International Electronic Journal of Mathematics Education
2023, 18(4), em0747
e-ISSN: 1306-3030
https://www.iejme.com

MODESTUM

Research Article **OPEN ACCESS**

Changes in students' mental constructions of function transformations through the APOS framework

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Citation: Yiğit Koyunkaya, M., & Boz-Yaman, B. (2023). Changes in students' mental constructions of function transformations through the APOS framework. *International Electronic Journal of Mathematics Education, 18*(4), em0747. https://doi.org/10.29333/iejme/13515

ARTICLE INFO

Received: 11 Mar. 2023
Accepted: 11 Jul. 2023

ABSTRACT

This study aims to examine the changes in students' mental constructions of function transformation based on the designed activities, classroom discussions, exercises (ACE) teaching cycle in the light of the action, process, object, schema (APOS) theoretical framework. The study was conducted by seven volunteer students who were enrolled in mathematics teacher education program. The data consisted of video records of the designed teaching sessions, students' worksheets and the researcher's field notes. The results indicated that ACE teaching cycle was effective in developing students' mental constructions of function transformation through sequential activities. Even though all the students initially did not reveal evidence for sufficient theoretical or practical knowledge on function transformation, which was evidence of their action level, through the process, it became evident that all the students provided evidence regarding at least their process level of function transformations. Some of the students also showed evidence related to their object levels of function transformations.

Keywords: APOS theory, ACE teaching cycle, function transformations

INTRODUCTION

Learning function transformation is part of learning the concept of function by providing new opportunities for students to use, reflect and discover knowledge about the subject (Lage & Gaisman, 2006). The topic is taught at different levels in secondary education in different countries (Bingham, 2007; Zazkis et al., 2003). For instance, in Türkiye, students are taught about function transformations in the 11th grade in which only transformations form of $f(x)+k, kf(x),$ and $f(x+k)$ for specific real number values of *k* forms for linear and quadratic functions are introduced (Ministry of National Education [MoNE], 2013). Similar to the Turkish national mathematics curriculum, curricula (e.g., Australia, England, etc.) and standards (e.g., National Council of Teachers of Mathematics [NCTM], 2000) in most other countries illuminate the importance of function transformation to strengthen learning and teaching of the concept of function. The Australian curriculum, assessment, and reporting authority [ACARA] (2008) states that "describe, interpret and sketch parabolas, hyperbolas, circles, and exponential functions and their transformations" (ACMNA267). NCTM (2000) and Common Core State Standards for Mathematics (2010) put great importance on function and function transformation. In addition to standards and teaching programs, some researchers examined learning and teaching of the concept of function from a broader perspective (Baker et al., 2001; Confrey & Smith, 1991; Cooney et al., 2010; Lage & Gaisman, 2006). For instance, Cooney et al. (2010) illuminated the importance of learning function transformation and propose combining and transforming functions is one of the five big ideas (the function concept, covariation and rate of change, families of functions, combining and transforming functions, and multiple representations of functions). This idea depends on the properties of addition, subtraction, multiplication, division, taking inverse and composition of other functions. When these computations and operations are applied, the graph of the functions can be transformed into another or appear as shrinking or stretching, horizontally or vertically (Baker et al., 2001; Confrey & Smith, 1991; Lage & Gaisman, 2006). Knowing or identifying the transformed form of a graph helps students understand the relations among various representations of functions such as algebraic and graphic forms. Although transforming the algebraic form into a graphic form (or vice versa) is very useful to understand the changes occurring in functions, it might be difficult for students to understand the changes in different forms and perform on it. Therefore, teaching function transformation is mainly related to breaking down the relations between graphs and formulas of function. In this context, it is important to state that students should be able to make a connection between the former formula of the given function and transformed one, additionally the graphical representations of parent functions. This topic is also one of the difficult topics for students to learn in high school (Faulkenberry, 2011), so teaching of the subject needs a different perspective to improve students' learning of the concept.

There are many researchers focused on learning or teaching of functions and function transformation considering APOS (Action-Process-Object-Schema) theoretical framework, which was adopted as theoretical background of this study (Breidenbach et al., 1992; Kabael, 2011; Maharaj, 2010; Martínez-Planell & Cruz Delgado, 2016; Trigueros & Martínez-Planell, 2010; Weber, 2002). APOS framework was derived from Piaget's work, specifically reflective abstraction, by considering individuals' mathematical knowledge development through the levels of the framework (Arnon et al., 2014). APOS theory is a well-known theory in the mathematics education community and is used to evaluate or improve students' learning and understanding of various mathematical concepts (Fuentealba et al., 2017; López et al., 2016; Man & Poon, 2014). Based on the theory, learning is related to the construct of knowledge and this construction could have occurred through an evolving mechanism between the levels and individuals' mental constructions. There are studies that focused on the students' learning of the concept of function and related concepts from APOS theory perspective. For example, Carlson (1998), Nyikahadzoyi (2006), and Reed (2007) identified that even successful university students remained at the action level regarding the concept of function. Additionally, Dubinsky and Wilson (2013) examined that high school students had no knowledge or at action level regarding function concept; after their designed study, most of them were able to evaluate functions using various representations, identifying functions according to given representations, and determine whether given equations were functions based on the function definition. Although students had an action level of understanding of APOS theory, Dubinsky and Wilson (2013) showed students could improve their knowledge and APOS levels on functions through a short term of instruction and were able to reach the process level at least. Asiala et al. (1997) claimed that carefully designed computer activities were reasonably effective in assisting students to "develop a relatively strong process conception of function and a graphical understanding of derivative" (p. 428). These researchers described a teaching cycle, which is called ACE (activities, classroom discussions, exercises), based on APOS theory to improve students' learning of a mathematical concept. Similarly, Siyepu (2013) identified that most of the students provided evidence regarding their action level, and he recommended that an ACE teaching cycle should be implemented in order to assist students to develop the required schema level. He stated that, in this process, students should be encouraged to "self-reflect by trying to identify their errors on their own during class discussions" (p. 590). Considering the existing research and importance of learning function transformation, the main goal of this study is to design a teaching cycle for function transformation using the APOS theory. In the light of this aim, APOS theory and ACE teaching cycle lead the researchers to design and apply the teaching procedure to answer the following research question:

RQ. Considering the teaching design of function transformations, how do the students' mental constructions of function transformations change in terms of APOS theory?

Theoretical Framework

Based on APOS theory, learning starts with the action level on a mathematical concept; and the first level is defined as any externally directed transformation of an existing physical or mental mathematical object to obtain other objects (Asiala et al., 1996). When an action is repeated, reflected upon, and/or combined with other actions, an action might be interiorized into a process (Arnon et al., 2014). The process level is under individuals' control, and it occurs without external stimulus, and individuals are able to describe, imagine or reverse the steps of a transformation without performing the steps explicitly (Asiala et al., 1996). Once individuals' practice with processes and accept them as a whole, they are able to realize that a process can be manipulated or modified. Then, they can encapsulate the process to form an object, which is the evidence of entering the object level. When a process can be encapsulated into a mental object, it can be de-encapsulated, as well (Arnon et al., 2014). In practice, this means that de-encapsulation allows students to go back to the process that gave rise to the object. If students reach this level of object conception, they have the ability to synthesize many related actions, processes, and mental objects to form a schema. A schema is a collection and a relation of the constructed actions, processes, objects, and other schemas that can be organized in a structured form (Arnon et al., 2014; Asiala et al., 1996). Even though it might be assumed that there is a linear progression from action through object levels for organization in schema, it might not be correct in every case. For instance, when individuals meet with unfamiliar situations, in order to make sense of the situations, the existing structures need to be reconstructed and the reconstruction might involve a nonlinear progression through ACE (Arnon et al., 2014).

In order to design and implement the instruction of a particular mathematical concept using APOS theory and the hypothesis on learning and teaching of the concept, a teaching cycle called ACE model was developed (Asiala et al., 1996) (**Figure 1**). The cycle consisted of three main components: The first one, activities, as the first step of the cycle are designed to develop students' mental constructions in the genetic decomposition by working cooperatively on the tasks. In this theory, a genetic decomposition describes and models how a schema (of a mathematical concept) could be developed and how the mental constructions of actions, processes, and objects could be used in the construction of this schema. Hence, following the designed genetic decomposition, teachers plan activities aiming to reach an APOS level regarding the related mathematical concepts; in the second one, students perform mathematical tasks by discussing the answers, listening to the explanations of the peers or the teacher regarding the mathematical concepts; In the last phase, the students strengthen their knowledge of the concept by completing this process and apply their constructed knowledge to solve standard problems related to the concept (Arnon et al., 2014). This process helps students build mental construction of concepts being studied and learn mathematics (Weller et al., 2003).

In order to design an ACE teaching cycle on a particular mathematical concept, researchers initially must design a genetic decomposition to foster students' mental constructions about the related concept. In this study, after identifying students' mental constructions regarding function transformation from APOS theory perspective (Boz-Yaman & Yiğit Koyunkaya, 2019), a genetic decomposition as well as an ACE teaching cycle for teaching function transformation was designed following the results of the previous study. Initially, the researchers constructed a genetic decomposition described in the method section and the teaching cycle was designed following the genetic decomposition.

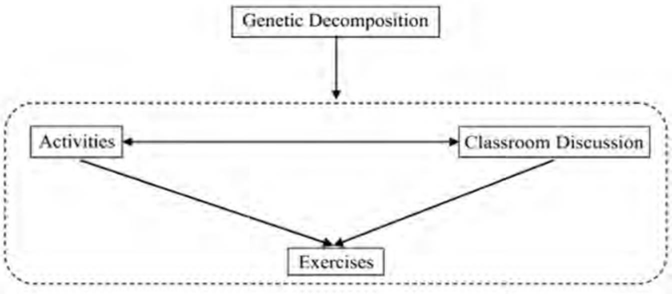

Figure 1. An ACE teaching cycle (adapted from Arnon et al., 2014, p. 58)

METHODS

This qualitative research was designed considering APOS theory framework for exploring participants' mental constructions and ACE teaching cycle for teaching the function transformations (Arnon et al., 2014; Asiala et al., 1996).

Participants

The participants of the study were seven (five female and two male) volunteer students. They were enrolled in a four-year secondary mathematics teacher education program at a state university in Türkiye, and they were at the end of their third year while participating in this research. There were nearly 15 students in the class, and they were asked to participate in the study. The seven students were selected based on their willingness to participate in the study. They took some mathematics courses (i.e., calculus 1 and calculus 2, abstract mathematics 1 and abstract mathematics 2, linear algebra 1, and linear algebra 2) as well as mathematics education courses (i.e., methods for teaching mathematics, technology-supported mathematics education, and curriculums in mathematics education) in their program. They had different academic achievements and their grades taken from mathematics courses, in which they were educated about the functions and related concepts were various. In order to ensure students' confidentiality, we assigned numbers for them from S1 (Student 1) to S7 (Student 7) through the paper.

Procedure

In our research, we designed three phases for constructing a process for teaching cycle regarding function transformation: **phase 1**-designing the instrument (this phase could be assumed as the pilot study of phase 2), **phase 2-**identifying students' existing APOS levels using the designed instrument as well as designing the genetic decomposition to construct phase 3, and **phase 3-**constructing ACE teaching cycle that was designed using the results of phase 1 and phase 2 and examining the effects of the teaching cycle on development of students' mental constructions regarding the function transformation. This paper is a part of a wider research, and we focused on the process and results obtained in the last phase (phase 3). The results of the first and second phases were presented in our published paper (Boz-Yaman & Yiğit Koyunkaya, 2019). All three phases are summarized in **Table 1**. The last phase is the main concern of this paper.

