Pedagogical Research
2023, 8(4), em0172
e-ISSN: 2468-4929
https://www.pedagogicalresearch.com

MODESTUM

Research Article

OPEN ACCESS

An investigation on the thinking structures and proof-writing levels of future mathematics teachers

Elif Aksan Kilicaslan [1*]

[1] Department of Mathematics Education, Fatih Faculty of Education, Trabzon University, Trabzon, TÜRKİYE
*Corresponding Author: eaksan@trabzon.edu.tr

Citation: Aksan Kilicaslan, E. (2023). An investigation on the thinking structures and proof-writing levels of future mathematics teachers. *Pedagogical Research, 8*(4), em0172. https://doi.org/10.29333/pr/13682

ARTICLE INFO

Received: 21 Mar. 2023

Accepted: 25 May 2023

ABSTRACT

In this study, it is aimed to reveal what kind of relationship exists between prospective teachers' thinking structures and their level of writing proofs. This study was carried out as a case study. The study was carried out with 82 prospective teachers enrolled in the Department of Secondary School Mathematics Teaching of a state university located in the Eastern Black Sea Region of Turkey. The first data collection is the mathematical processing instrument developed by Presmeg (1986). The second data collection was developed by the researcher and consists of five proofs that are also covered in the curriculum about the geometric shapes circle and circular region. As a result, when the thinking structures of the participants are examined, it can be understood that harmonic thinking is the most common thinking type among the study participants. When the proof-writing levels in this study are looked at, it becomes obvious that most of the prospective teachers are weak in terms of this skill. In this study, the participants' proof-writing levels and thinking structures were found to be correlated.

Keywords: mathematical processing instrument, proof-writing levels, prospective mathematics teachers, circle and circular region

INTRODUCTION

Thinking is defined as activating many mental processes to solve problems (McGillicuddy-DeLisi & Sigel, 1991). These mental processes are creative thinking, reasoning, and evaluation (Tekin et al., 2009). Thinking is a skill that can be improved. Mathematics education is a tool that improves thinking (Umay, 2003). Mathematical thinking can be described as the individual's access to new information by going through various mental processes with the aid of previously acquired mathematical knowledge (Alkan & Bukova-Guzel, 2005). Krutetskii (1976) divided people's thinking structures into three by how they handle mathematical knowledge. These are analytical, harmonic, and geometrical thinking. While people who adopt analytical thinking are prone to verbal items, they have low affinity for visual items. They prefer verbal and logical methods in solving problems. People who have harmonic thinking structures work equally with verbal and visual elements. They want to use both methods together in problem-solving. The last group, people who have geometrical thinking model, is good with visual elements. They need to use visual elements to solve problems (Kozhevnikov et al., 2002; Krutetskii, 1976).

Geometry is a branch of mathematics that deals with phenomena such as point, line, plane, planar shapes, space, spatial shapes and the relations between them and measurements of geometric shapes such as length, angle, area, and volume (Dursun & Coban, 2006). Learning geometry starts with making sense of the physical world and continues with higher-level geometrical thinking that develops in an inductive or deductive system (Ubuz, 1999). Among other, one important purpose of geometry education is to develop reasoning skills (Morali et al., 2006). One of the elements that serve the development of reasoning skill is proof. Proof is the process of judging as true or false a proposition or theorem or a statement in mathematics (Guven et al., 2005). Proof bears several mathematical meanings. However, in the broadest sense, proof is the systematic use of basic axioms, definitions, and theorems to make mathematically clear why a proposition is true (Bell, 1976). From pedagogical perspective, proofs allow students to develop their reasoning skills and increase their creativity by using various methods. The development of proof depends on individuals' acquiring different ways of logical thinking (Guven et al., 2005).

Background

The fact that human mind attends to varying degrees of achievement depending on the relationship between the visual-pictorial and verbal-logical components signals the existence of different mathematical thinking styles in mathematical activities (Krutetskii, 1976). It is seen that students with strong analytical thinking skills perform better at problem-solving compared to those with the dominant type of visual thinking style (Lean & Clements, 1981).

