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Non-Numerical Methods of Assessing Numerosity and the Existence of the Number Sense

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Abstract

In the literature on numerical cognition, the presence of the capacity to distinguish between numerosities by attending to the number of items, rather than continuous properties of stimuli that correlate with it, is commonly taken as sufficient indication of numerical abilities in cognitive agents. However, this literature does not take into account that there are non-numerical methods of assessing numerosity, which opens up the possibility that cognitive agents lacking numerical abilities may still be able to represent numerosity. In this paper, I distinguish between numerical and non-numerical methods of assessing numerosity and show that the most common models of the internal mechanisms of the so-called number sense rely on non-numerical methods, despite the claims of their proponents to the contrary. I conclude that, even if it is established that agents attend to numerosity, rather than continuous properties of stimuli correlated with it, an answer to the question of the existence of the number sense is still pending the investigation of a further issue, namely, whether the mechanisms the brain uses to assess numerosity qualify as numerical or non-numerical.

Keywords

numerosity, cardinality, number sense, numerical, counting, tallying

Despite the advances made in the field of numerical cognition in recent decades, the existence of the so-called "number sense" remains controversial. The number sense is usually described as the ability to perceive "numerical information in its nonsymbolic form" (Wilkey & Ansari, 2020, p. 76). Typical experiments probing the existence of the number sense consist of presenting participants with non-symbolic stimuli, such as two clouds of dots or two collections of snacks, and observing whether they are able to distinguish which has more elements (without being allowed to count, when participants are numerate humans).[1] The dominant view is that the ability to correctly identify the largest collection above chance level is evidence for the existence of the number sense.

One of the most influential arguments against this view claims that participants' behavior in experiments such as these are best explained by reference to continuous properties of stimuli, such as surface area and element size, rather than the number of discrete items (Gebuis et al., 2016; Leibovich et al., 2017). If this is the case, it is not necessary to postulate the existence of a number sense, since agents can succeed in such tasks above chance level by non-numerical

1) E.g., Hauser et al. (2000) tested rhesus monkeys with sets of apple slices; Antell and Keating (1983) tested 53-hour-old human neonates and Burr and Ross (2008) human adults with clouds of dots.

means. For example, considering that the area covered by a cloud of dots correlates with the number of dots in the cloud, participants may be able to identify the largest collection by attending to the covered area instead of the number of dots. In view of this, studies on the number sense have introduced controls to isolate the effect of continuous magnitudes that correlate with number.

One premise of this debate is that, if it is confirmed that participants are undoubtedly attending to the number of items in the stimulus—in one word, numerosity—to identify the largest collection, then they must have some sort of numerical ability—the number sense. That is why supporters of the existence of the number sense put much effort into showing that participants are truly relying on numerosity (e.g., Burr et al., 2018), whereas those who challenge this view concentrate their efforts on showing that continuous properties are more important (e.g., Gebuis et al., 2016; Leibovich et al., 2017; Mix et al., 2002; Yousif & Keil, 2020).

This premise, however, is false. As I show in this paper, there are non-numerical methods of evaluating numerosity. I show that numbers provide only one among several other methods of assessing numerosity. Therefore, evidence that participants are undoubtedly relying on numerosity to identify the largest collection does not necessarily imply that they have any kind of numerical ability, since the assessment of the numerosities involved can be achieved by a non-numerical method. I conclude that, even if it is confirmed that participants are undoubtedly attending to the number of items in the stimulus, this cannot be taken as sufficient indication of the presence of a number sense unless it is also shown that they do use a numerical method of assessing numerosity.[2]

The existence of non-numerical methods of assessing numerosity is rarely discussed in the literature. One exception is Núñez (2017), who distinguishes what he calls "quantical cognition" from numerical cognition, acknowledging the existence of non-numerical (quantical) processes through which cognitive agents can deal with numerosity. Núñez points out that the functions that enable quantical cognition are not numerical because they do not display the prototypical properties of numbers, such as exactness and symbolic reference. Núñez's view is confirmed by the analysis of what it means to be numerical conducted here.

This paper is meant to offer a conceptual contribution to the debate about the existence of the number sense. As pointed out by Núñez (2017) and acknowledged by Wilkey and Ansari (2020, p. 82), "[s]ome disagreements [about the existence of the number sense] may ultimately be a result of the varying ways that researchers define number, numerosity, quantity, and so on." Elsewhere I have analyzed the concepts of numerosity, numerousness, and number and their significance for the question of the existence of the number sense (dos Santos, 2022). In this paper, I analyze the concept of *being numerical*. The first step of this analysis consists of clearly distinguishing number from numerosity. This was already done in dos Santos (2022) and is recapitulated in Section 2, where I argue that number is best conceived of as a measurement scale of numerosity. In Section 3, I characterize methods of assessing numerosity that do not employ the numerical scale. I also present two necessary conditions for being numerical—precision and symbolic reference. Non-numerical methods of assessing numerosity are not precise and, if precise, are not symbolic. Once it is clear that there are various non-numerical methods of assessing numerosity, it becomes relevant to consider whether the mechanisms the so-called number sense uses to assess numerosity qualify as numerical. This is done in Section 4. As we will see, many models of the number sense postulate the existence of what are best seen as non-numerical mechanisms of assessing numerosity, despite the claims of their proponents to the contrary. Finally, in Section 5, I conclude that, in addition to the issue of which properties agents attend to when evaluating numerosities, whether continuous or discrete, the question of the existence of the number sense also depends on the issue of whether the mechanisms the brain uses to assess numerosity qualify as numerical or non-numerical.

Distinguishing Number From Numerosity

In this section, I am still not concerned with models of the number sense. The purpose of this section is to establish a conceptual distinction between number and numerosity. Noticing this distinction is the first step to understanding that

2) Certainly, the question of the existence of the number sense can still be answered negatively by showing that participants are not relying on numerosity at all, but rather relying on continuous properties that correlate with it, as Leibovich et al. (2017) and others have suggested.

there are non-numerical methods of assessing numerosity. Only later, in Section 4, this conceptual distinction is applied to analyze models of the number sense.

Numerosity is a technical term proper of the field of numerical cognition. More often than not, cognitive scientists do not present a definition of what they mean by this term. Those who do, however, introduce numerosity as a synonym of cardinality. For example, Butterworth (2005, p. 3) defines numerosity as "the number of things in a set" and explains: "[t]he term 'numerosity' is used here as the cognitive counterpart to the term 'cardinality' used by mathematicians and logicians." Following this practice, I use the terms 'numerosity' and 'cardinality' as synonyms in this paper.