Table 1. Summary of the phases

The Phases	Aim of the phase	Implementation process of the phase	Inferences derived from the phase
Phase 1: Designing instrument	Preparing an instrument that evaluated students' mental construction on function transformation based on APOS theory.	Instrument was applied with one high-ability 11th-grade high school student by conducting a semi-structured interview.	Based on interview results, we revised questions since 11th-graders basically used formulas related to function transformations & could not solve most of questions.
Phase 2: Identifying pre-service mathematics teachers' (PMTs) existing APOS levels using designed instrument (see Boz-Yaman & Yiğit Koyunkaya, 2019)	Examining revised instrument to identify PMTs' APOS levels on function transformation & construct a genetic decomposition regarding the concept.	Revision was completed by conducting a semi-structured interview with a pre-service mathematics teacher for distinguishing difference between high school & university student's perspectives. Final version of instrument was conducted with three PMTs to identify APOS levels on function transformation concept.	Following the results of these two phases, we designed the genetic decomposition that we presented in this paper and the content of ACE teaching cycle.
Phase 3: Examining ACE teaching cycle	Designing the ACE teaching cycle for function transformation. Implementing the designed teaching cycle to the students and observing development of their mental construction regarding function transformation.	ACE teaching cycle was conducted in three months. Cycle involves activities (designed by researchers), classroom discussions (conducted by one of researchers), & exercises (homework were assigned & examined by the researchers).	The results were examined and presented in detail in this paper.

Considering the results of the first and second phases, the designed intervention study was conducted iteratively as the last phase of the research. The designed teaching was completely different from teaching function transformation in ordinary mathematics classrooms in Türkiye. Generally, the teaching of function transformations is limited to giving formulas of the transformation and using the intersection or peak points on the graph. An example of function transformation from 11th grade mathematics textbook is given in **Figure 2**.

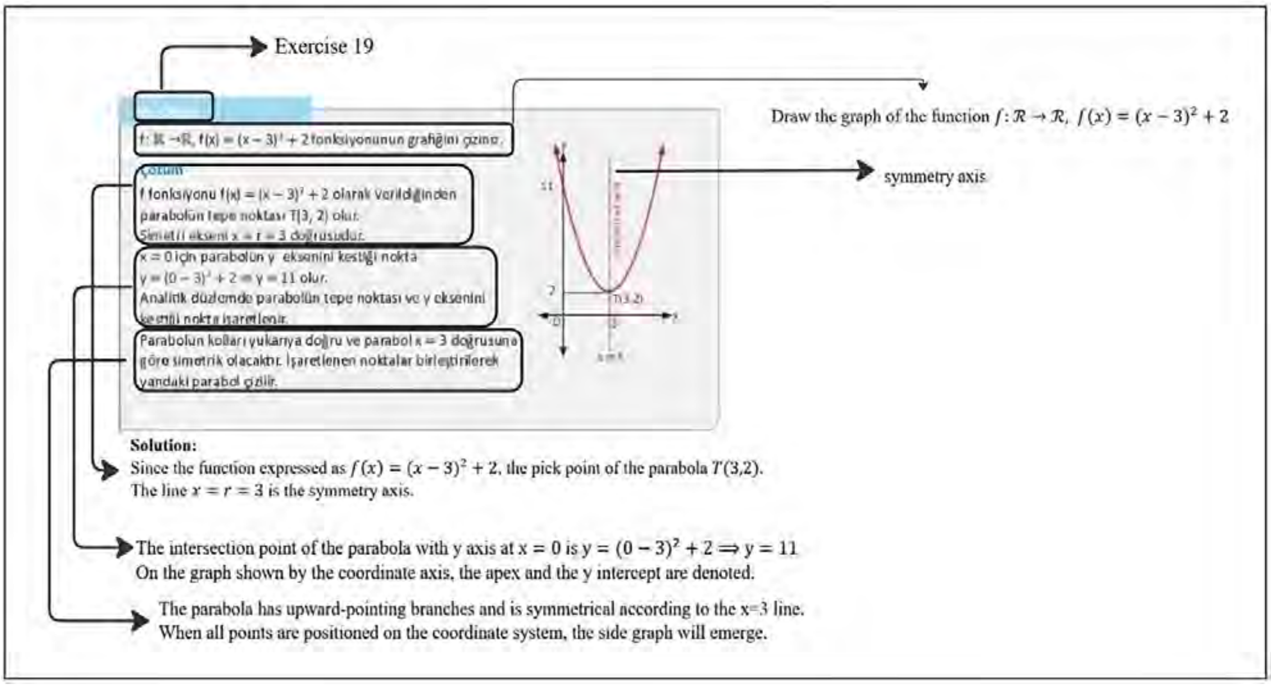

Figure 2. A screenshot from a textbook that is used in mathematics classrooms in Türkiye (Source: https://l24.im/H6je9v2)

In these process-oriented procedures, measurement and evaluations were conducted continuously. Interventions represent specific theoretical claims about teaching and learning and reflect an obligation to understanding the relationships among theory, designed artifacts, and practice. At the same time, research on specific interventions could contribute to theories of learning and teaching. In this sense, our research is constructed on APOS theory and ACE learning cycle on learning function transformations. Particularly, our design consisted of learning the linear, quadratic, root, absolute-value, logarithmic, and exponential functions transformation. In this paper, we focused on students' mental constructions regarding linear, quadratic, and root function transformations. Whole research process regarding design of teaching sessions for function transformation was given in **Figure 3**.

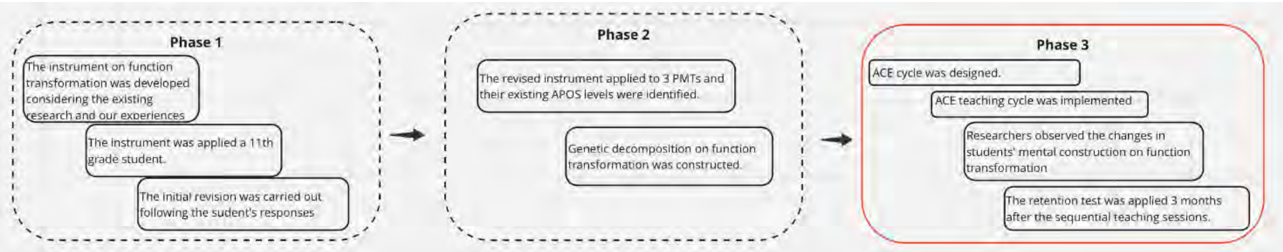

Figure 3. Stages of research process (Source: Authors' own elaboration)

Instrument (A Genetic Decomposition for Function Transformations)

A genetic decomposition is a model that is constructed to describe an individual's schema through the coordination and interactions among levels of APOS theory (Asiala et al., 1996). A genetic decomposition is constructed to investigate how an individual's mental constructions emerge and it is a unique model for any given concept and participants being studied (Asiala et al., 1997). It is also an important methodological component of APOS theory. In this study, the genetic decomposition was constructed considering the results of the phase 1 and phase 2 as mentioned in the previous section. In the constructed genetic decomposition, the first step is the actions taken on the Cartesian system. It is thought that it starts with the idea of moving the functions on the system. By using a given graphic of a function on the system, a learner is expected to move the graphic horizontally or vertically as well as stretching or shrinking it. In other words, the action step involves recognizing the movement on the graphic (for one specific coefficient individually–this situation represented as '*a*' value in the given **Figure 4**), realizing the meaning of the transformation of a function, and acting on the function, such as applying a rule to an algebraic formula or transforming it step-by-step using variables.

Considering the constructed genetic decomposition, a learner who has reached the process level, acts on the function internally rather than explicitly. S/he could think of a function transformation by manipulating them in some way based on his/her experiences or imagination regarding a concept. In this step, they could easily produce the same transformed quantity based on the original quantity repeated/reflected without making explicit calculations (for all coefficients at the same time-this situation represented as "all *u*" values in the given **Figure 4**).

As evidence, a learner could easily make a comment of movement of the graph on the coordinate system using her/his previous knowledge. S/he could know whether a graph of a function moves to the right or left direction as well as stretching and shrinking. Using knowledge constructed at the process level, a learner could encapsulate it to reach the object level. The encapsulation process helps a learner to see the function transformation as a dynamic process and operate on it. They could examine and transform the static entity. Evidence of object level involves recognizing and thinking about function transformation as a complete activity and operating and acting on these objects (functions transformations). They could also use the graphic of function transformation and reason about the movement by explaining how the transformation occurred using the function family of the transformed function. In the schema level, a learner could reveal evidence of constructed action, process and object levels by combining them to construct new knowledge. For instance, they could reason about the transformation of an inverse function using their existing knowledge. The schema level involves a learner's ability to reflect constructed knowledge into a new, but similar, concept. The whole process for quadratic function was summarized in **Figure 4**.

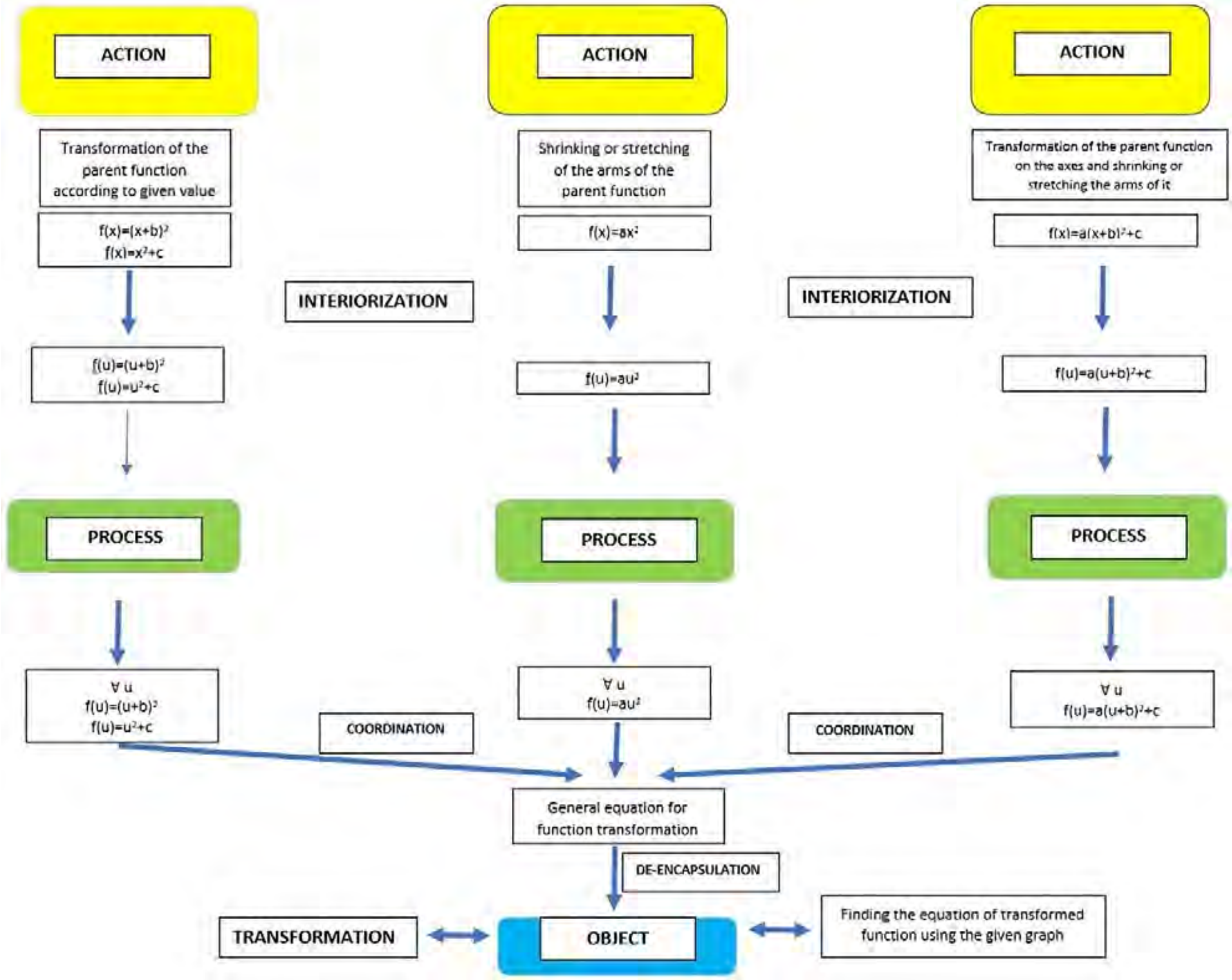

Figure 4. Genetic decomposition for function transformations (Source: Authors' own elaboration)

