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Teachers' Reflections on Their Knowledge and Practice of Teaching High School Functions

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Abstract

We report a characterization of the reflections made by high school mathematics teachers when discussing their resolutions of three numerical tasks related to the concept of function understood as a mathematical procept. Thematic analysis was used to code and categorize the reflections. Six teachers with at least 5 years of professional experience and an average age of 30 years participated. The results show that the teachers reflected on their actions in the tasks; their content knowledge; their teaching practice; and their professional training. As a result of these reflections, they recognized the need to improve their knowledge and practice of teaching functions in a more proceptual sense. The results contribute to deepen in how to make teachers reflect and incorporate a much more structural vision of functions in their teaching practices and, at the same time, incorporate the importance and role of their professional training in their reflections.

Keywords: Reflections, Teachers, Functions, Knowledge, Practice.

Reflexiones de Profesores sobre su Conocimiento y Práctica de Enseñanza de las Funciones en Bachillerato

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Resumen

Se reporta una caracterización de las reflexiones realizadas por profesores de matemáticas de bachillerato al conversar sobre sus resoluciones de tres tareas numéricas relacionadas con el concepto de función. Se usó el análisis temático para codificar y categorizar las reflexiones. Participaron seis profesores con al menos 5 años de experiencia profesional y una edad promedio de 30 años. Los resultados muestran que los profesores reflexionaron sobre sus acciones en las tareas; su conocimiento del contenido; su práctica de enseñanza y su formación profesional. Los resultados contribuyen a profundizar en cómo hacer para que los profesores reflexionen e incorporen una visión mucho más estructural de las funciones en sus prácticas de enseñanza y, al mismo tiempo, incorporen a sus reflexiones la importancia y papel de su formación profesional.

Palabras clave: Reflexión, Profesores, Funciones, Conocimiento, Práctica.

The calculus taught in schools at the pre-university and university levels is the result of the work of great mathematical thinkers, diverse ruptures and affiliations between forms of thought, such is the case of the ideas of infinity, limit and function, as well as their operability (Boyer, 1959; Edwards, 1979; Grattan-Guinness, 1980). In other words, calculus is the result of a long process of conceptual development that involved overcoming obstacles, many of which prevail in its teaching and learning (Artigue, 1995; Dreyfus et al, 2021; Hitt, 2003; Star and Smith, 2006; Thompson and Carlson, 2017).

The concept of function is a fundamental piece in the historical development of calculus and in all modern mathematics, its teaching and learning (Youschkevitch, 1976; Ponte, 1992; Zuccheri and Zudini, 2014). Studying functions is important "because they enable students to understand other mathematical ideas and connect ideas across different areas of mathematics" (NCTM, 2000, p.15). However, research reveals several deficiencies in students' knowledge of this concept (Adu-Gyamfi and Bossé, 2014; Bardini et al, 2014; Dreyfus, 2002; Dubinsky, 2013; Eisenberg, 2002; Hitt and Gonzalez-Martin, 2016; Oehrtman, Carlson and Thompson, 2008; Zandieh et al, 2017), and in teachers (Chesler, 2012; Hatisaru and Erbas, 2017; Hitt, 1994; Trevisan et al, 2020; Watson and Harel, 2013).

Hitt (2017) points out that research has provided important perspectives on the problems of calculus learning, but the problem persists on the teaching side. That is, the teaching of calculus in general and of functions, has focused on strengthening the procedural and to a lesser extent the conceptual, and even less on establishing relationships or transit between both parts. By way of example, many students who finish their calculus courses are competent in performing algebraic procedures to evaluate functions and to calculate derivatives and integrals; however, they do not achieve a conceptual understanding (Artigue, 2000). Espinoza-Vázquez et al. (2018) and Pino-Fan et al. (2019), report that the teacher's knowledge of functions corresponds to that of an operational, algebraic and correspondence meaning between sets and its school treatment is operational, thus limiting its understanding as a mathematical procept (Sfard, 1991).