In mathematics, the cardinality of a set refers to its "size," i.e., how many elements it has. Numbers provide one way of determining and expressing the cardinal size of a set, but it is a trivial fact in mathematics that the cardinality of a set can be determined and expressed without even mentioning numbers. The following example illustrates this point. Suppose that someone is organizing a meeting and wants to make sure that there are sufficient chairs for everyone. She can do this in at least two ways. By using numbers, she can count the people, count the chairs, and compare the outcomes; or, without even mentioning numbers, she can just ask people to sit down, each person in a single chair. If no person remains standing and no chair remains empty, she concludes immediately that both sets have the same size. If someone remains standing, then the set of people is larger than the set of chairs; inversely, if any chair remains empty, then the set of chairs is larger than the set of people. No number is involved in this procedure. It can be carried out recruiting only the notion of one-to-one correspondence or equinumerosity, which, despite its name, is defined without invoking numbers. (For a definition of equinumerosity, see Enderton (1977, p. 129).

As this example illustrates, cardinality can be assessed without using numbers by establishing a one-to-one correspondence between two collections. Let us call these the *target collection*, the one we want to have its cardinality determined, and the *model collection*, the one we use to represent the cardinality of the former. In the example above, the collection of chairs models the size of the collection of people (and vice versa[3]).

It may be argued that one-to-one correspondence just establishes that the two collections have the same cardinality, but does not reveal which cardinality they share. Supposedly, this could be done only by using numbers. However, even when we use numbers to assess and express cardinality, all we have is a comparison through one-to-one correspondence. Counting consists of establishing a one-to-one correspondence between the target collection and a collection of numbers. In counting, an initial segment of the sequence of natural numbers plays the role of model collection. For example, the collection {1, 2, 3, 4, 5, 6, 7, 8} models the size of the collection of planets orbiting the Sun because they share the same cardinality. But *which* cardinality do they share?

Set theory is the mathematical field where such questions are answered. What we see in set theory is that there is no privileged entity that could be called *the* cardinality of a set. In set theory, the cardinality of a set is established by definition and there are two different approaches to defining it. In one approach, the cardinality of a set A is defined as the class of all sets that can be put in one-to-one correspondence with A. For example, the cardinality of the set {a, b} is defined as the class of all pairs. In the other approach, one of the elements of the class of all sets that can be put in one-to-one correspondence with A is selected as a representative of the whole class, and then this selected set is said to be the cardinality of A. For example, once the set {0, 1} belongs to the class of all pairs (since it is a pair), it can be selected as a representative of this class and then be said to be the cardinality of any pair. The set {0, 1} happens to be the number two (according to von Neumann's definition) and then we can say that two is the cardinality of any pair (see Fraenkel et al., 1973, p. 96, for a more rigorous explanation). However, in principle we could have chosen any other pair as the representative of the class of all pairs, and then the cardinality of a pair could be expressed by a collection that is not a number.

More generally, we can use any set as a "yardstick" (model collection) to evaluate and express cardinalities. For example, I can use the set of pencils on my desk—let us call it P—for this purpose. Because there is an injective and non-surjective mapping from P onto the set of planets orbiting the Sun, I can say that the cardinality of the set of planets is greater than P's. The existence of a bijective mapping between P and the set of moons orbiting Mars allows me to say that the cardinality of the set of moons orbiting Mars is equal to P's. P is not a number, but I just used it

3) In principle, both collections can mutually play the role of *model* or *target* for each other. It is the intention of who is manipulating them that determines which is which.

to express cardinalities in the same way I could have used the number two. In other words, in the second set-theoretic approach mentioned above, I just selected a different, unusual set, P, to be the representative of the class of all pairs. One obvious advantage of using numbers is that almost everyone knows which set size two models, whereas only I know the cardinality P models. However, this does not prevent P from working as a "cardinality ruler."

At this point, it should be clear that numbers are just one among many possible cardinality rulers. In other words, numbers are just a commonly used measurement scale of cardinality (Stevens, 1939/2006, 1946). Cardinality (or numerosity) is the magnitude we measure by counting. Numbers provide one way of determining and representing cardinality, but there are also non-numerical ways of doing the same—in these ways, the model collection consists of elements other than numbers. Numbers provide a standard repository of model collections, but in principle we can use a variety of non-numerical collections to model cardinality (such as collections of chairs, pencils, beans, pebbles, etc.).

True enough, not all model collections are equally good. An important difference between using numbers and using other model collections is that numbers constitute a fully-fledged ratio scale for the measurement of cardinality. My set P of pencils can express only one cardinality with precision, whereas we have numbers to express every cardinality with precision. That number is a ratio scale for measuring cardinality was already noticed back in 1946 by S. S. Stevens, the influential psychophysicist who introduced the theory of measurement which is now standard in the field:

> Foremost among the ratio scales is the scale of number itself—cardinal number—the scale we use when we count such things as eggs, pennies, and apples. This scale of the numerosity of aggregates is so basic and so common that it is ordinarily not even mentioned in discussions of measurement (Stevens, 1946, p. 680).

Stevens is also responsible for introducing the term 'numerosity' (Stevens 1939/2006) as a synonym of cardinality. Although this term is still widely used in this sense in the literature on numerical cognition, Stevens's distinction between number (a ratio scale) and numerosity (the magnitude that ratio scale measures) seems to have been forgotten (dos Santos, 2022; Núñez, 2017).

Number is to numerosity as meter is to length and gram is to weight. It is easy to notice the difference between meter and gram—scales—and length and weight—magnitudes—because there are other scales for length and weight, such as feet and pounds. When it comes to cardinality, though, we do not have other scales, which makes the distinction less obvious. However, not having other scales for cardinality does not mean that there are not other, non-numerical ways of measuring it. For example, even if there were not any scale for length other than meter, we could still use, say, a stick to measure the length of a window, for example. By showing the stick, one could say "the window is that long." This is what happens when we use a non-numerical model collection to determine and express cardinality. Returning to the example above, suppose that the person who was organizing the meeting was able to provide exactly one chair for each person in the room. Then suppose that, after the meeting is finished and everyone is gone, someone comes in and asks: "how many people attended the meeting?" Even if the organizer had not counted the attendees, she could easily answer by pointing to the chairs: "that many."