ACE teaching cycle was designed using the constructed genetic decomposition. The cycle was divided into three parts as it is suggested in the theory. In the activities part, students worked on the designed sequenced activities using paper and pencil and dynamic geometry software, GeoGebra. In the existing research, in this part, researchers particularly used ISETL or similar programs to help students for developing mental constructions regarding a mathematical concept. We chose a GeoGebra software, which includes both algebraic and graphic windows, that could help students both see the graphic of transformed functions as well as the equation of it, simultaneously. Even though we chose a new software (different from the existing studies) to improve students' mental constructions, the research results illuminated that GeoGebra helped the students reach a higher APOS level.

The sequenced activities were designed by considering the framework as well as the mental construction of the participants, who took part in Boz-Yaman & Yiğit Koyunkaya's (2019) study and were divided into four parts. The activities were sequenced by considering function types such as linear, parabola and root functions (for an example of the quadratic function transformation see **Appendix A**).

During the activities, they had a chance to discuss their ideas with the whole class while one of the researchers led the class discussion. In detail, the process included those students solved the given question by paper-pencil or using GeoGebra individually after they participated in the classroom discussion for explaining the following methods and strategies they used for the solutions of each question. In detail, for each activity related to each function transformation, students initially solved the activity on paper and then work on similar activities using GeoGebra. This design was intentionally conducted since ACE teaching cycle recommended the use of technology for solving the activities (Asiala et al., 1997). For each activity, students discussed their responses, and, in this process, we encouraged them to reveal evidence regarding the function transformation for each family by leading the discussion by posing questions regarding their mental constructions in the discussion process. The activities and class discussions gave them an opportunity to share their thoughts and to reflect their work. After conducting activities and classroom discussions, for each function transformation type, we gave homework as exercises as suggested in ACE cycle. In the following weeks, we began the lesson by discussing their responses regarding the homework.

Data Collection and Analysis

The data consisted of video recordings of the classroom discussions, students' worksheets, and the instructor's field notes. Students' worksheets and video recordings gave us evidence for the students' mental construction of a particular function transformation, taking field notes after each teaching session gave us an opportunity for considering students' responses as well as difficulties in the process. All the collected data (video records of teaching sessions, worksheets, and field notes) was used to examine students' mental constructions of function transformation as well as the effects of ACE teaching cycle. The data was collected in three weeks, and two-hour teaching sessions were conducted each week. The whole teaching sessions were recorded using two video cameras. The first camera records the whiteboard for analyzing the students' responses in depth, and the second one is used for observing the whole classroom for examining the class discussions (the students and the instructor). Three months after the completion of the teaching intervention, we applied a retention test to evaluate the effectiveness of the designed teaching cycle. In particular, we asked the participants to solve similar activities to examine their mental constructions regarding each function transformation.

In this study, the video records were analyzed by using Powell et al.'s (2003) video analysis method, that includes seven nonlinear phases: (1) viewing attentively the video data; (2) describing the video data; (3) identifying critical events, (4) transcribing, (5) coding, (6) constructing storyline, and (7) composing narrative. Initially, as the researchers, we watched each video record two times separately and described the events in the video by using time coded descriptions and dividing the whole video into 10 minutes-timeline. In this description, we specifically focused on each observed event that occurred in the classroom. Then, we identified the critical events that included evidence regarding students' mental constructions about the function transformation and we transcribed the identified scripts. The coding was completed in the light of the levels of APOS theory, and we particularly focused on descriptive coding structure. After this point, we constructed a storyline and composed narratives to describe the findings. While completing these seven steps, we used the tables to organize the data and each column of the table represented one step in terms of the video analysis phases. For instance, in the teaching sessions about the quadratic functions (particularly between 20-30 minutes), students were asked to draw a graph of $f(x) = ½(x-1)^2 + 1$. In the discussion, one of the students responded that she initially drew $f(x) = x^2$, then she transformed this graph to 1 unit right on the x-axes, then shrunk the graph by considering coefficient ½. Particularly, she claimed that for the values that x takes, the value of y will decrease compared to the main function in this process. Finally, she indicated that she moved the graph 1 unit up on the *y*-axes. In other words, she transformed the function by considering the parent function. Her response to this question was identified as a critical event because she initially (before the teaching) drew the graphs step by step by giving the value to identify the intersection points on the axes (which was her action level of the concept). Her response gave evidence regarding the development of her mental constructions since she could reflect and repeat the transformation as she learned through the process. After constructing a storyline for this student, we composed her narrative regarding her development of APOS levels in terms of quadratic functions.

For the worksheets that included students' answers regarding the exercise about function transformation, we used descriptive analysis and examined the documents considering the levels of APOS theory. We identified each student's mental structures on function transformation based on their answers for each question. For each function family, we analyzed and found the students' responses as action level when students draw the graph of the function by giving *x* and *y* values or finding peak points as they previously taught. We have observed process level evidence when the students used the logic of transformation and observed the movement of the function on the x and y-axis based on the parent function to draw the graphs. Object level indications were rarely observed while the students produced the formula of the expected function by analyzing the transformed functions' graphs. Schema level is the highest level, which we did not observe; however, if it would appear, we may identify evidence on this level if the students integrated the knowledge of the transformation of the functions as a whole and used this logic for solving problems in different areas of mathematics. For instance, when we asked students to draw a graph of $f(x) = (x-1)^2$ we analyzed how students drew the function. While they drew it step by step or giving a value to identify the intersection points on the axes, we coded their levels as action; but if they directly transform the $f(x) = x^2$ function on the *x*-axes, we coded these students' levels as a process since they were able to imagine, repeat or reflect their actions internally. Then, by using systematic tables, we compared and contrasted each student's mental constructions through the phases to identify the changes in his/her APOS levels of function transformation. In addition, we did cross-case comparison for all students' answers regarding each question to obtain a general view of their mental constructions.

We also used the video records of class discussions and field notes to support students' responses as well as identified APOS levels through the research. Particularly, we triangulated students' responses on the worksheet, video records of classroom discussions, and field notes to support each data source. Doing such a triangulation ensured the trustworthiness of the study. In addition, to ensure the reliability of the study, we analyzed the whole data separately and then we met and discussed until we had a consensus on the analysis throughout the process. We also informed the students about their confidentiality and anonymity (Patton, 2002).

RESULTS

In the results section, we initially presented findings of students' previous knowledge of function and function transformation before the effects of the teaching cycles to reveal change in their mental constructions regarding the concept. Then, regarding the analysis of teaching sessions, we shared the results of students' mental constructions related to each specific function transformation, respectively linear, quadratic, and root functions.

What Students Knew About Function Transformations?

While teaching a new mathematical concept, it is important to investigate the learners' prior knowledge to strengthen the building of knowledge related to the new concept. Particularly, teachers could identify the level of knowledge (if it exists) to evaluate and measure the effectiveness of the intended teaching. From this point of view, the students in the study were initially asked open-ended questions about their existing knowledge of function transformations, and the results showed that they had limited knowledge related to the concept. It was obvious that even though they were familiar with and used the function transformation before, they did not have knowledge about labeling the concept or the procedure.

When students were asked about how a function could be transformed into a new function by using the graphs, they proposed different responses: They claimed that it is possible to transform the function by taking the symmetry of the defined function, changing the domain of a function (like restraining the domain), or applying a shifting (or moving) process to a function. In this question, one of the students (S3) proposed a different way: She could obtain a transformed function by taking a derivative of it. In addition, students indicated that they could find a new function by using a graph of a given function by shifting, rotating, taking symmetry, decomposing, restraining to the domain, compressing, or dilating. A few students left this question unanswered or answered it as "I do not know".

In the classroom discussion, in response to the question, it was also observed that most of the students did not answer the questions about the interpretation of how a function is transformed by using formula and graph. The discussion revealed that they had limited knowledge and considered the equations and some rules regarding the function transformation rather than explaining the concept in a conceptual way. At this point, although the students did not answer or gave limited answers to open-ended questions about function transformations, the classroom discussion was effective in revealing, developing, or revising the ideas/thoughts on transformation. It was observed that conducting a classroom discussion, which is a component of the ACE teaching cycle framework, had positive effects on the first phase of implementation.

Examining Function Transformations through Activities, Classroom Discussions, and Exercises

Linear functions transformation

While designing the teaching session, we began to teach linear functions *f* to remind the students of the meaning of function transformation because based on our experiences, we inferred that this function type was the most fundamental and user-friendly functions for the students. Considering the functions (*f: R→R*) $f(x) = x + b, f(x) = ax$ and $f(x) = ax + b$ considering both negative and positive coefficients a and b in real numbers set, students were required to draw and investigate the linear function transformations by using paper-pencil and GeoGebra tasks, and then classroom discussion was conducted for each task, respectively.

The students were asked first to draw the graph of $f(x) = x + 1$ by using the parent function $f(x) = x$, and we examined how they considered the parent function while drawing the given one. It was found that most of the students (S2, S3, S4, and S6) plotted the graph by choosing the way of giving a value to *x* or by finding the points, where *y* intercepts. Even though we reminded them of using the parent function for the transformation, the students preferred to use the methods that they used to apply. We thought that they drew the graphs using their own ways because it was simpler to do or because they did not internalize the concept of function transformation. However, S6 drew the graph by applying one unit shift to the function, but he was not able to draw the parent function correctly and could not obtain the expected graph (**Figure 5**). He discovered that the formula provides shifting movement information for the parent function. Like S6, the other students (S1, S5, and S7) said that when they drew the function $f(x) = x + 1$, they considered the movement of the parent function as we aimed in designing the process.