Table 1. Number of participants

Grade	Number of participants
1	24
2	22
3	20
4	19

It is known that students resort to visual solution strategies to decode the data in problems that they have not experienced before and find complex while they tend to use analytical strategies in dealing with easier problems (Lowrie & Kay, 2001). It is added that students make more effort and have difficulty in managing visual processes than analytical processes (Presmeg, 2006). In short, the literature reveals that different thinking structures decide on the approaches to problem-solving (Delice & Sevimli, 2011; Krutetskii, 1976; Lean & Clements, 1981; Lowrie & Kay, 2001; Presmeg, 2006; Tasova, 2011; Turan, 2011).

Problem-solving refers to elimination of the problem by utilizing the processes of reasoning and proof with the aid of certain operations. There is a relationship between problem-solving skills and the concept of providing proof (Ozdemir & Ozdemir, 2018). According to Waring (2000), the development of students' proof conception is handled at six levels. These levels are proof level 0 (students at this level are unaware of proof), proof level 1 (students with this level are aware of proof, but they regard proof as verification by checking a few cases), proof level 2 (students at this level regard proof as checking with different or randomly selected examples. Also, they take a general example as verification). Proof level 3 (students at this level cannot construct a proof alone, but they can understand the solution of a proof with a certain difficulty level), proof level 4 (students at this level are aware of the need for proofs and they can understand the formation of proofs and use a limited number of familiar proofs). Proof level 5 (students at this level can construct some formal proofs and the proofs they have just seen). In this model, students usually fall into lower-level proof categories (McCrone & Martin, 2004). On the contrary, teachers and pre-service teachers have a limited understanding of what writing proof is, and they perceive experimental assumptions as valid proofs while having difficulties in making proof (Varghese, 2008).

Objective

Previous research suggests that teacher candidates do quite bad at writing proofs. Proving is a process comprised of many skills. Prospective teachers are supposed to learn and use proofs in most undergraduate courses. When it comes to schools, the ability to prove underpins the teaching curricula. Still, identifying the factors that determine the prospective teachers' levels of writing proofs may justify the activities that can be undertaken to elevate these levels. In this direction, thinking, which is a phase of the proof-writing process, can be associated with proof writing. In this study, it is aimed to reveal what kind of relationship exists between prospective teachers' thinking structures and their level of writing proofs. For this purpose, answer was sought to the following research questions.

1. What are the prospective mathematics teachers' thinking structures like?
2. What are the prospective mathematics teachers' levels of writing proofs?
3. What relationship exists between the prospective mathematics teachers' thinking structures and their levels of writing proofs?

METHOD

This study was carried out as a case study. This research method aims to collect in-depth information about the research topic and to understand the construct from all aspects. Special case studies are particularly suitable for individual studies. In those studies, selected case is reviewed in depth and comprehensively. The data are collected in a systematic way and attempt is made to discover the relationship between the variables (Cepni, 2006).

Study Group

The study was carried out with 82 prospective teachers enrolled in the Department of Secondary School Mathematics Teaching of a state university located in the Eastern Black Sea Region of Turkey. The breakdown of the participants is given by their grade year in **Table 1**.

Data Collection Tools

Mathematical processing instrument

In this study, two data collection tools were used. The first one is mathematical processing instrument (PI) developed by Presmeg (1986). It is made up of three parts intended to measure the thinking structures determined within the framework of Krutetskii (1976) mentioned earlier. It was designed to find out teachers and students' preferences of visual and non-visual methods in solving non-routine mathematics problems. The tool consists of three parts as A, B, and C. Part-A is solely targeted at students and part-C for teachers, but the other part, that is part-B, is designed for both students and teachers. There are 12 problems in part-B and there are six problems in each of part-A and part-C. Since this study was conducted with prospective mathematics teachers, part-B and part-C were used excluding the first part. MPI provides a list of solutions with three to six probable solutions to each of the 18 problems in parts B and C. An answer key follows on which respondents mark the option, which corresponds to their own solution in the list. One of the items in part-B and part-C are shown in **Figure 1** as an example.