The common mistake of conflating number, the scale, and numerosity, the magnitude, is understandable. Because numbers are our only scale for the measurement of cardinality, we usually refer to it indirectly, via its numerical measurement. For example, in the sentence 'the number of planets in the Solar System is eight,' the expression 'the number of planets' is used in place of 'the cardinality of the set of planets.' This is a clear case of metonymy, the figure of speech in which a word is used in place of another associated with it. A similar case of metonymy occurs when we say that "the square footage of the house is 1,200 sq ft." In this case, the expression 'the square footage of the house' is used in place of 'the area of the house,' which is the property that is being measured in square feet. Thus, when in a study on numerical cognition an infant is said to be able to perceive "the number of dots on the screen," this may be charitably interpreted as meaning that the infant is able to perceive the numerosity[4] of the set of dots on the screen.

4) More precisely, we should say that the infant is able to perceive the numerousness of the set of dots on the screen. 'Numerousness' was introduced as a technical term by Stevens in the same paper where he introduced the term 'numerosity' (Stevens, 1939/2006). Numerousness refers to the subjective perception of numerosity. See dos Santos (2022).

Certainly, the infant is not seeing numbers, as we do not see square feet when we attend to the area of a house. Scales are not the sort of thing that can be perceived.

If the distinction drawn in this section between number as a measurement scale and cardinality/numerosity as the magnitude numbers measure is correct, then the argument according to which the number sense is numerical because it allows for the perception of number is not tenable. What is perceived through the so-called number sense is the magnitude, i.e., numerosity. Whether the brain uses a numerical scale for assessing numerosity is a different question. I address this question in Section 4. Before that, though, let us go deeper into the distinction between numerical and non-numerical methods of assessing cardinality.

Numerical and Non-Numerical Methods of Assessing Cardinality

In this section, I am not yet concerned with models of the number sense. The purpose of this section is to establish a conceptual distinction between numerical and non-numerical methods of assessing numerosity. Only later, in Section 4, this conceptual distinction is applied to analyze models of the number sense.

As we saw above, establishing one-to-one correspondences between target collections and non-numerical model collections is a non-numerical method of assessing numerosity. The nature of the elements belonging to the model collection, however, is not the only difference between numerical and non-numerical methods. There is also an important procedural difference.

The counting procedure is governed by five rules, the so-called Counting Principles, as defined by Gelman and Gallistel (1978). The Counting Principles are listed in Table 1. Non-numerical one-to-one correspondence—let us call it *tallying* for short—is governed by only three of these principles, namely, One-to-one correspondence, Order irrelevance of Items and Abstraction. In their tallying versions, these principles say, respectively, that "each item of the target collection must be paired with a single item of the model collection," "the order in which items of the target collection are paired with items of the model collection is irrelevant," and "tallying applies to any collection of all sorts of objects."

Table 1

Counting Principles (Gelman & Gallistel, 1978)

Principle	Description
One-to-one Correspondence	Each item of the counted collection must be paired with one and only one number word.
Stable Order of Counting Words	The order in which number words are used for tagging items must follow a stable order, i.e., the order must be kept constant across different counting events.
Order Irrelevance of Items	The order in which items are paired with number words is irrelevant, i.e., the order may change across different counting events.
Abstraction	Counting applies to any collection of all sorts of objects, including sets formed by physical objects or ideas, and heterogeneous sets, formed by a combination of different kinds of objects.
Cardinality	The number word used for tagging the last item of a collection represents the cardinality of the whole collection.

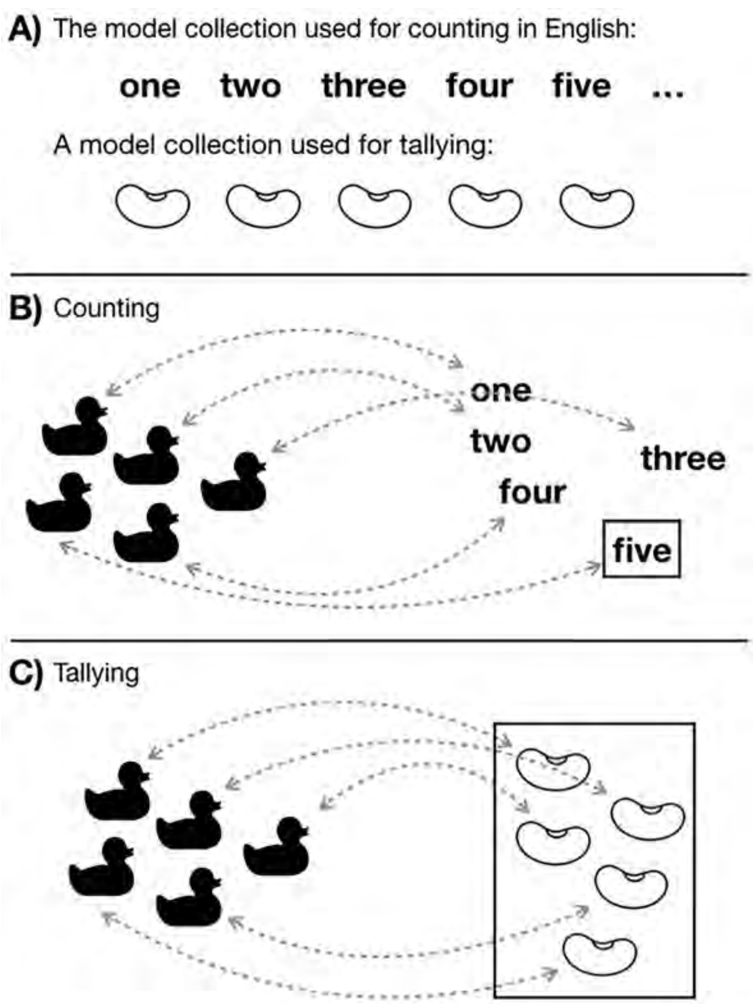
The two remaining principles—Stable Order of Counting Words and Cardinality—do not have a tallying version, since they do not apply to tallying. It would be counterproductive to use the items of a model collection in a stable order across different tallying events, since the resulting model collection will be the same, no matter the order. What is more, model collections used for tallying often consist of hardly distinguishable items, such as pebbles, sticks, or beans, which could make a principle such as Stable Order very difficult to follow. For the same reason, the Cardinality principle does

not apply to tallying, since the last item of the model collection tallied with the last item of the target collection may be hardly distinguishable from the items used previously in the process. In tallying, it is the whole model collection that represents the cardinality of the target collection.