A simple but significant exercise that the authors of this paper have carried out with several mathematics students at the end of their first or second year of university calculus courses, as well as with high school calculus teachers

and students, is to ask them why one can be sure that when adding or multiplying two functions the result is another function? Most students and professors do not usually give an adequate and conclusive answer to this question. In other words, they evidence fragility in the theoretical-conceptual and operational foundations of functions, they omit the closure property of real numbers. Functions are then assumed and operated as formulas, leaving aside their dual sense and their foundations. "In order to speak about mathematical objects, we must be able to deal with products of some processes without bothering about the processes themselves" (Sfard, 1991, p.10).

Research has shown that the way a teacher teaches mathematical content and, therefore, the way a student learns it is closely linked to the type of mathematical knowledge the teacher possesses about that content (Ball et al., 2008; Charalambous and Pitta-Pantazi, 2016). It has also been shown that the quality of teaching practice does not depend only on the teacher's content knowledge; it also depends on the type of learning experiences and opportunities about his or her teaching that he or she may have (Steele et al., 2013). For example, Dubinsky and Wilson (2013) assert that through appropriate pedagogy it is possible for high school students with low performance in mathematics to reach an understanding of the function like high performance students, and even to that of teachers in training.

On the other hand, reflection is fundamental to develop knowledge in mathematics teachers from their practice (Arcavi, 2016; Ponte and Chapman, 2016; Preciado-Babb et al., 2015; Rassmusen, 2016; Saylor and Johnson, 2014). However, research has shown that it is complex to make teachers reflect (Smith, 2015; Saylor and Johnson, 2014); moreover, when it is achieved that they reflect on their practice, they do so by placing their attention on teaching rather than on learning, including their own (Chamoso, Cáceres and Azcárate, 2012).

Reflection, according to Dewey (1993), is the action based on "the active, persistent and careful consideration of any belief or supposed form of knowledge in the light of the foundations that support it" (p. 9), therefore, it implies a (conscious) questioning of personal experiences to establish relationships between thoughts and actions, whose starting point is usually a disturbing or interesting problem or phenomenon with which the individual is trying to deal (Shön, 1983). In this sense, for a reflective process to take place among teachers about their knowledge of the content and their teaching practice, it is essential to place them in front of a situation that demands them

to question themselves about it. Thus, the type of reflection referred to in this research is that referred to by Dewey (1993) and which, in the case of teachers, is singularized in what Shön (1983) calls Reflection-on-action, which implies a conscious thought about the action or experience after it has taken place.

Duality in the understanding of the concept of function

One of the reasons for the complexity of mathematical knowledge is that, for the most part, concepts assume a dual role, as processes (operational conception) and as objects (structural conception), depending on the situation or people's level of conceptualization (Sfard, 1991; Tall, 1997). In Mathematics Education it is agreed that these two ways of conceiving a mathematical concept are complementary and that it is essential to favor the transition between them in order to achieve understanding.

Interpreting a notion as a process implies regarding it as a potential rather than an actual entity, which comes into existence upon request in a sequence of actions. Thus, whereas the structural conception is static instantaneous, and integrative, the operational is dynamic, sequential, and detailed (Sfard, 1991, p.4).

From the above perspective, understanding the function concept involves recognizing and dealing with its dual nature, without neglecting that "in the process of concept formation, operational conceptions would precede the structural" (Sfard, 1991, p.10), as well as "APOS stages of understanding the function concept" (Dubinsky and Wilson, 2013, p. 5). In this regard, Sfard (1991) proposes a spiral model with three phases for conceptual development in mathematics: Interiorization-Condensation-Reification. In the first phase there must be a process performed on familiar objects, in the second phase the idea of turning this process into an autonomous entity must be conceived, and in the third phase the ability to see the new entity as an integrated whole must be acquired.