B-10 (4 solutions) If you place a cheese on a pan of a scale and three quarters of a cheese and a three quarter kilogram weight on the other, the pans balance. How much does a cheese weigh?

C-5 (6 solutions) Two candles have diffrent lengths and thicknesses. The long one can burn three and a half hours, the short one five hours. After burning for two hours, the candles are equal in lenght. What was the ratio of the short candle's height originally?

Figure 1. Sample questions for parts B & C (Presmeg, 1986)

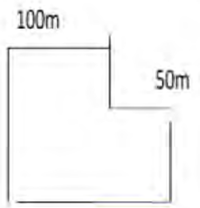
2. In the circle, an exterior angle is equal to half the absolute value of the differences of the arcs it sees.

Figure 2. A proof situation in proof-writing test (Presmeg, 1986)

B2- (Solution 1): Calculate the distance between begining and ending by imaning the path taken by ballons

I found than the distance would be 150m

B2- (Solution 2): I drew a diagram representing the path taken by the balloon and I found the distance between the start and end points



Distance= 100m+50m=150 m

B2- (Solution 3): In order to solve this questions, informaton that is important for the solution. I paid attention (without imagining the path the balloon took). Thus, the distance between the start and destination points is 150m.

E: Visual method

H: Non-visual method

	1	2	3	4	5	6
B1	E	E	H			
B2	E	E	H			
B3	H	H	E	E		
B4	E	E	H			
B5	E	E	H	H		
B6	H	E	E	H	H	
B7	E	E	H			
B8	H	E	E	H		
B9	E	E	H			
B10	E	E	H	H		
B11	H	E	E			
B12	H	E	E	H		

Figure 3. Solution & scoring criteria in answer key to question B-2 (Presmeg, 1986)

Geometry proof-writing test

This tool was developed by the researcher and consists of five proofs that are also covered in the curriculum about the geometric shapes circle and circular region. During the development of the data collection tools, two proficient academicians were consulted for expert opinion within the scope of validity studies. One of the experts is a mathematics teacher and the other is a lecturer specializing in the field of geometry. Thanks to the experts' feedback, the spelling and mathematical errors in the form were corrected. A proof situation in the proof-writing test is shown in **Figure 2**.

Analysis of Data

Analysis of mathematical processing instrument

The version of MPI used in this study is comprised of two parts including a total of 18 questions. In addition, the instrument features in a list of solutions with three to six possible solutions to each of the 18 questions. There is also an answer key so that respondents can mark the option showing their own solution. Each of the problems can be solved with visual or non-visual methods. For respondents who apply an authentic solution option, the list provides a space for an open-ended answer. In the analysis of the MPI, each visual solution is scored two points and each non-visual solution is scored as zero, regardless of whether the answer is correct or wrong. Problems solved with a somewhat ambiguous method gets 1 point. The lowest overall score in the MPI is 0 and the highest score is 36. **Figure 3** shows the solution and scoring criteria in the answer key to question B-2 as an example.

Figure 4 demonstrates the respondent's answer to B-2 and the score of two points given to the solution as it was resolved by using the visual method.

B2- A balloon rises 200m from its location east. It descends 100m after that. It then moves 50m further east and finally lansd straight on the ground. How far is this balloon from the starting point?