When the students' answers were examined, it became evident that some of the students (S2, S3, and S4) continued to draw the graph by using x and y values step by step in which they could mostly feel comfortable in using the method. The four students (S1, S5, S6, and S7) presented evidence for the process level by drawing the graph considering the parent function and transformation ideas. In the class discussion, one of the students (S6) who volunteered to solve the question on the board, firstly drew the parent function and then stated that when a number was added to the variable *x* (she implied that the adding a constant to the function, i.e., $y = f(x) + c$, where c is a constant), it shifted the graph up or down on the *y*-axis. Considering the answer, the discussion was focused on the new position of the transformed function using the parent function. Although the students had no trouble drawing the linear function using the technique they used to apply, the requirement of performing the transformation helped them to comprehend how this function moved along the axes. They were therefore asked to connect the changes that would happen in the formula of a linear function moving on axes with the changes in the function's graph, and how these linkages were exposed was discussed both in class and on the worksheets.

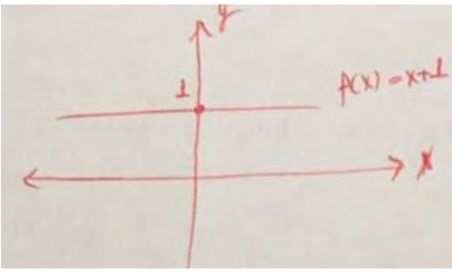

Figure 5. S6's answer for drawing $f(x) = x + 1$ (Source: Field study)

Through the process, during the classroom discussions, students came up with the idea that a transformed linear function could be explained using not only the movements on the y-axis but also movement on the *x*-axis. While explaining the transformation of $f(x) = x - 1$, the students concluded that "$f(x) = x$ function either moved one unit down on the *y*-axis or moved 1-unit right side on the *x*-axis." With this justification, it was clear that when it came to function transformations, students understood that they were able to do many transformations in a particular order to get the desired function graph rather than memorizing specific patterns. In light of the parent function being used as intended for the linear function, the conclusion demonstrated that students had started to grasp the transformation notion.

In the next questions, which the students were required to apply stretching idea (in the form of $f(x) = ax,$ where a is a constant), most of them (S1, S3, S4, S5, and S7) used the method of giving values for *x* and y to draw the function $f(x) = 2x$ without using transformation (**Figure 6**). For instance, S1 drew the graph by giving values for x to find the *y* values and finding the coordinates of the points on the graph. However, during the classroom discussion, they used the slope of the line for an explanation of how they could transform the function. They rationalized the explanation of the graph of the transformed function by comparing the values occurred based on the changes in the slope with the parent function. By changes in the constant of *x*, they were able to control the result and see how the slope increases as the constant increases, bringing the graph closer to the y axis. Most of the students contributed to this conclusion through GeoGebra tasks and class discussions: however, only S7 combined some facts, such as the slope of the line and its movement around the origin. Since she repeated the action (the idea that the function gets larger when the constant of *x* gets larger and rotates the line about the origin), interiorized the relationships between numbers and graphs (constant of x and graph of the line), and finally condensed the steps of transformation ideas by combining rotation and slope, she provided evidence of her object level of function transformation.

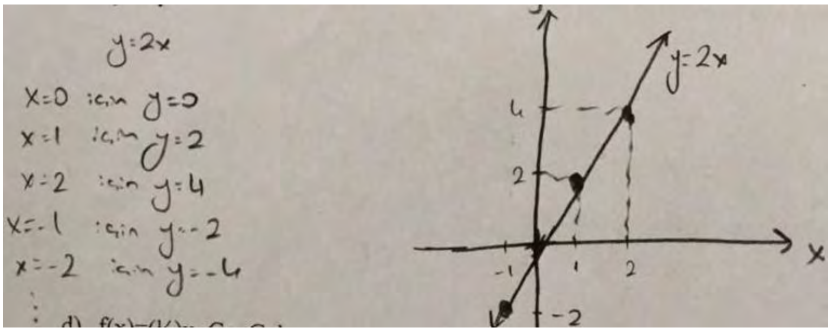

Figure 6. S1's answers for drawing $f(x) = 2x$ (Source: Field study)

In the last type of questions, which required considering the combination of shifting and stretching, participants were asked to use the transformation rules to draw the given formula of the function. Only two students (S7 and S4) were able to transform the function $f(x) = ax + b$, according to parent function $f(x) = x$ by considering the signs of a and b coefficients. S7 draw $f(x) = 2x + 1$ by using the parent function $f(x) = x$ (**Figure 7**). Particularly, S7 repeated the mental construction regarding how the function transforms based on the changes on coefficients. She wrote that "I used the parent function *f(x)=x*. Then, I initially draw the graph of $f(x) = 2x$ and then I draw the graph by adding one into the drawn graph $f(x) = 2x$". It became evident S7 revealed evidence regarding the object level by reflecting, repeating and encapsulating the actions to form an object related to the function transformation (**Figure 7**). It was possible to see from both the graphic drawing and the explanation that the student reflected each piece of information looked at in the previous questions, repeated them step by step, and then coordinated them together to reveal a for constructing the relationship between each new movement in the graph of the function and the change of the formula.

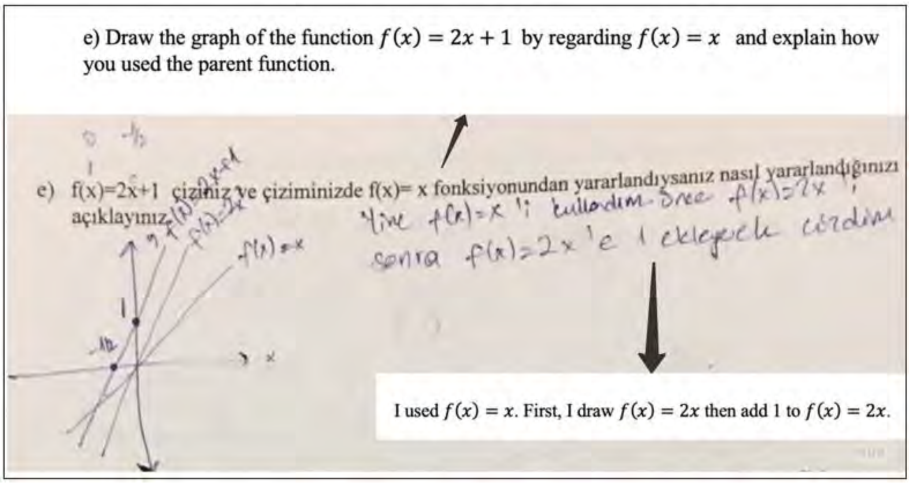

Figure 7. S7's answer for drawing $f(x) = 2x + 1$ (Source: Field study)

Since manipulating and interpreting a linear function were simple for students, they were encouraged to explore ideas about transformation during the teaching sessions on linear function transformation. However, not all students achieved the aimed mental constructions regarding the concept. In detail, only two of the students changed their mental construction so they were able to draw the graphs through the expected way. For the generalization of the transformed function $f(x) = ax + b$, some of students presented evidence regarding the process level and they reflected their reasoning and mental structures about x and y-axis movements (see **Table 2**).

Table 2. Observed levels & evidence for linear function

Student	Evidence for levels
S1	S1 mostly presented evidence regarding her action level while drawing linear function by using a transformation approach. In a couple of the questions, she realized the meaning of transformation of the function (especially the y-axis movement of the function). However, she was unable to interiorize and reflect the idea of transformation when searching for formulas from the graph. In summary, her mental constructions did not change in this process.
S2	Although he initially utilized certain values for the x and y coordinates to draw the given function, he later began to use the slope to interpret how the graph was approaching or moving away from the y axis. He provided evidence regarding his action level. Later, through the class discussions, he interiorized transformation, recapitulated the concepts of slope with a graph, and developed the mental structures through object level. For example, in order to draw the graph of $f(x) = 2x$, he used two process collaboratively: (1) slope of the function, (2) where the function located according to parent function, that is two units downward on the y axis. In this question he encapsulated the two different knowledge that he collected from the beginning of the investigation of linear function transformation applications, then used that knowledge from backward to construct the formula of the given graph.
S3	At the beginning she provided evidence of her action level regarding the linear function transformations. She did not prefer to use the transformation approach even though they were asked to use it. However, after class discussion and GeoGebra tasks, she realized how to observe the transformation of the linear function and her mental constructions were developed into the object level. For the finding of the given graph's formula, she collected information which was gathered through the sequential teaching session and encapsulated them to produce the formula of the given graph.
S4	Although she began to explore notions for transformation and used these concepts to deduce formulas from the given graphs, she was unable to demonstrate decapsulation her actions and did not reveal the object level evidence. Almost all of the questions she preferred to use specific x values to find out y corresponding and draw the function. But only for the drawing $f(x) = 2x + 1$, she used a transformation approach step by step and explained through transformation. However, she held to her previous expertise of sketching the linear function knowledge and insisted on applying the formula $y - y_1 = m(x - x_1)$.
S5	She used what she already knew, but she didn't get the idea behind the drawing transformation that was asked for. Therefore, she kept writing about what she already knew and showed evidence regarding her action level throughout the whole session.
S6	He used a general linear function formula for all question both to draw graph of function & finding out formula of given graphs.
S7	She was able to construct graph directly, & she mentioned that she took parent function into account when interpreting axes' movement & stretching/shrinking. She repeated & executed transformation operations on each function one-by-one to generate needed graph, &she internalized these actions for each drawing question. However, for extracting the formula from the supplied graph of the function, she resorted to return to basic and use $y - y_1 = m(x - x_1)$approach. This led us to believe that she was unable to provide evidence of the transformation's encapsulation and that she only repeated the most fundamental information she possessed.

Students were given three graphs and given instructions to use the concepts of function transformation to derive the algebraic form of the graphed function. They stated that these tasks were more difficult for them than the previous ones. Although, most of them (S1, S4, S5, S6, S7) solved these questions by using equation $y - y_1 = m(x - x_1)$, a few of them (S2 and S3) preferred to solve it by using the parent function $f(x) = x$ and interpreting how transformed function $f(x) = ax + b$ was affected from the value of coefficients a and b. Students typically chose to implement procedures that they knew previously rather than considering the transformation of the graph on the axes to determine the formula of it.

In general, as shown in **Table 2**, some students' mental structures changed as a result of the carefully planned to teach sessions, allowing for connecting the idea of translation rules and drawing activities. In detail, it was deduced that they repacked their mental structures to find the formula of the functions when the graphs were given, which were also evidence regarding their object level. Students in this question rethought the transformation methods from the reverse perspective, which means they observed the transformation's outcome when they looked at the graph, tried to relate the parent function and graph, and examined each step of the transformation while also analyzing the formula and graph. Repacking thought processes, utilizing repetition, and interiorizing ideas were all used to handle the examination process. To be more precise, two students (S2 and S3) began with evidence from the action level before beginning to use slope through paper-pencil drawings. GeoGebra software also supported them in observing changes in the slope and graph simultaneously so that they could draw conclusions about transformation and understand the impact of slope on the formula and graph. Additionally, they explained how they interpreted the connections between a graph and a formula based on a function transformation.

We can conclude that the students had limited knowledge about transformation, and they considered x and y values to draw the graph of a transformed function at the beginning. After teaching sessions, they presented evidence regarding expected approaches like positioning the function through axes and transforming the function by using parent function as aimed in the teaching. The sequential teaching method, which includes classroom discussions and the use of GeoGebra tasks during sessions, was believed to be the reason for these alterations in the mental structures. Despite this, the teaching sessions were acceptable but ineffective. Even though applying a linear function is the first ACE cyclical session of the investigation into function transformation, the analysis and discussion of quadratic and root functions continued so that students could accumulate the transformation thought process throughout whole sessions.