This brings us to two structural differences between counting and tallying. First, in counting, the model collection consists of easily distinguishable items, which are used in a fixed order; in tallying, by contrast, the model collection may consist of hardly distinguishable, unordered items. Second, in counting, each item of the ordered model collection corresponds to a cardinality—that is, each number (or number word) is associated with a certain set size (a magnitude). In tallying, by contrast, single items are not associated, *per se*, with any cardinality; only whole model collections represent cardinality. Figure 1 illustrates these structural differences.

Figure 1

Structural Differences Between Counting and Tallying



Note. In A), the model collection of English number words, used for counting, is compared with a model collection consisting of beans used for tallying. The former is ordered and its elements are easily distinguishable from each other; the collection of beans, by contrast, is unordered and its elements are hardly distinguishable from each other. In B), the collection of rubber ducks was counted; the number word 'five,' which was the last item of the model collection paired with an item of the target collection, represents the cardinality of the whole collection of rubber ducks. In C), a model collection of beans was used to assess the cardinality of the collection of rubber ducks; no single bean can represent the cardinality of the whole target collection, since they are indistinguishable from each other; only the whole collection of beans can do.

This second difference explains why counting is so superior to tallying as a way of denoting and expressing cardinality (although both are equally precise). Since counting uses an ordered model collection—numbers or the sequence of number words—in every counting event where the target collection has the same size, the last number word used is always the same. Thus, that number word becomes associated with that specific cardinality. Furthermore, the model collection used in counting is standard: within a linguistic community, everyone uses the same sequence of words. This facilitates communication, since every numerate person knows this model collection as well as the position each number word occupies in the sequence and, consequently, the set size it represents. Thus, mentioning a number word suffices for referring to a cardinal size. Surely, the use of numbers words is not essential here; body-part counting systems (such as those described in De Vries, 1994, and Owens et al., 2018, obtain the same benefits by using an ordered model collection of body parts (fingers, arm, shoulder, and so on). What is essential here is the use of a standard ordered model collection. Tallying systems, which use unordered model collections, cannot do the same. By using non-numerical one-to-one correspondence, if one wants to refer to a cardinal size, she needs to display a whole model collection (recall the example of chairs and people in Section 2, where the meeting organizer points to the collection of chairs to refer to the cardinality of the set of attendees).

The fact that each number or number word represents a cardinal size is commonly referred to as the symbolic character of number, often mentioned as an essential property of numerical systems (e.g., Núñez, 2017). Each number word is a symbol for a precise cardinal size. Tallying does not use symbols, in the specific sense of what a symbol is. The standard definition of 'symbol' in semiotics is due to Charles Sander Peirce. According to Peirce (1994, §2.247), a sign is anything that can be used to represent something else. There are three types of signs: icons, indices, and symbols. An icon is a sign whose association with its signifier is established by a relationship of resemblance. For example, this sign ☾ is an icon that refers to the moon, because its form resembles a half-moon. An index, in turn, is a sign whose association with its signifier is direct or natural because it is produced by the signifier. For example, smoke is an index of fire, and the moonlight is an index of the moon. A symbol, by contrast, is a sign whose association with its signifier is arbitrary; it is established by a cultural convention. For example, the word 'moon' is a symbol; it does not resemble the moon in any sense, nor is it produced or influenced by the moon. It is associated with the moon by mere linguistic convention. It is easy to see that the relationship between a number word and a cardinal size is symbolic, whereas the relationship between, say, a model collection of beans and the cardinal size it represents is iconic. For example, in Figure 1C, the collection of beans represents the cardinal size of the collection of rubber ducks by instantiating it (iconic resemblance), whereas in Figure 1B 'five' refers symbolically to the same cardinal size (arbitrary association).

Symbolic reference is an essential property of numerical systems because the elements of numerical model collections usually cannot refer to the cardinality each one of them represents by iconic or indexical means. For example, the word 'five' in English, the thumb in the Kombai body-part counting system (De Vries, 1994) and the set $\{\{\{\{\{\emptyset\}\}\}\}\}$ (the number five in Zermelo's definition) do not resemble, nor are they intrinsically related in some natural way with the cardinality they represent.[5] Thus, the only way of associating them with their signifier is by arbitrary association, i.e., symbolically.

Once a numerical system is established, however, the association between the elements of its model collection and cardinal sizes is no longer totally arbitrary. According to Wiese (2003), these associations follow a mechanism she calls "system-dependent linking." This mode of semantic association consists of using the relationships the symbols bear with each other within the symbolic system to model certain relationships between the objects we want to refer to in the world. For example, Wiese considers the English sentence 'The dog bites the rat.' The positions of the symbols 'the dog' and 'the rat' in the sentence are used to model a relationship between the dog and the rat in the world. In the sentence, 'the dog' comes before 'the rat', and from this fact we know that the dog is the agent and the rat is the patient of the action in the world. A symbol-to-symbol relationship models the object-to-object relationship we want to refer to.[6]

The same mode of semantic linking takes place when we use numbers to represent cardinality. The position numbers occupy within the natural-number structure is used to model the relationship of relative size between collections.

5) As noted by an anonymous reviewer, Zermelo's definition of the natural numbers "might be interpreted as having some iconic flavor" to our eyes, since the couples of parentheses are somewhat analogous to tallies. However, from *within* set theory, the cardinality of the Zermelo sets is always the same—one—and therefore there is no resemblance, in this sense, between a Zermelo number (greater than one) and the cardinality it represents.

Through system-dependent linking, the relationship “comes before” (predecessor) between numbers translates into the relationship “has less elements than” between collections, and the relationship “comes after” (successor) between numbers translates into the relationship “has more elements than” between collections (Wiese, 2003). This is also true of sequences of number words or body parts, where the position words/body parts occupy in the sequence is equally used to model the relationships “has more/less elements than” between collections.

Wiese points out that numbers are used not only to express cardinality, but also to assign names. This is the case, for example, when we call bus lines by numbers. In such cases, the symbolic association between numbers and the objects they designate is also established by means of system-dependent linking. However, in such cases the relationship exploited within the system of natural numbers is not the one of successor/predecessor, but the relationship of identity. “[W]hen we distinguish different bus lines by different numbers,” Wiese (2003, p. 385) explains, “the numerical relation ‘=’ (or ‘≠’) is associated with the empirical relation ‘is identical (or non-identical) with’.” Thus, we know that the bus line 311 is different from the bus line 305 because 311 is different from 305, but no size relationship can be inferred from this.[7] Numbers can be cardinality rulers only when the successor relationship is the one used to establish system-dependent links with collections (I come back to this point below, when discussing Laurence and Margolis’s (2007) model of the number sense).