In the case of the function concept, the three previous phases can be interpreted as follows (Sfard, 1991, pp. 19-20):

- a) Interiorization. It is when the idea of variable is learned and the ability of using a formula to find values of the "dependent" variable is acquired.
- b) Condensation. It is when function is considered, the more capable the person becomes of playing with a mapping as a whole, without

actually looking into its specific values, the more advanced in the process of condensation he or she should be regarded. Eventually, the learner can investigate functions, draw their graphs, combine couples of functions.

- c) Reification. The things on which these functions act - are other things of the same kind, i.e., also functions.

The objective of our research was to analyse the reflections of high school mathematics teachers in relation to their knowledge and practice of teaching functions, in particular, on the teaching of the addition of functions, when they talk about their processes of solving tasks that involve mobilizing the function as a procept. In addition, to analyse whether the phases of internalization, condensation and reification are evidenced by the teachers in their resolutions. In this way, information is provided on the type of reflections that teachers make on their knowledge and teaching practice about the concept of function. This is particularly important because depending on the type of reflection of the teachers, they could identify or situate the problems or potential learning problems of their students from their own practice and knowledge of the content, which, in turn, has a positive impact on their professional development.

Methodology

The research is qualitative with a descriptive approach and an exploratory case study design was used to analyse the reflections of high school teachers on their knowledge and practice of teaching functions.

Participants

The participants were 6 high school mathematics teachers (3 men and 3 women) with more than five years of experience teaching algebra, geometry, and calculus. In addition, all had professional training in mathematics teaching from a public university in Mexico. The teachers ranged in age from 28 to 32 years old. Their collaboration was voluntary and without any type of compensation. It was explained to them that their work would consist of solving and talking collectively about the resolution of precalculus tasks. The

resolution and conversation were carried out in a 2-hour virtual session through the Microsoft Teams platform.

Instrument and data collection

The data collection instrument consisted of three tasks on functional relationships given through tabular representations (numerical tables). Its design was such that the first to the third task corresponded to the phases of internalization, condensation, and reification, respectively. Thus, for example, the resolution of the third task could be seen as an "immediate" consequence of the answers given to the first and second tasks, i.e., it implied the reification, comprehension, and treatment of the function as a concept. The tasks (T1, T2, T3) are shown below.

Task 1. Let t and c be two variable quantities with the values shown in Table 1, determine for which value of t , the quantity $c = 10,201$.

Table 1
Variation of the quantities t and c .

t	1	2	3	4	...
c	1	4	9	16	...

Task 2. Let t and c be two variable quantities with the values shown in Table 2, determine for which value of t , the quantity $c = 10,302$.

Table 2
Variation of the quantities t and c .

t	1	2	3	4	...
c	2	6	12	20	...

Task 3. Let t and c be two variable quantities with the values shown in Table 3, determine for which value of t , the quantity $c = 10,440$.

Table 3
Variation of the quantities t and c .

t	1	2	3	4	...
c	3	10	21	36	...

The data were obtained through the sheets of each teacher's answer to the three tasks, which were shared during the session, and the audio and video recording of the collective conversation among the participants about what each one did in the tasks. The implementation of the tasks was in parts, i.e., first task 1 was given and once it was completed by all, the answers were shared and there was a free conversation about the thoughts and procedures used in its resolution. Then task 2 was implemented and we proceeded in the same way until we reached task 3. It is worth mentioning that the teachers always had their task resolutions at their disposal for any consult or analysis.

After sharing and discussing task 3, and observing that no teacher solved it as a sum of functions, they were asked to answer the following question: Why do you think that everyone tried to construct the function without using the results obtained in the previous tasks? Subsequently, they were asked to comment on the scope or limitations of the three tasks for teaching the concept of function and its operability, for example, teaching the addition of functions, given that the function represented in Table 3 can be seen as the sum of the functions corresponding to the values in Tables 1 and 2.