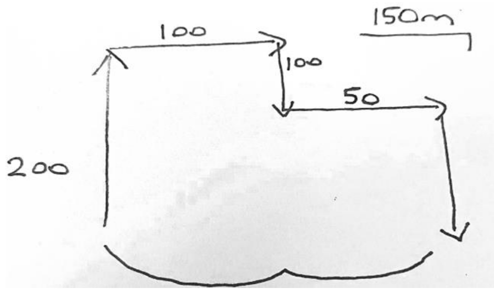


Figure 4. Respondent's answer to B-2 & score of two points given to solution (Presmeg, 1986)

4. The product of the lengths of the diagonals of a cyclic quadrilateral is equal o the product of the lengths of its opposite sides.

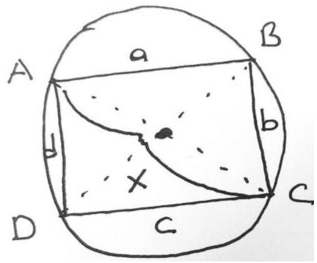


Figure 5. Participant answer gets one point (Senk, 1983)

As a result of the analysis of the data obtained from MPI, the students' thinking preferences were grouped in three categories within the rationale of Krutetskii. In other words, they were classified as analytical, geometrical, and harmonic thinkers based on their scores. Even though it is difficult to draw sharp borders between the students' analytical, harmonic, or geometrical thinking styles, different methods are available for assessing the scores obtained from the MPI. In the current study, the method of Tasova (2011) was used. Tasova (2011) calculated the difference (distribution range) between the largest and smallest value in the MCI data and divided the distribution range into three to identify the class range in a way representing the three distinct thinking structures. Then, he used the class range to determine the reference values for each thinking structure, that is the lowest and highest scores for each category.

In this study, the students' scores ranged between six and 33 points. The distribution range was 33-6=27 and thus the class range was found as 27/3=9. The upper limit of the analytical thinking structure, which is the bottom group, is calculated by summing the class range value and the smallest value obtained from the instrument. The lower limit is the smallest score obtained from the MSA. By following this sequence of transactions, the lower and upper limits were calculated for the other thinking structures. As a result, those with scores of six to 15 were rated as analytical thinkers, 16 to 25 were rated as harmonic thinkers, and those obtaining 26 to 33 scores were rated as geometrical thinkers.

Analysis of geometry proof-writing test

The assessment of geometry proof-writing test was done according to Senk's (1983) analysis method. In her study, Senk (1983) scored questions of proving, as follows: one point was given for the correct drawing, two points for the placement of the given data, and one point for writing the proof. In the current study, the respondents obtained one point for drawing only the shape of the proof situation, two points for placing some of the given data appropriately, three points for placing all the given data appropriately, and four points for placing all the given data and writing the proof completely. The top score for each question was four points. It means that a student who can write all the proofs correctly can get a maximum of 20 points from this test. Apart from this, application of Senk's (1983) criteria for determining the level of writing proofs in this study suggests that a respondent who gets a total score of five or below from the test is considered unsuccessful in writing proofs. On the contrary, a respondent who gets a proof score of 20 or above is considered very successful in writing proofs. The rating scale continues as following: Scores between five and 10 points refer to low success in writing proofs and scores from 10 to 15 points refer to intermediate success in writing proofs. Overall, students with a proof score of up to 10 points are considered low achievers while those obtaining higher scores are considered high achievers.

Answers for sample scoring are displayed in **Figure 5** and **Figure 6**. In the sample in **Figure 5**, one point was given to the respondent as nothing except for the illustration of the given situation was drawn. In the sample in **Figure 6**, three points were given as the situation was illustrated along with full placement of the given data, but a complete explanation of the proof situation was missing.

1. In a circle the center is equidistant from the congruent chords.

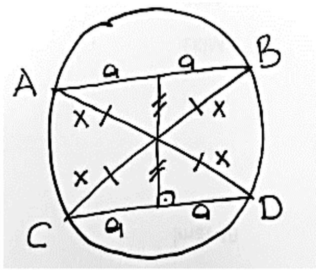


Figure 6. Participant answer gets three point (Senk, 1983)

RESULTS

In this section, the study findings are presented under separate headings as "identifying prospective teachers' thinking structures", "identifying prospective teachers' proof-writing levels", "identifying relationship between prospective teachers' thinking structures and their proof-writing levels".

