polygons.*

Lecturer: *Have you studied sigma form? Try to pay attention to the area of polygon 1, polygon 2, polygon 3, and so on. Do you see the connection?*

Student (MT): *hmmm (student thinks then smiles), yes Ma'am*

The answers of students with high abilities (MT) with the help of HLT can be seen from the development of models used by students to solve problems until they find formal concepts as in Table 4:

	Students determine the approximate surface area of the fish pond if it is divided into 8 polygons
	Students are able to approximate the surface area of a fish pond by dividing the area into n polygons
	Students determine the formula for the surface area of a fish pond

$L = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$ $\Delta x = \frac{b-a}{n}$ $x_i = a + i.\Delta x$	Students generalize the formula for the area under the curve

Table 4: *Student Answers (MT) During the Discovery Process in Activity 2*

Retrospective Analysis

Based on the experimental results, for activity 1 there are several changes to the HLT that have been designed, namely adding two hypotheses and anticipation, namely 1) when students cannot see the relationship between many partitions with the best approximation. The anticipation that is carried out is a probing question, how can the area outside the flat area which is still the surface area of the fish pond become smaller? 2) When students are not able to see that the area of all flat areas (polygons) is the same as the surface area of a fish pond when the area in the partition is infinite. The anticipation that is carried out is a probing question, what if the surface of the fish pond is divided into 100 flat areas, 1 million flat areas, and so on? When is the area of the entire plane equal to the surface area of the fish pond? For activity 2, there are several changes to the HLT that have been designed, namely anticipating the first question, namely 1) when students are not yet able to determine the length of the polygon. The anticipation is a probing question, pay attention to the graph of the function and one of the polygons, if the area of origin is known, can the length of the polygon be determined? For question 2, you need to add hypothesis and anticipation, namely when students are confused about dividing an area into n polygons. The anticipation is a probing question, can you describe some of the partitions and determine their sizes? Anticipation of the second hypothesis was changed to a more specific probing question. Haven't you learned sigma notation? Take a look, is there a relationship between the areas of polygon 1, polygon 2, and so on? For question 4, anticipation is changed to be more specific, pay attention to the answer to question 3, what if the function is arbitrary f(x) and the limit is from a to b?

DISCUSSION AND CONCLUSIONS

This research describes how students understand the area under the curve and how the designed HLT can facilitate students to discover the concept of the area under the curve. The area under the curve is the basis for understanding Definite Integrals or Riemann Integrals. This integral is built from the Riemann sum which is the algebraic sum of the area under the curve (Varberg et al., 2016). Based on the research results, it can be seen that students have difficulty determining the area under the curve. Students are only able to approximate the surface area of the pool by using a flat plane or a polygon. This is because students are accustomed to calculating area using formulas without knowing how the concept of area was discovered.

The existence of intervention in the form of RME-based HLT can help students discover the concept of area under the curve. This is in accordance with the opinion of Clement and Sarama (2004) that learning flow can increase students' understanding of concepts. Through contextual

problems given, students develop their knowledge by gradually discovering formal mathematical concepts. In accordance with the opinion of Gravemeijer (2020), during the discovery process, student knowledge is formed and developed. According to Vygotsky, a social constructivist figure (Santrock, 2008), children actually already have rich concepts but are not yet systematic and organized, so external intervention is needed, one of which is RME-based HLT. In accordance with the opinion of Cárcamo et al (2019), RME-based HLT has a great opportunity to improve students' reasoning abilities and make Integral Learning more meaningful. This is also supported by research by Aziza (2020) which recommends that Calculus learning must be connected to students' real lives, which is one of the characteristics of RME.

There were several changes to the draft HLT that was designed, this is in accordance with the opinion of Simon & Tzur (2004) that the HLT must be modified regularly and through an iterative process. HLT must be adapted to student characteristics and be dynamic. In line with Gravemeijer and Cobb (2006), there is a reflective relationship between learning theory and its implementation and the development of HLT. In other words, HLT can change based on the experiments carried out. Of course, this is a challenge for lecturers in designing HLT that suits student characteristics.

Based on the research results, it can be concluded that students have difficulty determining the area of an area whose sides are curved lines. The existence of HLT helps students discover the concept of the area under the curve. There are several revisions to the HLT, namely the addition of two hypotheses and their anticipation in activity 1. For activity 2 there is a change in anticipation when students are not able to determine the size of the polygon in the first question, namely "Pay attention to the graph of the function and one of the polygons, if the area of origin is known, what is the length of the polygon can it be determined?" Subsequent revisions to the anticipation of questions 2, questions 3, and questions 4 were changed to be more specific. Students with low abilities tend to need quite a long time and quite a lot of experiments to come to this conclusion and tend to use words. Medium and high students tend to be relatively fast and draw conclusions using symbols. Likewise, when formulating the area under the curve, high and medium ability students with the help of HLT tend to formulate it more quickly.

A limitation of this research is that the research subjects only involved three students with heterogeneous abilities. For further research, it is recommended to expand the research subject and also designs HLT on other Definite Integral topics such as Riemann sums, definitions, and properties of definite integrals including the Fundamental Theorem of Calculus (FTC). The hope is that Integral learning will certainly become more meaningful and effective.

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