Quadratic functions transformation

As it was expected in the designing of the teaching session, we observed the effects of the process and techniques the students learned in the linear function transformation on drawing the graphs of the quadratic functions even though drawing the graphs of quadratic functions is more complex than drawing the graphs of linear functions as they stated. In the teaching cycle, the students initially were asked to draw the graph of parent function $f(x) = x^2$. Then, the students were asked to draw the graph of a couple of questions, which were formed as $f(x) = x^2 + c$ and $f(x) = (x + b)^2$. In these transformations, almost all of the students easily noticed the direction of the transformation according to the coefficient c or b and they shifted /translated the function on the axes. On the contrary to drawing the graphs of linear functions, most of the students drew the graphs based on translation directly on both the *y*-axis and *x*-axis and explained their reasoning regarding their actions instead of giving values to find the intersection points. This result was considered as one of the critical points of sequential teaching and it shows that students' mental structures of quadratic functions regarding the transformation on both axes were at the process level by reflecting or imagining the action they encountered in the teaching of the linear functions on the quadratic functions. At this point, while some students directly drew the transformation of the parent function on the *x*-axis according to the coefficient *b*, two students (S4 and S7) first drew the parent function and then drew the transformed function using the parent function and explained that they considered how the changes in the parent function to draw the transformed function. S7 wrote that "we used the parent function $f(x) = x^2$ to draw the graph of $f(x) = (x - 1)^2\dots$ if we subtracted one from *x* values the graph of $f(x) = (x - 1)^2$ is formed. In that case, the graph is shifted one unit to the right on the *x*-axes". Her imagination about shifting and directly drawing the graph using the parent function was evidence regarding the process level (**Figure 8**). Only one student (S2) drew the graph by giving values and provided evidence regarding his action level.

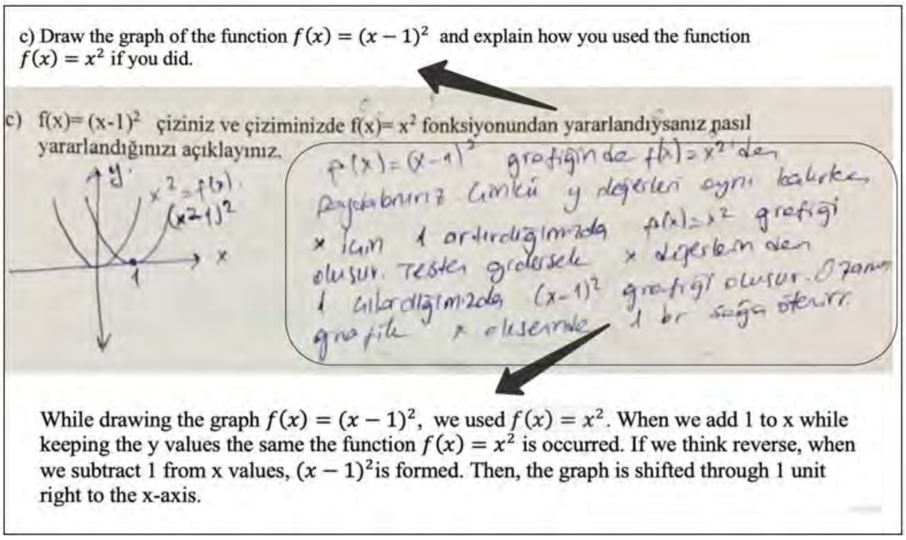

Figure 8. Answer of S7 for drawing graph of $f(x) = (x - 1)^2$ (Source: Field study)

It is deduced that the most critical point in the transformation of quadratic functions was the transformations that occurred based on the changes of the coefficient a (the function transformation in the form of $f(x) = ax^2$). At this point, it was observed that the students were able to reason about the concept of a slope by combining them with the mental structures they constructed in the teaching about linear function transformations, and some of them also often draw the graphs by giving a value to find a specific point on the graph. For instance, while S1 stated that the graph approaches the *y*-axes (**Figure 9**), she said that "the slope has increased" and explained this approach by indicating that the slope of the graph she drew is greater than the slope of the parent function at a specific point.

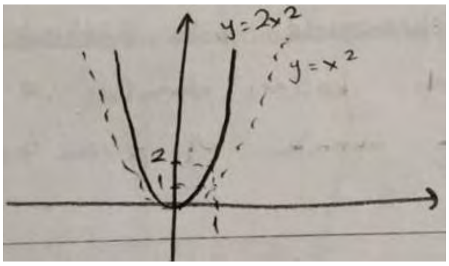

Figure 9. S1’s examples of quadratic function translations (Source: Field study)

It is deduced that she thought about the tangent line of the quadratic function at a specific point to compare the given transformation with the parent function, which showed that she could mentally operate on the graph. Her imagination regarding the slope as well as her drawing and statements revealed that she was able to reflect her actions, which was also evidence of the process level regarding this type of transformation.

Some of the students (S2 and S3) stated that while drawing these graphs, they intuitively predicted that the graph would actually shrink or stretch according to the parent function, but they indicated that they gave a value to see a specific point on the graph to make sure about it. For example, while drawing the graph of function $f(x) = 2x^2$, four students (S1, S4, S6, and S7) initially drew the parent function and then found the transformed function based on that function and indicated that the arms of the parent function closer to the *y*-axes (or stretching farther away from *x*-axes) using the argument regarding the slope. Particularly, even though quadratic function is a curve not a line and slope are not applicable for a curve unless draw a tangent line at a point, it is thought that they imagined the tangent line or linear function transformation to reason about shrinking or stretching of the parent function based on the given coefficient a. The rest of the students (S2, S3, and S5) drew the graph of the transformed function directly and stated that the drawn graph is closer to the y-axis compared to the parent function and provided evidence regarding their process level as expected. Particularly, it was observed that S2 began to mentally operate on the graph by imagining the transformation and directly drew the graph by explaining the reasoning of it.

In the quadratic function transformations, when students were given the questions in which all the coefficients a, b and c were included, most of the students directly interpreted and drew the transformed form of the function in their drawings using their previous knowledge (**Figure 10**). In these questions, S2’s responses revealed his interiorization; while he was drawing the graphs by giving values, he began to imagine the transformation in his mind and directly graph the transformed function through the process. Some of the students drew each function transformation step by step and were successful in drawing the transformed function. For instance, S4 was able to transform the function step by step considering the parent function. She stated that “I drew the graph by combining the previous information I learned and taking x^2 as a reference” (**Figure 10**).

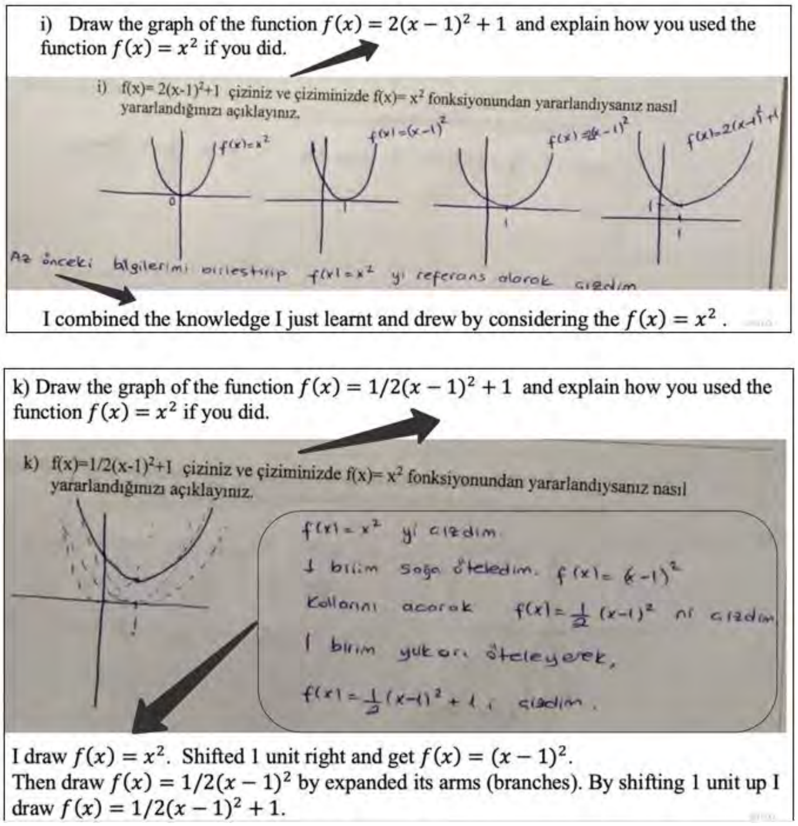

Figure 10. S4’s two different responses to questions including *a*, *b*, and *c* coefficients (Source: Field study)

During the implementations of the quadratic functions, it was deduced that the completed implications enabled students to improve their knowledge of function transformation cumulatively. In the drawing of the graph of the function in this form, it was observed that the students transformed the function in different priority order based on the given coefficients. This situation might be related to the students' mental constructions regarding the transformations; for example, some of the students first consider the coefficients *b* and *c* and transform the graph on the *x*-axis and *y*-axis, and then shrink or stretch the graph considering the coefficient a. For instance, S5 stated that "the transformed function was obtained by moving the parent function 5 unit on the left direction of the *x*-axes and one unit down on the y axes; and taking the symmetry of it on the *x*-axes and the arms approaches to the *y*-axes, they are narrower". Some students first considered the *b* (transformation on the *x*-axis), then the *a* (shrinking and stretching) and finally the *c* (transformation on the *y*-axes) to draw the transformed function step by step (**Figure 11**).

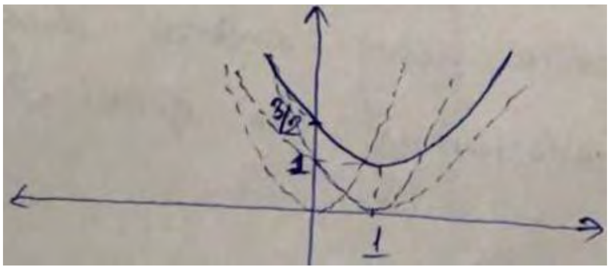

Figure 11. S6's examples in which he considered first *b*, then *a*, and finally *c* (Source: Field study)

To sum up, students were asked to generalize transformation of the graph of the function (*f:R→R*) $f(x) = a(x+b)^2 + c$ according to the positive and negative coefficients *a*, *b*, and *c* in real number sets. All students were able to explain how the function transformed based on the coefficients using their knowledge developed through the cycles. In general, especially as the process progressed, students were able to construct conceptual knowledge regarding the transformation on-axis as well as stretching and shrinking the arms of the graph considering the coefficients of the functions. In other words, they were able to combine and compare their mental constructions about the functions and transformation and all of them revealed evidence related to their process level.