Identifying Prospective Teachers' Thinking Structures

In this study, it was found out that 29 of the 82 teacher candidates have dominant analytical thinking skills, 44 have harmonic thinking, and the other nine have dominant geometrical thinking skills.

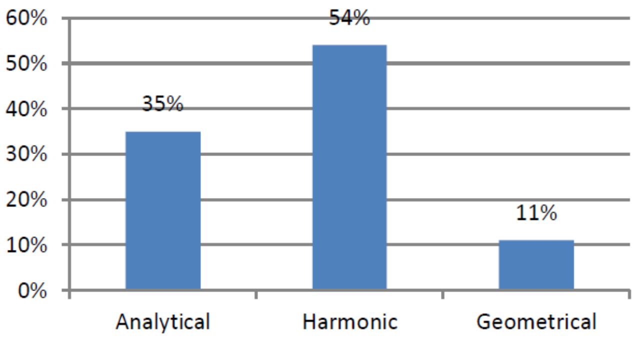


Figure 7. Participants' thinking skills (Source: Author's own elaboration)

As seen in **Figure 7**, 54% of the participants are characterized with harmonic thinking, 35% with analytical thinking, and 11% geometrical thinking structure. **Table 2** details the frequency and corresponding percentage of each MPI score obtained by the prospective teachers symbolizing their thinking styles.

Table 2. Details frequency & corresponding percentage of each MPI score

Thinking structure	MPI score	Frequency	Percentage (%)
Analytical	6	7	9
	8	5	6
	10	6	7
	12	6	7
	13	3	4
	14	2	2
Harmonic	16	10	12
	18	14	17
	20	11	14
	21	6	7
	22	3	4
Geometrical	27	2	2
	28	3	4
	30	3	4
	33	1	1

As **Table 2** shows, seven students got six points, five got eight points, six students got 10 and 12 points, respectively, three students got 13 points, and two got 14 points. In the middle band, 16 points were obtained by 10 students, 18 points by 14 students, 20 by 11 students, 21 points by six students, and 22 points were obtained by three students. In the last and the highest band, two students got 27 points, three got 28 and 30 points, respectively, and one student got 33 points from the test.

Table 3. Participants' overall scores in proof-writing test

Level	Proof-writing score	Frequency	Percentage (%)
Unsuccessful	3	3	4
	4	4	5
Low success	6	7	9
	7	9	11
	8	11	13
	9	12	14
Intermediate success	11	7	9
	12	11	13
	13	9	11
	14	2	2
Very successful	16	3	4
	18	3	4
	19	1	1
Total	140	82	100

Table 4. Frequencies representing each proof-writing level by thinking structures

Thinking structure	Number of unsuccessful students	Number of students with low success	Number of students with intermediate success	Number of very successful students
Analytical	7 (9%)	22 (27%)		
Harmonic	-	17 (21%)	27 (33%)	
Geometrical	-	-	2 (2%)	7 (9%)
Total	7 (9%)	39 (48%)	29 (35%)	7 (9%)

Table 5. Pearson's correlation coefficient between two sets of data

		Thinking structure	Proof-writing level
Thinking structure	Pearson's correlation	1	.912
	p-value		.000
Proof-writing level	Pearson's correlation	.912	1
	p-value	.000	

Identifying Prospective Teachers' Proof-Writing Levels

The participants' overall scores in proof-writing test are displayed in **Table 3**.

It was found out that three students obtained three points and four obtained four points from the test, both of which remain below the success threshold. In other words, seven prospective teachers were found to be low achievers scoring five or less. In the next band, seven students got six points, nine got seven points, 11 got eight points, and 12 students got nine points. A total of 39 students were found to have low levels of success (scores from five to 10). Next up, 11 points were recorded by seven students, 12 points by 11 students, 13 points by nine students, and 14 points were obtained by two students, constituting the intermediate level of success. There were 29 students regarded moderately successful in writing proofs (scores from 10 to 15). In last group, that is high achievers in proof-writing, there were seven students all of whom got 16 points or above.