Up to this point, we have seen two methods of assessing cardinality and two differences between them. The methods are counting (numerical) and tallying (non-numerical). One difference between them is procedural: counting is governed by five rules (the counting principles), whereas tallying is governed by only three of these. The other difference pertains to semiotics: the numerical method uses symbols for representing cardinal sizes, whereas tallying uses iconic representations. But there is an important similarity between the two: both yield exact representations of cardinal sizes. Each numeral, number or model collection represents exactly and precisely one cardinal size. Are there imprecise methods of representing cardinality? The answer is yes.

In the literature on numerical cognition, the accumulator has been proposed as an imprecise device for the representation of numerosity (Gallistel & Gelman, 2000; Meck & Church, 1983). The accumulator is an imprecise form of tallying, where one-to-one correspondence is replaced by one-to-*some* correspondence. The accumulator can be thought of as a metaphorical beaker into which a cupful of water is poured for each item in the target collection. The resulting water level represents the cardinality of the collection: the bigger the collection, the higher the final level of water in the beaker. Just as tallying systems, the accumulator represents numerosity iconically. Imprecision is added to the system whenever there is any variability in the amount of water each cup contains. Small differences in the amount of water add up, so that large numerosities are represented with less precision than small numerosities. For example, a collection of three items and a collection of two items produce markedly different final levels, since the variability two or three cups of water add to the system is not large enough to make the resulting levels in the beaker look similar. With larger collections, though, these small variations accumulate and then the final levels of water produced by collections of, say, 23 and 24 items become hardly distinguishable.

The accumulator owes its plausibility as a model of the number sense from the fact that it captures the observation that the capacity of the number sense to distinguish between two numerosities decreases as the ratio between them approaches 1:1. The imprecision of the number sense makes room for an often-mentioned objection to its numerical character. I address the argument from imprecision and Clarke and Beck’s (2021) response to it in the next section. For now, it is sufficient to recognize that the accumulator uses an imprecise non-numerical method of representing cardinality/numerosity (in other words, an imprecise form of tallying).

In this section, we saw three methods of assessing and representing cardinality: counting, tallying, and accumulation. We also saw two essential properties of numbers with regard to their use as cardinality rulers: they are precise—each number represents a single cardinality precisely—and they do so by symbolic reference, relying on the successor

6) Notice that the choice of a symbol-to-symbol relationship to model a certain object-to-object relationship is arbitrary. For example, the positional relationship that in English means the agent-patient relationship can mean, in other languages, a different relationship between objects or no relationship at all.

7) This does not mean that properties of numbers other than identity cannot be used to represent other properties of bus lines. For example, as an anonymous reviewer pointed out, proximity between numbers can be used to represent proximity between the areas covered by the buses.

relationship to model the “has more/less elements than” relationship between collections. Given these criteria, among the methods considered here only counting (with number words, body parts or any other ordered model collection) qualifies as numerical. Tallying is not numerical because it does not satisfy the second criterion (it is iconic) and accumulation is not numerical because it does not satisfy both criteria (it is imprecise and iconic). Table 2 summarizes the conclusions drawn in this section.

Table 2

Comparison of Three Different Methods of Assessing Cardinality

Method	Precise	Form of representation	Procedural rules	Numerical
Tallying	yes	iconic	1-to-1 correspondence, Order Irrelevance, Abstraction	No
Counting	yes	symbolic	1-to-1 correspondence, Stable Order, Order Irrelevance, Abstraction, Cardinality	Yes
Accumulation	no	iconic	1-to-some correspondence, Order Irrelevance, Abstraction	No

Note. This list is not meant to be exhaustive.

The Numerical Character of the Number Sense

There are two ways of viewing the number sense as numerical: (1) the number sense may be said to be numerical because it putatively allows for the perception of numbers; or, (2) it may be said to be numerical because it putatively uses numbers (or representations thereof) to assess and represent numerosities.

According to the first point of view, numbers are perceptual properties of perceptual collections, and the number sense represents number just as perception represents other perceptual properties, such as color and distance. In the final paragraph of Section 2, I mentioned that this view is mistaken, since it is conflating the scale (number) with the magnitude (numerosity). Number is a measurement scale of numerosity, and measurement scales are not the sort of thing that could be perceived. Just as we do not perceive square feet, but area, we do not perceive numbers, but numerosity.[8] And, since perception of numerosity may rely on non-numerical methods, perception of numerosity does not imply, *per se*, any numerical ability nor the possession of numerical concepts.

The second point of view does not make this mistake; it admits that the magnitude the number sense represents is numerosity, and it claims that the number sense is numerical because it uses numbers (or representations thereof) to represent numerosity. However, researchers who adopt this point of view often make another mistake. They uncritically assume that every method of assessing and representing numerosity qualifies as a numerical method. In fact, there is no clear understanding in the literature of the existence of non-numerical methods of doing this.[9]

These points of view are rarely clearly articulated in the literature. Two exceptions are Clarke and Beck (2021), who take the first point of view, and Gallistel (2018), who takes the second. Gallistel assumes “the existence of numbers in the brain—in animals from insects to humans” (Gallistel, 2018, p. 1) and claims that these mental numbers are used to assess and represent numerosities. He writes:

> Measurement processes establish numerical reference by mapping from magnitudes in the world (numerosities, lengths, durations and so on) to the numbers with which we represent those magnitudes [in the brain] (Gallistel, 2018, p. 1).

8) More precisely, numerousness; see Footnote 2.

9) This is illustrated, for example, by Gallistel and Gelman’s (1992) belief that accumulation is a form of counting. More on this below.

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Clarke and Beck (2021), by contrast, deny that numerosity is the magnitude the so-called number sense represents. They claim that 'numerosity' is an ill-defined term (but they do not consider Stevens's, 1939/2006, definition; see Dutilh Novaes & dos Santos, 2021). According to them, the numerical character of the number sense comes from the fact that it represents number. Therefore, the internal mechanisms the number sense uses to assess and represent what they call "number" are irrelevant, since the numerical character of the number sense comes from its function. In other words, in their opinion it does not matter how numbers are represented in the mind; what matters is that the mind represents number.