In the next step, students were given the graph of the transformed function and asked to find the algebraic equations of it. It was observed that students spent more time on these questions in thinking, interpreting, and finding the equation of the functions. First, students were given the graph of $f(x) = (x-2)^2$, and all of them explained this graph as a 2-units transformed form of the parent function in the right direction on the x-axis. In the second question, students were given the graph of the function $f(x) = (x+1)^2 + 3$, six of the students were used the methods taught in the teaching cycles to find the equation of the function and only one student (S4) used the formula regarding the graph including one point and peak point. Similarly, when students were given the graph of the functions $f(x) = 3(x+1)^2 + 1$ and $f(x) = 3x^2 + 3$, the same students (S4) found the equation using the general formula of an equation and peak point, and the remaining six students considered the transformation on the axes and stretching and shrinking to find the equation. In general, students were able to directly notice the transformations on the axis considering the coefficients *b* and *c* as well as using the point on the graph and general formula of an equation in order to find the coefficient *a*. In other words, students preferred to identify the equation by finding the coefficient c first, then the coefficient b, and finally the coefficient *a*. Their responses revealed evidence regarding how they were able to use their mental mechanism to deduce about how the parent function transformed on the graph while finding the equation. Students' ability in interpreting the transformation revealed that their mental constructions of function transformation were developed, and they could easily draw the graph when the equation was given as well as they were able to find the equation when its graph was given, which was evidence of their process level. Only one student (S3) revealed evidence regarding her object level of quadratic function transformations. Through the process, she indicated she intuitively reasoned about the graph of function when the equation was given, and she directly imagined how the graph was transformed in her mind and found the equation without any calculation when the graph was given. Her ability in transforming between these contexts showed that she reached the object level by encapsulating her constructed process.

While students drew the graph by giving values on the equation, they started to imagine and intuitively reason about how the function moved on the axes as well as stretched or shrunk based on the parent function. In their own words, students indicated that they solved these questions using a formula (including one point on the graph and the peak point) or they could not solve those because the questions were difficult and complex for them. Additionally, students emphasized that the process allowed them to learn the concept easily and the method is more effective and easier for both understanding the function transformation, drawing the graphs, and finding the equations of the given graphs. **Table 3** summarized their responses regarding the questions as well as showed how each student's mental construction was changed through the process.

Table 3. Observed levels & evidence for quadratic function

Student	Evidence for levels
S1	She was able to reflect and repeat her actions and provided evidence of her imagination about the transformed function. For the $f(x) = a(x + b)^2 + c$ transformation, she was able to interpret how the graph was transformed on the axes and find the coefficients *b* and *c* and she found the coefficient *a* by using the slope of the graph.
S2	Even though he was provided evidence regarding action level in the first questions, his responses revealed that he imagined and repeated his actions through the process. As an example, he drew the graph of function by giving values as seen in the first example. He indicated that since the values in the image set were larger. The gap between the arms was less. In his second response, he directly drew the graph and provided evidence regarding his imagination of the transformed function.
S3	At the beginning she provided evidence of her process level regarding the quadratic function transformations. She was able to directly draw the graph of the transformed function and imagine the stretching and shrinking. She indicated that "I already know intuitively that the parabola should approach the y-axis." Through process, she showed evidence of her object level; when a graph was given, she directly wrote algebraic form of graph, her previous answer revealed that she had an ability in transforming between these contexts.
S4	She provided evidence of her process level by directly drawing the graph of the given function and coordinating the x and y values on her mind while drawing the stretching/shrinking. For the shrinking/stretching, in her own words she indicated that "It became a more open-armed parabola as each x-value went to a smaller y-value. As an example, she drew the graph on a system step by step and stated that "I drew $f(x) = x^2$ first. I shifted this 1 unit to the right $f(x) = (x - 1)^2$. I drew $f(x) = 1/2\,(x - 1)^2$with open arms (branches). I drew $f(x) = 1/2\,(x - 1)^2 + 1$ by shifting it 1 unit up. She provided evidence of the process level when the equation was given. However, she needed to consider the peak point and equation while identifying the equation using the graph as given. She stated that "I wrote an equation using a vertex and a crossing point".
S5	She was able to directly draw the graph, and she indicated that she considered the parent function while interpreting about the moving on the axes and stretching/shrinking. Through the process, she provided evidence of her process level.
S6	He was able to directly draw graph of transformed function. He directly reasoned about transformations on axes. He used slope while interpreting about stretching or shrinking. He provided evidence regarding his process level. For example, he directly drew graph of $f(x) = 2(x - 1)^2 + 1$ and stated that "The graph of $f(x) = (x - 1)^2$was the graph of $x^{2,}$ which shifted 1 unit to the right. It means that the slope of the graph of $f(x) = 2(x - 1)^2$ is increasing, that is, the arms are getting closer together. At the end, I shifted 1 unit up."
S7	She directly drew the graph of the transformed function and used the coefficients while interpreting the transformed function. She provided evidence regarding how she reflected her actions to reason about the transformed function.

Root functions transformation

As with linear and quadratic function transformations, the design process for analyzing the root function transformation involved analyzing the change of each coefficient ($f(x) = a\sqrt{x + c} + d$) based on the parent function ($f(x) = \sqrt{x}$). First of all, students were asked to draw the graph of the root function after the discussion about its domain and range. At this point, while five students drew the graph immediately, S3 and S7 assumed that the root function is the inverse of the quadratic function and drew the graph using this argument. We deduced that almost all students had a background in conceptual understanding of the root function, and they were able to draw a graph of it.

As intended in the teaching cycle, the students easily explained the transformation of the $f(x) = \sqrt{x} + d$ and $f(x) = \sqrt{x + c}$ (for example,$f(x) = \sqrt{x} + 1$, $f(x) = \sqrt{x + 4}$), as given the function shifted on the *y*-axis and *x*-axis considering the coefficients *d* and *c*. Students knew how the *d* and *c* coefficients of a function affected how the graph moved along the *x*- or *y*-axis because they had been trained many times. S3 claimed that she repeated the drawing she learned in quadratic function shifting. Even though S7 gave the same explanation, after using the shifting process to make sure the result was correct, she gave values on the function to check how it changed and how similar it was to the previous function. During the process, students who internalized the translation movement rather than generating rule-based drawings had an idea of how the graphics could be seen and had started to acquire their own knowledge about transformation. This demonstrated that they provided evidence regarding their object level. Although in the classroom discussion, just one student (S1) stated that she gave values for checking the shifted graph to see whether it was correct. Under these conditions, the applications of the previous function transformation helped them improve their conceptual understanding and they shift the root function on both axes. Therefore, we assumed that all students reached the process level and some of them could provide object level evidence for shifting the x and y-axis by reflecting their actions on the graph, encapsulating the processes they had as a mental construction been expected in the sequential teaching procedure.

We observed that students presented different perspectives for the root function transformation when considering the coefficient a. While examining the coefficient a, students usually reason about approaching or moving on the *x*- and *y*-axis, stretching or shrinking the arms of the function, and making explanations regarding the slope of it. While drawing the function $f(x) = 2\sqrt{x} - 1$, S2 considered the coefficient two as "*[the function*] approaches the y axis more based on the parent function $\sqrt{x}$, then I moved it one unit down on the y axes" (**Figure 12**). This response showed us that S2 was able to imagine and reflect the mental construction regarding the transformation, which was evidence of the process level.

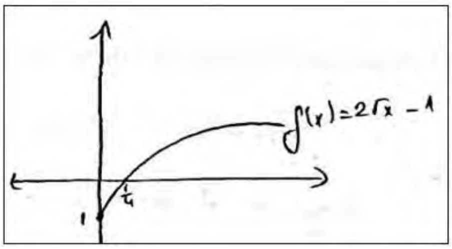

Figure 12. S2's answer for $f(x) = 2\sqrt{x} - 1$ (Source: Field study)

Additionally, S1 and S6 explained the form of transformation produced by considering the coefficient *a* for the root function, they used the idea of slope that was observed while examining the first exercise, a linear function. S1's example for this question was "the slope is getting bigger based on that is approaching *y*-axis and moving one unit down on the *y*-axis". S6 also built and strengthened from the beginning of the linear function analysis to explain how steep the function is or how close it approaches the *x*-axis. Finally, students mostly preferred to use the terms 'shrinking or stretching the arms' for root functions, where this explanation was especially used while explaining quadratic function transformation. Whereas, although there was only one extension of the root function that can be defined as the 'arm', the expression 'expanding arms' was used because it was the mental constructions they transferred from the quadratic function. S4 stated that the arms of the transformed function were expanding, and the graph was moved 1 unit down (**Figure 13**). However, in the classroom discussion, she explained that the arms of the root function were stretching.

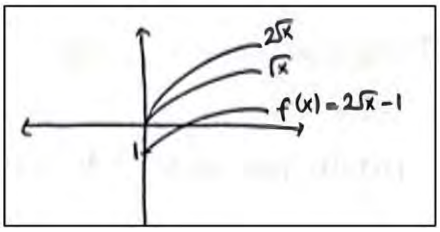

Figure 13. S4's answer for $f(x) = 2\sqrt{x} - 1$ (Source: Field study)

By considering the students' responses, students reached the process level when examining coefficient a, as evidenced by the mental constructions of the transformed function graphs they demonstrated, including the use of slope thoughts, step-by-step explanations, repeated iterations of the parent function, and, as a final step, the recapsulating of the use of previously covered transformation strategies in the root function as a result of the sequential teaching cycle.

The step-by-step drawings of the transformed function might be a result of the sequential teaching cycle because S3 presented evidence regarding action level in the previous examination on function, and she internalized the concept of transformation so that she reached the process level (**Figure 14**).

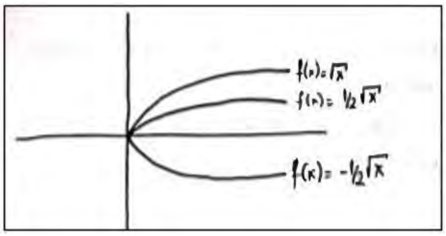

Figure 14. S3's step-by-step drawing (Source: Field study)

During the examination of the $f(x) = a\sqrt{bx + c} + d$ function, the most difficult part of the examination was the coefficient b for the students. For example, S5 drew the graph gradually in the same coordinate system (**Figure 15**) and **Figure 15** might present the students struggled with drawing and explaining of the coefficient that is placed in the root. Although they were very successful when the coefficient was placed outside of the root, they had difficulties in explaining the changes in the function that occurred based on the coefficient b. This situation was evidence for not reaching the object level.

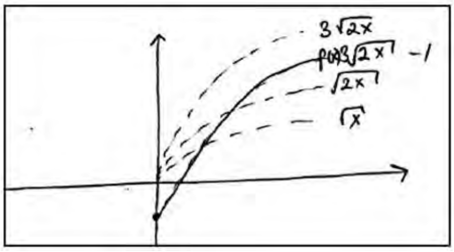

Figure 15. S5's answer of $(x) = 3\sqrt{2x} - 1$ (Source: Field study)

When students were asked to give a generalization for the coefficients *a*, *b*, *c*, and *d* that explained changing of the graph of $f(x) = a\sqrt{bx + c} + d$ based on the parent function $f(x) = \sqrt{x}$, it was concluded that all students were able to construct the conceptual meaning about the transformation, which was related to the coefficients *c* and *d* as shifting on *x*- and *y*-axes; the coefficients a and b as either stretching/shrinking or approaching or moving away from the axes. These results proved that students reached the process level by repeating steps and applying the idea when it was necessary. Moreover, in the classroom discussion it was observed that students started to think about the aforementioned method, and additionally as S6 explained "now, when [I] see function equations, can easily visualize the graph step by step by considering the coefficients"

All students were able to easily write the formula of the graph of transformed root function. It was observed that while obtaining the formula of the graph they first identified the amount and types of shifting on the x and y axes then used one point on the graph. The answer of S6 given in **Figure 16** was an example for this explanation. S6 realized that the graph was shifted two units on the *x*-axis considering the coefficient *c*. After that, in order to check whether the graph performed any stretching or shrinking, he selected the point (3, 1), used the general formula for the root function, found the coefficient *a* as one and finally he obtained the formula of the transformed root function graph.