Identifying Relationship Between Prospective Teachers' Thinking Structures and Their Proof-Writing Levels

The frequencies representing each proof-writing level by thinking structures of the prospective teachers are given in **Table 4**.

Table 4 shows that seven of the prospective teachers with analytical thinking structure were unsuccessful and 22 had low success in writing proofs. Of the prospective teachers with harmonic thinking structure, 17 had low success and 27 had intermediate success in the same task. Lastly, two prospective teachers with geometrical thinking structure could write proofs at intermediate level and seven could write at a high level.

In order to explore the relationship between the prospective teachers' thinking styles and their proof-writing levels, Pearson's correlation coefficient between the two sets of data was analyzed by using SPSS. **Table 5** displays the results of this analysis.

According to **Table 5**, the coefficient of significance between the thinking structures and proof-writing levels of the participants was found to be 0.000. These values were significant at $p<0.01$. It is also noticeable that Pearson's correlation coefficient was 0.912. Since this value is statistically positive, a positive relationship appeared between the thinking structures and proof-writing levels of the students. Also, this value implies a statistically higher level of positive correlation as it is between 0.71 and 0.99 (Büyüköztürk, 2006).

DISCUSSION AND CONCLUSIONS

This section is devoted to interpreting the findings regarding the prospective teachers' thinking structures and their proof-writing levels and the relationship between these two variables.

When the thinking structures of the participants are examined, it can be understood that harmonic thinking is the most common thinking type among the study participants. This type is followed by analytical and geometrical thinking types, respectively. As known, harmonic thinkers are predisposed to both verbal and visual elements equally. They are ready to apply both methods together in problem-solving. Geometrical thinkers are prone to visual elements. Analytical thinkers apt to choose verbal items, while they are unlikely to use visual items. It can be inferred that the prospective teachers in this study tend to co-use visual and verbal elements in problem-solving. Other studies revealed that the most common thinking structure among teacher candidates is analytical thinking and the least common one is geometrical thinking (Delice & Sevimli, 2012; Galindo-Moralez, 1994). The ranking of the geometrical thinking structure for this target group can be explained with the heavier emphasis on algebra teaching in the undergraduate curricula. Besides, while separate curricula were taught in mathematics and geometry courses until the year 2015, the two courses were mingled in one single curriculum then. This amendment in 2015 might have affected the distance to geometry. Another explanation for the recessive geometrical thinking structure could be the teaching method used for preparing the students for the university entrance examination. In this teaching method, students have to reach the result in the shortest time possible if they want to succeed in the exam; thus, visual elements are ignored as a part of the teaching method.

When the proof-writing levels in this study are looked at, it becomes obvious that most of the prospective teachers are weak in terms of this skill. The second largest group is comprised of those with an intermediate success level. Only a small number of participants exhibit a high level of proof-writing. Similarly, Varghese (2008) remarkably found that teachers and prospective teachers faced difficulty in constructing proofs, as well as having a limited understanding of what writing a proof is and they perceived experimental assumptions as valid proofs. The related literature also reveals that students' proving levels are generally low (Dickerson, 2008; McCrone & Martin, 2004; Sevgi & Orman, 2020). This seems to be an expected result as the classroom practices of the teachers directly affect the students.