> To suppose otherwise is to confuse what the system is doing (e.g., functioning to track and represent numbers—the computational level description with which we are concerned) for a specific account of how it does this (an algorithmic level description (Marr, 1982)) (Clarke & Beck, 2021, Section 2).

However, as we have seen, to suppose that number is the perceptual property the so-called "number sense" represents is to confuse the measurement scale for the magnitude it measures. Contrary to Clarke and Beck's opinion, 'numerosity' is a well-defined term, and when we clearly distinguish number (the scale) from numerosity (the magnitude), we realize that purely computational models of the number sense should be seen as neutral regarding its numerical character. Even if we assume that it is the function of a cognitive system that determines its conceptual content, the observation that the function of the so-called "number sense" is to represent numerosity only means that its contents are numerosity representations. But, given that there are numerical and non-numerical methods of building numerosity representations, it may well be that the so-called "number sense" builds numerosity representations by non-numerical means. If this is so, it is correct to characterize it as a sense of *numerosity*, but not a sense of *number*, since numbers would not be involved in it at all.

Computational models of cognitive systems describe the problem the system aims at solving and provide a mathematical function specifying the relationship between input and output (given stimuli x, behavior y is displayed). At this level, cognitive systems are taken as black boxes; no claim is made about internal processes, the methods it uses to transform inputs into outputs (Marr, 1982/2010). Thus, computational models of the number sense may, at best, demonstrate that it represents numerosity, but they are silent about the mechanisms used to build such representations, which is the most relevant point when it comes to assessing its numerical character. This is why we need to consider algorithmic models in order to determine whether the so-called number sense is really numerical.

True enough, certain computational models may provide some clues on the algorithmic level. For example, the model presented by Cheyette and Piantadosi (2020) implies that a single system may be responsible for both subitizing and estimation (in contrast to the dominant view, according to which these competences are implemented by two different systems; Feigenson et al., 2004). In the same vein, Cheyette et al. (2021) claim that numerosity perception may emerge from a system dedicated to the identification and representation of individual objects in a scene. Both models put constraints on the mechanisms that could assess numerosity, but say nothing about the specific methods used to represent numerosity.

This is what the algorithmic level is meant to do. Algorithmic models describe the strategies and processes a cognitive system uses to solve a problem (Marr, 1982/2010). A few algorithmic models of the number sense have been proposed. In the following, I discuss some of them in order to illustrate which aspects must be considered when judging whether the proposed mechanisms qualify as numerical.

The Numerical Character of Algorithmic Models of the Number Sense

To date, the prevailing view is that the ability to perceive numerosities is implemented by two different systems: one system for numerosities smaller than four or five, whose fast and accurate perception is called "subitizing," and one system for larger numerosities, whose fast and only approximate perception is called "estimation" (Feigenson et al., 2004; Gilmore et al., 2018; Knops, 2020).

According to the most common algorithmic models of subitizing, this ability relies on a domain-general mechanism for individuation of multiple objects in parallel, called the object tracking system (OTS) (Trick & Pylyshyn, 1994). This mechanism is believed to consist of three or four "memory slots" or "object files" onto which objects individuated in

stimuli are mapped. This limited storage capacity implies that the OTS is not able to track more than three or four objects in parallel—that is why subitizing is also limited at three or four. It is easy to see that subitizing represents numerosity by establishing a one-to-one correspondence between objects in the stimulus and the OTS's memory slots: each object is allocated to one slot (up to three or four objects), and the resulting "map" represents the cardinality of the collection iconically, reproducing the operation of tallying techniques (Wiese, 2003, p. 387). Some researchers acknowledge that OTS representations cannot count as representations of number:

> The object tracking system . . . is thought to represent numerical information only in an implicit way: in this system, there is no summary representation of 'two'; instead, infants form a mental model of two objects by recruiting two attentional indexes or 'object files' (Izard et al., 2008, p. 281).

In line with the conclusion drawn in Section 3, that one-to-one correspondence (tallying) is non-numerical, Simon (1997) contends that subitizing in infants and non-human animals is best conceived of as a non-numerical ability, because discrimination of numerosities within the subitizing range demands nothing more than establishing one-to-one mappings and making same/different discriminations. To illustrate Simon's point, let these diamonds represent the OTS's memory slots of a subject: ◇◇◇◇. When the subject is presented with a collection of two objects, two slots are occupied and this information is stored in its memory: ◆◆◇◇. After that, when presented with a collection of three objects, three slots are occupied: ◆◆◆◇. To tell the difference between their numerosities, the only thing the subject has to do is to compare whether ◆◆◇◇ and ◆◆◆◇ are the same or different. The subject does not need to have any numerical understanding for doing so, since the task can be solved by one-to-one correspondence. Strictly speaking, then, subitizing is non-numerical. Subitizing becomes numerical only when numerate humans map OTS's working memory representations onto the number scale and express the evaluated cardinality by means of number words. But for innumerate humans and non-human animals, which do not make this last step, subitizing is non-numerical (judging by the OTS model; I consider Laurence and Margolis's (2007) alternative model of subitizing below).

The most common algorithmic model of estimation is the already mentioned accumulator. At the implementation level, the metaphorical "cupful of water" translates into neural activity: each object in the stimulus produces a variable quantity of neural activity, which is normalized to an approximately constant quantity. These normalized, approximately constant quantities are then summed by "accumulation neurons." The level of activation in these neurons corresponds to the level of water in the metaphorical beaker: the higher the activation, the higher the numerosity represented (Dehaene & Changeux, 1993; Dehaene, 2011, p. 251). Due to small variations in the normalized amount of activation produced by each object in the stimulus, the representation of numerosity is imprecise. Not only because of its imprecision, but also because accumulation neurons represent numerosity iconically (Wiese, 2003, p. 387), the accumulator does not qualify as numerical, as we saw in Section 3. Indeed, the iconic nature of the accumulator is recognized in the literature, albeit through the adjective 'analog.'

> Unlike the computer, it [the brain] does not rely on a digital code, but on a continuous quantitative internal representation. The brain is not a logical machine, but an analog device. Randy Gallistel has expressed this conclusion with remarkable simplicity: "In effect, the nervous system inverts the representational convention whereby numbers are used to represent linear magnitudes. Instead of using number to represent magnitude, the rat [like the Homo sapiens!] uses magnitude to represent number" (Dehaene, 2011, p. 220).