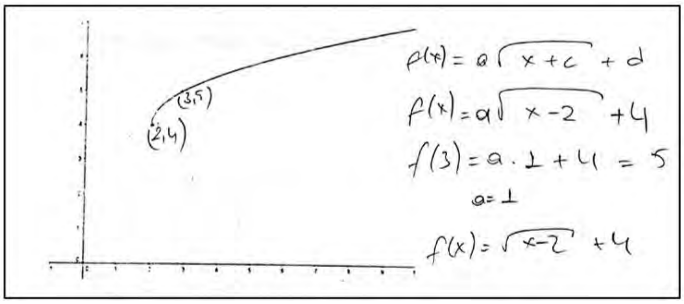

Figure 16. S6's way of obtaining formula of root function (Source: Field study)

In general, although drawing the root function was more difficult than the quadratic function, it was observed that students drew them easily and found the formula of given graphs (**Table 4**). Despite the fact that students admitted that before the designed sequential teaching they drew the graphs by giving values for *x* and *y* or they even could not draw it. However, after the sequential teaching they were able to consider and interpret the idea of the relation between the parent function and the transformed one so that they step by step construct the transformed graphs. While examining the root function transformation, it was observed that students recalled the information they learned through the teaching cycles about the linear and quadratic functions and used them. This is also very strong evidence for the effectiveness of the designed sequential teaching cycles. It is believed that, using GeoGebra besides paper-pencil tasks helped students observe the visual changes depending on the coefficients and examination supported to develop conceptual understanding. During the classroom discussion, one of the students commented that "I have no idea about this topic, yet I learnt very well now" and presented an unmodified opinion about sequential teaching. At the same time, in exercises given as homework for each function transformation as part of the ACE teaching cycle, students solved the questions by considering the transformation procedures (shifting and approaching to the axes), in a way similar to the ones done during the classroom applications, and they did not use any methods that were considered preferred by them before the sequential learning.

Table 4. Observed levels & evidence for root function

Student	Evidence for levels
S1	As intended, she developed her mental constructions regarding the function transformation. She visualized the graph, then used the parent function's coefficients to construct the graph. She reached the process level by assimilating the rules in the process and continuously applying them. But she checked her drawn graph using the given *x* values to get the *y* values.
S2	He applied the transformation rules to the parent function in order to construct the graph. He stated that he was able to determine how each coefficient of the function influences the movement of the graph of the parent function. He used the previously taught transformation rules often and was able to imagine the graph. However, he said that he preferred to utilize values (like x=0) of x and y to construct the function's graph, regardless of how complex the function was. This evidence led us to deduce that his mental structures regarding transformation on root function is at the process level.
S3	At the very beginning, she connected the root function with quadratic function therefore we deduced that she presented evidence regarding her object level. However, in the other questions she continued with the transformation procedure by repeating her actions. She processed step by step application by using parent function for instance, she drew $f(x) = \sqrt{x}$ first, then applied each coefficient to this parent function and represent the graph of each step to reach the aimed graph.
S4	S4 applied the transformation procedure repeatedly to the root function. She realized the connection between the coefficient and the coordinate system positioning of the graph. Moreover, she discovered the graph's formula easily. She effectively reached the process level, but we could not observe sufficient evidence for her object level. According to classroom discussion, S4 confessed that these drawings were easy because she transferred the transformation idea to the root function. She said that "the branch of the function is expanded that means gets close to *y*-axis." This confession is considered as evidence of the process level since she repeated the ideas and applied to each function types.
S5	S5 followed rules for transformation that she repeated through sequential learning, allowing her to easily identify effect of coefficient on the function's graph. She utilized the parent function and successively applied the transformation rules to draw the desired graph. These behaviors provide evidence for the process level. Through the drawing, she visualized and reproduced her activities.
S6	S6 became more professional so that it could use translation rules to draw the function's graph. He figured out how each coefficient affects the function's graph. When given a graph, he can figure out where the graph is in relation to the parent function. This lets him write the formula for the function's graph. These things are thought to show that S6's mental structure is at the process level. The sequential learning allowed him to connect each function to another. When he was asked to explain the GeoGebra drawing considering the transformation rules he stated that "This shifts the function down by 2 units and to the left by 1 unit. Furthermore, the decline in slope causes the branch in the graph to be brought to x axis (get closer to x axis). And then the function's symmetry is obtained." In the classroom discussion S6 explained transformation on the x axis as "I tried to get a value which make inside of the root ($f(x) = \sqrt{x-4}$) is zero. So, the parent function should move 4 units to the right on the x axis." This kind of explanation is considered as evidence of his process level. In addition, he said that he would have kept drawing the functions by considering the particular values for x and related y values even if the sequential learning had not investigated this approach.
S7	At the very beginning she connected root function with quadratic function. This graph is an evidence of process level because she encapsulated & repeated the knowledge about transformation on root & quadratic functions. However, for the following the questions, S7 drew the graph of the function step by step considering the transformation rule. S7 shows evidence of the process level.

What Happened After Three Months?

Three months after the teaching sessions, the students were asked to answer four tasks in order to evaluate their persistence of mental constructions of function transformations. In the first task, students were required to answer an open-ended question related to their knowledge of the meaning of function transformation, and in the rest of the tasks, they were asked to draw the graphs of linear, quadratic, and root function transformations and explain their reasoning about the drawings. Even though three months had passed, students were able to reflect the mental structures they developed through the teaching cycles on given tasks. For instance, S1, S2, S3, S6, and S7 initially explained the transformation on the *x*a- nd *y*-axes according to any given value. S3 answered the question by addressing the transformation on the axes, symmetry, and changes of the slope. In particular, she explained that the transformation of the function $y = f(x)$ in k units is $y = f(x) + k$ and indicated the movements (up and down) on the *y*-axis according to the positive and negative values of k. Similarly, she explained the transformation of $y = f(x)$ on the *x*-axis would be similar to $y = f(x + k)$ and indicated the movements (right and left) according to the positive and negative values of *k*. Some of the students (S2, S3, S6, and S7) explained the changes that occurred regarding the coefficient a by referring to the concept of slope. Specifically, S2 and S7 explained that the slope increases when the coefficient *a* increases, S3 explained this case as "The slope at any point in a function is related to the coefficient of the *x*. When the coefficient of *x* changes, the function approaches or moves away from the axes depending on the situation". S2 explained how the function approaches the axes according to the cases, where the coefficient of *x* is between zero and one and is greater than one. In addition, S1, S3, and S7 indicated that if the coefficient *a* is negative, its symmetry according to the *x*-axis would be taken in drawing the transformed function. S5 answered this question as "the process of writing different functions in terms of each other by considering a required/asked function rule is called function transformation". Even though S4 drew the graphs of functions, she did not write an explanation for this question. Comparing the students' answers in this process with their answers given before the teaching cycles, it was clearly seen that they built conceptual knowledge regarding the concept and their mental constructions of the meaning of function transformation have significantly developed y.

In the second task, students were asked to draw and explain the graph of the function $f(x) = -2x + 1$. S2, S5, and S6 drew the graph of the function directly and explained in detail how the parent function transformed on the axes. S1, S3, S4, and S7 drew the graph step by step as seen in **Figure 17**. Particularly, they first took the symmetry of the parent function, then they drew the function $f(x) = -2x$ and explained this situation regarding the increases of the slope of the graph. They transformed the function one unit upward on the y-axis and finally drew the graph of the function transformation.

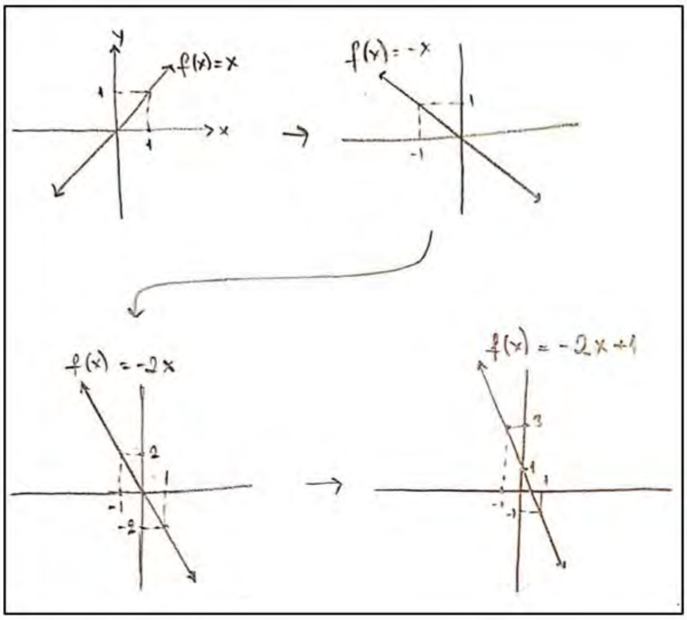

Figure 17. S3's answers for the linear transformation task (Source: Field study)

Similarly, in drawing the graphs of the functions $f(x) = {}^{1}/_{2}\,(x-1)^2 + 1$ and $f(x) = 3\sqrt{x-1} - 2$, students generally used the methods they learned through the teaching cycles to reason about the transformations on the axes and to determine the changes of the arms of the graphed function. For instance, while drawing the quadratic function transformations, some of the students drew the graph step by step on a single graph, some of them directly drew the graph of the desired one and others drew the graph step by step using different graphs. S1 initially drew the parent function, transformed it on the x-axis, stretched the arms of the function according to the coefficient a, and moved it on the y axis as the last step. It was identified that S1 always used the same order in all the tasks during the teaching cycles of quadratic and root functions; that is, she initially moved the parent function on the x-axis, then considered the approach or moving away of the arms to the axes and finally moved the graph on the y-axis. While some of the students gradually drew the graph of the root function task using different graphs, some of them drew it directly as shown in **Figure 18** and explained their actions in detail. For example, S2 indicated that the parent function increases starting from the origin, and so he transformed the function one unit in the right direction on the *x*-axis; increased the slope according to the coefficient three, and finally transformed the graph two unit down on the y-axis.