In this study, the participants' proof-writing levels and thinking structures were found to be correlated. According to the results, lower proof-writing levels relate to analytical thinking structure, intermediate level of proof-writing indicates harmonic thinking structure. Lastly, prospective teachers with the highest level of proof writing embody geometrical thinking skills. However, Lean and Clements (1981) concluded that the students with analytical thinking type performed better at problem-solving compared to their peers with the dominant visual thinking type. In contrast, Lowrie and Kay (2001) pointed out that the students implemented visual problem-solving strategies to conceive the data in problems, which were new and complex to them, whereas they tended to use analytical strategies in addressing easier problems. Presmeg (2006) stated that the students exerted more efforts and had difficulty in managing visual processes than analytical processes. These results emphasize the need to attach more importance to works aimed at improving visuality and visual skills. As an example, algebraic/symbol-using skills and their integration with visual skills can be practiced.

Funding: No funding source is reported for this study.

Ethical statement: Author stated that the study was approved by Trabzon University with Ethical Approval E-81614018-000-2200045656/8.11.2022).

Declaration of interest: No conflict of interest is declared by the author.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the author.

REFERENCES

Alkan, H., & Bukova-Guzel, E. (2005). Öğretmen adaylarında matematiksel düşünmenin gelişimi [Development of mathematical thinking in teacher candidates]. *Gazi Üniversitesi Gazi Eğitim Fakültesi Dergisi* [*Journal of Gazi University Gazi Education Faculty*], *25*(3), 221-236.

Bell, A. W. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics, 7*(1/2), 23-40. https://doi.org/10.1007/BF00144356

Büyüköztürk, Ş. (2006). *Sosyal bilimler için veri analizi el kitabı* [Manual of data analysis for social sciences](6. Edn.). Pegem.

Cepni, S. (2005). *Araştırma ve proje çalışmalarına giriş* [*Introduction to research and project work*]. Üçyol Kültür Merkezi [Üçyol Cultural Center].

Delice, A., & Sevimli, E. (2011). İntegral kavramının öğretiminde konu sıralamasının kavram imgeleri bağlamında incelenmesi; belirli ve belirsiz integraller [Examining the subject order in the context of concept images in the teaching of the concept of integral; definite and indefinite integrals]. *Pamukkale Üniversitesi Eğitim Fakültesi Dergisi* [*Pamukkale University Faculty of Education Journal*], *30*(30), 51-62.

Delice, A., & Sevimli, E. (2012). Analiz dersi öğrencilerinin integral hacim hesabı problemlerindeki çözüm süreçlerinin düşünme yapısı farklılıkları bağlamında değerlendirilmesi [Evaluation of analysis course students' solution processes in integral volume calculation problems in the context of thinking structure differences]. *Marmara Üniversitesi Atatürk Eğitim Fakültesi Eğitim Bilimleri Dergisi* [*Marmara University Ataturk Faculty of Education Journal of Educational Sciences*], *36*(36), 95-113.

Dickerson, D. S. (2008). *High school mathematics teachers' understandings of the purposes of mathematical proof* [PhD thesis, Syracuse University].

Dursun, S., 6 Coban, A. (2006). Geometri dersinin lise programları ve ÖSS soruları açısından değerlendirilmesi [Evaluation of geometry course in terms of high school programs and OSS questions]. C.Ü. *Sosyal Bilimler Dergisi* [*C.U. Journal of Social Sciences*], *30*(2), 213-221.

Galindo-Morales, E. (1994). *Visualization in the calculus class: Relationship between cognitive style, gender, and use of technology* [Unpublished PhD thesis]. The Ohio State University.

Guven, B., Celik, D., & Karatas, V. I. (2005). High school students' sufficiencies of doing mathematical proofs. *Journal of Contemporary Education, 30*, 319-332.

Kozhevnikov, M., Hegarty, M., & Mayer, R. E. (2002). Revising the visualizer-verbalizer dimension: Evidence for two types of visualizers. *Cognition and Instruction, 20*(1), 47-77. https://doi.org/10.1207/S1532690XCI2001_3

Krutetskii, V. (1976). *Psychology of mathematical abilities in schoolchildren*. The University of Chicago Press.