In this passage, Dehaene and Gallistel make the mistake of confounding number with numerosity. Correcting the final sentence, we could say that "instead of using number to represent magnitude, the rat, like the Homo sapiens, uses magnitude to represent magnitude." In the accumulator model, one magnitude—level of activation—is used to represent another magnitude — numerosity. This is the analog or iconic aspect of the accumulator, which contrasts with the symbolic mode of representation of numbers. Accordingly, the cognitive system responsible for estimation is often described as the "Analog Number System" (ANS). This is a misnomer, however, since there are no such things as "analog numbers;" numbers can represent numerosity only symbolically.

To be fair, in Dehaene and Changeux's (1993) model of the ANS, there is a final step that translates analog representations of numerosity into something closer to symbolic representations. In addition to accumulation neurons,

the model includes neurons that function as "numerosity detectors;" they receive input from accumulation neurons and fire only when their level of activation surpasses a certain threshold. Thus, certain neurons will be tuned to fire when the activity in accumulation neurons corresponds to approximately five, whereas other will be tuned to fire when the activity corresponds to approximately six, and so on. Each numerosity detector has a preferred numerosity, but they also fire with neighbor numerosities (their activity decreases as the distance from their preferred numerosity increases), reflecting the fact that estimation is imprecise. These numerosity detectors correspond to the "number neurons" found in the brain of monkeys (Nieder, 2016; Nieder & Miller, 2003).

> "Number neurons" or "numerosity detectors" resemble numbers in one aspect, namely, in that they represent numerosity symbolically. But "number neurons" differ from number in that they are not precise; they fire when presented to neighbor numerosities, whereas each number represents a single numerosity. Since the so-called "number neurons" do not satisfy one of the two criteria for being numerical (precision), the designation "number neuron" is misleading.

Nothing new here. The imprecision of the so-called number sense is often-mentioned by those who deny its numerical character (e.g., Carey, 2009, Marshall, 2018, and Núñez, 2017). As Clarke and Beck (2021, Section 5) put it, "the argument from imprecision" is "perhaps the most prominent critique of the ANS's capacity to represent number." In defense of the numerical character of the ANS, Clarke and Beck (2021) suggest what would be a good refutation of the argument from imprecision, were they not conflating number and numerosity. They compare the perception of what they call "number"—more properly called numerosity, as we saw above—with the perception of other magnitudes such as distance. Distances, just like numerosities, are exact in nature. Although the perceptual system represents distances with different degrees of imprecision, this does not prevent us from acknowledging that the perceptual system does represent distances. The same should go for numerosities; even when the number sense represents numerosities imprecisely, it is still representing numerosity. So far, so good. However, since Clarke and Beck do not distinguish number from numerosity, they wrongly conclude that the ANS should be said to represent *number*, even if imprecisely. This does not follow when we take into account that the magnitude the ANS represents is *numerosity*, and that number is a measurement scale thereof. In particular, number is an exact measurement scale. Thus, although it may be correct to say that the ANS represents numerosity, even if imprecisely, the very fact that it represents numerosity imprecisely shows that it is not using numbers as its measurement scale. Otherwise, its representations of numerosity would be exact.

Precision is not a necessary condition for numerosity representation, but it is a necessary condition for any method of representing numerosity that is to be classified as numerical. Clarke and Beck's response to the argument from imprecision does not change this requirement. Once we understand the difference between number (scale) and numerosity (magnitude), we understand that what the argument from imprecision shows is that the imprecision of the ANS implies that it does not use a numerical method, otherwise it would be precise. If this is so, "Approximate Number System," a common alternative reading of the acronym ANS, is also a misnomer, although this acronym may remain unchanged if it is taken to stand for "Approximate Numerosity System."

Whereas Dehaene and Changeux's accumulator-based model of the ANS does not qualify as numerical, Gallistel and Gelman (1992) provide an interpretation of the accumulator in which it is claimed to be both symbolic and precise and, therefore, numerical. Accordingly, they claim that the accumulator implements all the five counting principles:

> The operation of this mechanism [the accumulator] conforms to the principles that define counting processes (Gelman & Gallistel, 1978). The mechanism pairs states of the accumulator (numerons) with the items in the set being counted. The pairing is one-one, because the pulse former gates a burst of impulses to the accumulator once and only once for each item in the enumerated set. The order in which the states of the accumulator are used is stable (does not vary from one count to the next), because the ordering of magnitudes is isomorphic to the ordering of numbers. The final state of the accumulator—the final numeron in the sequence of numerons used in a count—is used as a representative of the numerosity of the set (the cardinal principle). Also, the current content of the accumulator is used as a representative of the numerosity of the set so far counted in decision processes that involve comparing a current count to a remembered count (Gallistel & Gelman, 1992, pp. 51-52).

Gallistel and Gelman define numerons as "mental representatives of numerosities," and take numerons to be symbols (Gallistel & Gelman, 1992, p. 44). But, as made clear in the above passage, numerons are states of the accumulator. As such, they are not symbols, but icons. For this sole reason, their accumulator does not qualify as numerical. The iconic nature of numerons makes all the difference for the Stable Order and Cardinality principles. Stable Order says that elements of the model collection are paired with elements of the target collection in a stable order across counting events. The model collection here consists of pulses which, in Gallistel and Gelman's precise accumulator, are exactly equal to each other. Since the pulses are completely indistinguishable, it does not make sense to say that they are being used in a stable order. In fact, the order does not matter here. But notice that they say that it is the states of the accumulator, rather than the pulses, which form a stable order. However, such modified statement is no longer the Stable Order principle, since the Stable Order principle governs the order of use of the items of the model collection, and not its intermediate states. They commit a similar misinterpretation with regard to the Cardinality principle. They say that the final state of the accumulator represents the cardinality of the target collection. But the Cardinality Principle says that it is the last item of the model collection used, and not the whole model collection, that represents the cardinality of the target collection. If the Cardinality principle were to be followed by Gallistel and Gelman's accumulator, the final pulse should represent the cardinality of the target collection. But since it is indistinguishable from the others, it cannot do this.