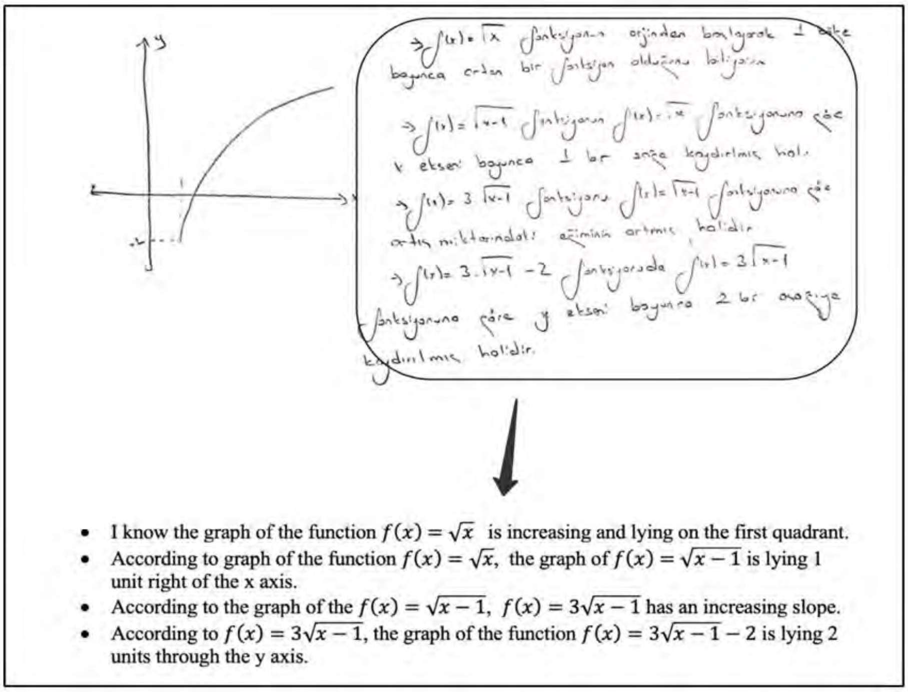

Figure 18. S2's answer for the root function task (Source: Field study)

Students' answers to the tasks revealed that they were still able to reflect, which means developed, on the conceptual knowledge they learned through the designed teaching cycle conducted three months before. This evidence revealed that their mental constructions, which indicated their permanent APOS levels as well, regarding the function transformation have developed (mostly from action level to process level). In summary, it could be deduced that the designed teaching cycle was effective to support the development of their knowledge regarding function transformation.

DISCUSSION AND CONCLUSIONS

This study illuminated how the students' mental constructions of function transformations could change through designing teaching sessions in the light of APOS theoretical framework. The results revealed that at the beginning, the students had practical knowledge on function transformation and drew the graph of the transformed function by giving values or using the formulas related to the function transformation. In other words, they could reveal evidence regarding their action level (Arnon et al., 2014) by providing external stimuli such as words, values, or operations. Through the sessions, the students were able to transform the strategies they learned while drawing the graph of function transformations. From drawing the graph of linear functions through quadratic functions, the students easily identified the transformation of the parent function as they were taught in the study as well as they were able to make interpretations about the position of the transformed function using the terms of shrinking or stretching in regards to the axes. Their actions on the graph of function transformation such as repeating, reflecting, or transforming showed the evidence of their process levels regarding the transformations as it was expected in the designed teaching cycle. Encapsulation and de-encapsulation were some of the most significant steps in learning a concept regarding APOS theory (Arnon et al., 2014), and only few students identified the equation of the transformed function by using a given graph and applying the strategies they had learned in the designed teaching. In other words, they easily went back to process level and synthesized constructed actions and processes to find equation of the transformed function. In summary, this study revealed that even though the function transformations are mainly taught by using formulas in Turkish mathematics curriculum, students had learned to transform the functions by using parent functions and changed their mental constructions through teaching sessions.

In the teaching sessions for each function transformation, given the sequential tasks considering the coefficients as well as using the GeoGebra tasks were helpful to change students' mental constructions. In detail, focusing on several types of sequential tasks for each function transformation as well as spending more time in the designed cycle could foster change of students' mental structures at the process and object levels (Maharaj, 2013), and so students were able to deduce what types of actions, operations or strategies they should use step by step to draw a graph or find the equation of transformed functions. Sequential teaching involves building a new knowledge on the prior one and the importance of the prior knowledge in learning mathematics is a well-known phenomenon. An et al. (2004) underlined that "using prior knowledge not only helps students to review and reinforce the knowledge being taught, but also helps them to picture mathematics as an integrated whole rather than as separate knowledge" (p. 165). Similarly, Martin and Towers (2016) examine the prior knowledge through Pirie-Kieren model for the dynamical growth of mathematical understanding and suggest teachers to identify existing understandings for a particular topic and to design pedagogical actions that facilitate the engagement with a new context. Therefore, the growth of understanding could be observed through the interactions with the contexts, materials and with other students and teachers (Kieren et al., 1999), where this idea aligned with the findings obtained through the sequential teaching in the current study.

Throughout the cycles, we also observed some key points that were related to changes in students' mental structures. For instance, using the strategy "looking for the slope of the function" in linear function transformation provided an opportunity to students for understanding the meaning of the transformation on the graph. It became evident that using the slope of the function allowed students to understand the idea of shrinking or stretching, in particular in quadratic and root functions, through approaching a line to the *x*- or *y*-axis. The idea of slope is used as prior knowledge to construct an understanding for identifying quadratic and root functions' transformation. It is thought that using the slope of the graph was students' previous process related to drawing a graph before attending the cycle; so, students could construct the process level regarding stretching and shrinking by synthesizing or coordinating their existing processes as Asiala et al. (1996) and Dubinsky (1991) indicated. One of the other key points in the changes was using GeoGebra tasks, as well. Through the teaching cycles, we observed that solving the tasks in GeoGebra allowed them to verify their predictions (de Villiers, 1998) regarding the transformation, which also could support the changes through process level by repeating the actions on the technological environment.

The genetic decomposition of the function transformation guided us to design the sequential teaching cycles; however, while designing the sequential tasks in each function type, we only considered the function transformation in the form of $f(x) + k, kf(x), f(kx)$ and $f(x+k)$. However, we did not consider the function transformation, which has inordinate coefficients such as $f(x) = ax^2 + bx + c$. As a following step of this study, one could extend the designed teaching cycle by adding these types of tasks and could examine how students transfer the functions in the form of $f(x) + k, kf(x), f(kx)$, and $f(x+k)$ to reason about the transformation. In general, we initially taught to transform the function on the y axis, then on the x-axis, and finally stretch or shrink the function based on the coefficient in front of the x, x^2, or $\sqrt{x}$. Students' responses showed us that they differently used the coefficients while transforming the functions on the graph even though we taught the transformation in the order of *y*-axis, *x*-axis, and shrinking/stretching. We did not explicitly examine their actions about using the coefficients as well as their relation to the development of mental structures of function transformation. However, their responses brought a new research question to identify the role of using the coefficients in the development. One could design and analyze a hypothetical learning trajectory, which is "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (Simon, 1995, p. 133) to examine how students engage the orders of the coefficients in the learning of function transformation. In addition, while designing the sequential teaching cycles, we did not consider constructing relationships among different types of functions. The discussion about the relation could trigger future research to consider the question of whether it is possible to see the schema level in constructing the relationship between linear and absolute value function transformation as well as quadratic and root function. According to the students' responses and examples, it became clear that the development of the mental constructions related to the relationships among the function transformation, and we thought that this development should be examined in future research to support the learning of function transformation.

Author contributions: All authors have sufficiently contributed to the study and agreed with the results and conclusions.

Funding: No funding source is reported for this study.

Ethical statement: The authors stated that consent was obtained from the participants before the research was conducted; and they were informed about the research, anonymity and their rights about withdrawing from the study without adverse repercussions.

Declaration of interest: No conflict of interest is declared by authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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APPENDIX A: QUADRATIC FUNCTION TRANSFORMATIONS

1. Draw the graph of $f(x) = x^2$. Examine how we could obtain the following functions by using this parent function.
   a. Draw the graph of $f(x) = x^2 - 1$. If you used the function $f(x) = x^2$ in your drawing, explain how you used it.
   b. Draw the graph of $f(x) = x^2$+1 by using GeoGebra. Explain that what kind of transformation might be done in the function $f(x) = x^2$to obtain this function.
   c. Draw the graph of $f(x) = (x-1)^2$. If you used the function $f(x) = x^2$ in your drawing, explain how you used it.
   d. Draw the graph of $f(x) = (x+1)^2$by using GeoGebra. Explain that what kind of transformation might be done in the function $f(x) = x^2$ to obtain this function.
   e. Draw the graph of $f(x) = 2x^2$. Explain that what kind of transformation might be done in the function $f(x) = x^2$ to obtain this function.
   f. Draw the graph of $f(x) = -2x^2$by using GeoGebra. Explain that what kind of transformation might be done in the function $f(x) = x^2$ to obtain this function.
   g. Draw the graph of $f(x) = \frac{1}{2}x^2$. Explain that what kind of transformation might be done in the function $f(x) = x^2$ to obtain this function.
   h. Draw the graph of $f(x) = -\frac{1}{2}x^2$ by using GeoGebra. Explain that what kind of transformation might be done in the function $f(x) = x^2$ to obtain this function.
   i. Draw the graph of $f(x) = 2(x-1)^2 + 1$. Explain that what kind of transformation might be done in the function $f(x) = x^2$to obtain this function.
   j. Draw the graph of $f(x) = -3(x+5)^2 - 1$ by using GeoGebra. Explain that what kind of transformation might be done in the function $f(x) = x^2$ to obtain this function.
   k. Draw the graph of $f(x) = \frac{1}{2}(x-1)^2 + 1$. Explain that what kind of transformation might be done in the function $f(x) = x^2$ to obtain this function.
   l. Draw the graph of $f(x) = -\frac{1}{3}(x+5)^2 - 1$ by using GeoGebra. Explain that what kind of transformation might be done in the function $f(x) = x^2$ to obtain this function.
   m. In the given function $g(x) = a(bx+c)^2 + d$, consider the positive and negative values of the coefficients a, b, c and d and make a generalization about it.
2. Find the following function by using the given graphs and explain your reasoning.

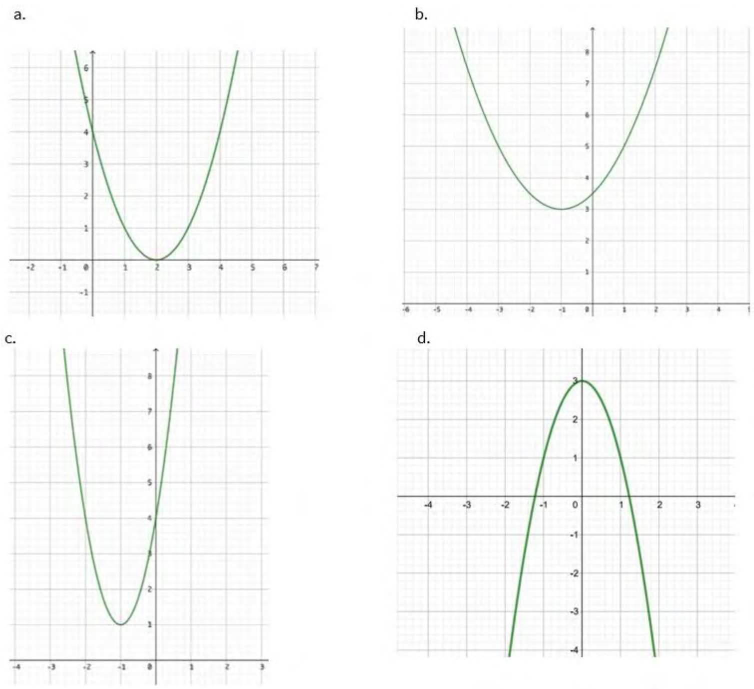

Figure A1. Examples of functions (examination of quadratic functions [examining the coefficients in the given functions and graphs] were carried out similarly in the linear and root functions transformations in the same order & also, the sheets of activities, exercises, and including the tasks given after three months for all function transformations were also similar to this given sequential questions) (Source: Authors' own elaboration)