Lean, G., & Clements, M. A. (1981). Spatial ability, visual imagery, and mathematical performance. *Educational Studies in Mathematics, 12*(3), 267-299. https://doi.org/10.1007/BF00311060

Lowrie, T., & Kay, R. (2001). Relationship between visual and nonvisual solution methods and difficulty in elementary mathematics. *The Journal of Educational Research, 94*(4), 248-255. https://doi.org/10.1080/00220670109598758

McCrone, S. M. S., & Martin, T. S. (2004). Assessing high school students' understanding of geometric proof. *Canadian Journal of Math, Science & Technology Education, 4*(2), 223-242. https://doi.org/10.1080/14926150409556607

McGillicuddy-DeLisi, A. V., & Sigel, I. E. (1991). Family environments and children's representational thinking. *Advances in Reading/Language Research, 5*, 63-90.

Morali, S., Ugurel, I., Turnuklu, E., & Yesildere, S. (2006). Matematik öğretmen adaylarının ispat yapmaya yönelik görüşleri [Opinions of pre-service mathematics teachers on proving]. *Kastamonu Eğitim Dergisi* [*Kastamonu Journal of Education*]*, 14*(1), 147-160.

Ozdemir, F., & Ozdemir, H. (2018). Investigation of high school students' opinions on constructing proof and skill perceptions on problem-solving. *Journal of Academic Social Sciences, 6*(71), 405-419.

Presmeg, N. C. (1986). Visualisation in high school mathematics. *For the Learning of Mathematics, 6*(3), 42-46.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In Á. Gutiérrez, & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 205-235). Brill. https://doi.org/10.1163/9789087901127_009

Senk, S. L. (1983). *Proof-writing achievement and Van Hiele levels among secondary school geometry students* [PhD thesis, The University of Chicago].

Sevgi, S., & Orman, F. (2022). Eighth grade students' views about giving proof and their proof abilities in the geometry and measurement. *International Journal of Mathematical Education in Science and Technology, 53*(2), 467-490. https://doi.org/10.1080/0020739X.2020.1782493

Tasova, H. I. (2011). *Matematik öğretmen adaylarının modelleme etkinlikleri ve performansı sürecinde düşünme ve görselleme becerilerinin incelenmesi* [*Examining the thinking and visualization skills of pre-service mathematics teachers during modeling activities and performance*] [Doctoral dissertation, Marmara University].

Tekin, M., Ozmutlu, I., & Erhan, S. E. (2009). Özel yetenek sınavlarına katılan öğrencilerin karar verme ve düşünme stillerinin incelenmesi [Examining the decision-making and thinking styles of students who participated in special talent exams]. *Beden Eğitimi ve Spor Bilimleri Dergisi* [*Journal of Physical Education and Sport Sciences*], *11*(3), 42-56.

Turan, A. O. (2011). *12. sınıf öğrencilerinin analitik geometrideki temsil geçişlerinin Krutetskii düşünme yapıları bağlamında Incelenmesi* [*Investigation of 12th grade students' representation transitions in analytic geometry in the context of Krutetskii thinking structures*] [Doctoral dissertation, Marmara University].

Ubuz, B. (1999). 10. ve 11. sınıf öğrencilerinin temel geometri konularındaki hataları ve kavram yanılgıları (10th and 11thgrade students errors and misconceptions on basic geometry) [Errors and misconceptions of 10th and 11th grade students on basic geometry subjects (10th and 11thgrade students errors and misconceptions on basic geometry)]. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi* [*Journal of Hacettepe University Faculty of Education*]*, 16-17*, 95-104.

Umay, A. (2003). Matematiksel muhakeme yeteneği [Mathematical reasoning ability]. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi* [*Journal of Hacettepe University Faculty of Education*]*, 24*(24), 234-243.

Varghese, T. (2009). Secondary-level student teachers' conceptions of mathematical proof. *IUMPST: The Journal, 1*.

Waring, S. (2000). *Can you prove it? Developing concepts of proof in primary and secondary schools*. The Mathematical Association.