A remark is in order. If Gallistel and Gelman's precise accumulator is supplemented with a final step, of the sort found in Dehaene and Changeux's (1993) model, which translates analog (iconic) representations of numerosity into symbolic representations, then it could qualify as numerical more easily. The only additional requirement is that these "numerosity detectors" be precise: the final level of the precise accumulator should be translated into a symbol representing an exact cardinality. Such a device would be both precise and symbolic and, therefore, qualify as numerical. This shows that counting is not the only numerical method of assessing cardinality. A precise tallying device coupled with a precise means of translating icons into symbols can do as well. Notice that the translation of icons into symbols does not need to involve counting. For example, referring back to the metaphorical water accumulator, a graduated beaker can do this without counting. This truly deserves the qualification of numerical because the symbols in the beaker resemble numerals in that their relationship with cardinalities is established through a similar kind of system-dependent linking: the relationship the signs engraved on the beaker have with each other (higher/lower) models the relationship between cardinalities (larger/smaller). The difficulty, when it comes to modeling the number sense, is that such a mechanism could hardly account for the variability observed in the estimation of numerosity.

Another model of the number sense worth considering here is the one proposed by Laurence and Margolis (Laurence & Margolis, 2007; Margolis, 2020; Margolis & Laurence, 2008). Their model is confined to subitizing and they claim that it is truly numerical. Margolis (2020) calls it the "small number system." In Margolis's words, this system "includes a small stock of discrete representations that are causally responsive to particular numerical quantities and that have the function of responding to these quantities" (Margolis, 2020, p. 114). The system contains one symbol for each numerosity up to four elements, and is implemented through a neural network whose inputs come from the OTS. The network has four input nodes and four output nodes. The connections between them are weighted so that the activation of any single input node (corresponding to one object indexed by the OTS) suffices to activate the "one" output node, the activation of any two input nodes suffices to activate the "two" output node, and so on. "Under this arrangement, *n* individuated entities would cause the activation of a unique symbol corresponding to that precise quantity" (Margolis, 2020, p. 114).

The "small number system" is both precise and symbolic. This system would qualify as numerical, were it not modeling a different, non-numerical relationship. Recall from Section 3 that, when numbers are used as symbols to represent numerosities, the sign-to-sign relationships predecessor/successor are used to model the relationships "has less/more elements than" between collections. The symbols of the "small number system," however, do not bear predecessor/successor relationships between them, nor do they bear a similar relationship that could be exploited for the same purpose, such as the higher/lower relationship between symbols engraved on a graduated beaker. Margolis (2020, pp. 114-115) recognizes this disanalogy between the "small number system" and numbers:

> Notice that on this minimal account, the symbols for small numerical quantities need not be inherently ordered, and there need not be a procedure that ensures that three is represented as more than two, or two as more than one (unlike conventional counting terms).

Since the symbols used in this system are not ordered, there is no sign-to-sign relationship in this system which could be used to model the relationships “has more/less elements than” between cardinalities. What Laurence and Margolis’s system is modeling is the relationship of identity between numerosities. The system simply assigns a name to each numerosity (up to four), in the same way that we assign numerals as names to bus lines. Different symbols (names) go to different numerosities, and that is all. From this, the agent cannot infer anything about the size of the collections, except that they are equal or different. In terms of the classification of measurement scales used in measurement theory, the scale Laurence and Margolis’s system uses is *nominal*, whereas numbers are a *rational* scale (Stevens, 1946). In Laurence and Margolis’s scale, the relationship of identity between symbols is used to model the relationship of identity between numerosities, so that two numerosities that receive different names are different and two numerosities that receive the same name are equal. That is all the cognitive agent needs to know. No numerical ability is required.

In this section, I applied the criteria for being numerical we saw in Sections 2 and 3 to judge whether models of the so-called “number sense” qualify as numerical. I concluded that, among the models analyzed, only a precise accumulator combined with precise numerosity detectors would qualify as truly numerical. A model such as this has never been proposed, though, since it is unable to account for the imprecision of estimation. The more realistic models of the number sense considered here do not qualify as numerical.

Conclusion

I started this paper by showing that number is best conceived of as a measurement scale of cardinality. Cardinality, a.k.a. numerosity, is the magnitude. Consequently, if there exists a number sense, it does not allow for the perception of number, but numerosity, since what we perceive through our senses are magnitudes, and not their measurement scales. Thus, “number sense” is a misnomer, and it is best described as a sense of *numerosity*.

Perhaps the expression “number sense” could still be used to describe a sense of numerosity if representations of numerosity were built through numerical means. However, as we have seen, it is possible to build numerosity representations by non-numerical means and, in fact, most of the algorithmic models proposed for the so-called number sense do not display the essential features of numerical methods. Whereas symbolic reference and precision in the representation of the relationship “larger/smaller than” between cardinalities are the hallmarks of numerical methods, the algorithmic models of the so-called number sense use iconic or imprecise methods.

These conclusions have an important consequence for the debate about the existence of the number sense. This debate has focused on establishing whether cognitive agents rely on continuous or discrete properties (numerosity) of stimuli in order to distinguish the size of collections. The question of the existence of the number system can be answered negatively if it turns out that agents rely exclusively on continuous properties. However, if the final conclusion is that agents do rely on numerosity, then, I submit, the question of the existence of the number sense remains open. After all, agents may be using non-numerical methods to assess and represent numerosity. Only conceptual clarity (of the sort put forward here) and the further empirical investigation of the mechanisms the brain uses to assess and represent numerosity can answer this latter question.

Conceptual clarity should not be seen as mere philosophical pedantry. Evidence for the existence of a capacity to perceive numerosity accumulates at the same time that correlation studies about the relationship between this capacity and symbolic arithmetical skills are inconclusive or find only a weak correlation (Chen & Li, 2014; Fazio et al., 2014; Qiu et al., 2021; Schneider et al., 2017). Insofar as the literature regards both sets of abilities as numerical, the observation that there may be only a weak correlation between them is an unexpected outcome. One explanation for this may be that what is called the “number sense” is, in fact, a non-numerical capacity to track numerosities. This capacity may allow for the perception of numerosity, but if it does so by relying on non-numerical methods, it comes as no surprise that it is not strongly correlated with truly numerical abilities. True enough, the acquired ability to count relies on the genetically evolved ability to perceive numerosity: we could not count if we were not able to subitize singletons

(since we could not establish one-to-one correspondences) and it could be difficult for us to endow high numerals with meaning if we were not able to estimate the numerosity of large collections (Dehaene, 2011). This relationship between the two sets of abilities may explain the weak, rather than zero, correlation between them.

Rather than mere pedantry, conceptual analysis can shed light on relevant empirical questions.

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