Data Analysis

The identification and analysis of the reflections was carried out from the transcriptions of the video recordings using the method of thematic analysis (Braun & Clarke, 2006), because this method allows identifying categories of verbal reflections in a group. Therefore, the data were reviewed trying to find cases that could be considered as examples of general and specific reflections of the participants on their knowledge and practice of teaching functions in conversational interaction.

After categorizing the reflections, a researcher familiar with the method of thematic analysis and reflective teaching provided feedback on the categories obtained. This opinion was considered to refine and clarify the categories obtained. The resulting categories were as follows: 1) Reflections on task actions; 2) Reflections on content knowledge; 3) Reflections on teaching practice and 4) Reflections on professional training. Table 4 shows some excerpts from the conversations that account for each type of the above-mentioned reflections.

Table 4
Types of teacher reflections

Types of reflection	Conversational extract
1) Actions on tasks	<p>T3: Looking closely at my data and what I have written, adding the first two algebraic expressions together yields the expression for Task 3!</p> <p>T1: Before T2 commented, I had not related that the new sum was the sum of the previous ones.</p> <p>T4: worked the situations as unique, I did not check if there was any relationship between them.</p>
2) Content knowledge	<p>T3: I think this was missing from my knowledge of function sum, to work it from a tabular representation and see that sum.</p> <p>T2: To encourage this type of exercise, it must first be part of the teacher's knowledge.</p> <p>T5: I find that my knowledge of function operations improves with this type of experience.</p>
3) Teaching practice	<p>T4: I think that the habit of working on exercises that were different from each other made me not realize the relationship between the tasks.</p> <p>T3: The tasks that are always proposed to the student are to solve, not to think and analyze if there is any relationship between them. Generally, the problems are isolated.</p> <p>T5: The sum of functions I only work with the idea of adding algebraic expressions.</p> <p>T2: Many times, in our practice we forget the important aspects of mathematical concepts.</p> <p>T5: Spaces for reflection such as this one allows us to take up those ideas that we are leaving out of our practice.</p>
4) Professional training	<p>T2: When we studied to be teachers we proposed many things, we planned, we had ideas of what and how to make the student generate learning, but in the field,</p>

Table 4 (continue)
Types of teacher reflections

Types of reflection	Conversational extract
	there are many factors that you cannot control, one is time.
	T3: Now that T3 comments, in undergrad we were asked to favor different representations for a concept to be understood more.
	T1: Yes, we needed this "refreshing" of our training, on what to look for with the problems. We should not give up reflection.
	T4: This experience was a "refreshing of our memory" and our objective as teachers. We have "forgotten" the look with which we left the university. The system absorbs us.

To identify whether the teachers went through the phases of internalization, condensation and reification of the function concept when solving the tasks, their procedures and arguments used in each of the tasks were analyzed based on the characteristics of each phase (Sfard, 1991). Images 1 and 2 are examples of how the teachers solved task 3 without using the functions previously obtained by them in tasks 1 and 2.

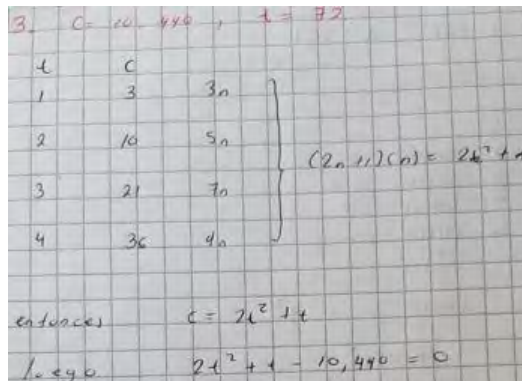


Figure 1. Resolution of Task 3 by teacher T1 through the study of patterns.

Handwritten work on grid paper. At the top, it says 'Tabla 3' and 'C=10440' and 'diferencia f'. Below this is a table with columns labeled 't' and values 1, 2, 3, 4, and an ellipsis. The corresponding values in the second row are 3, 10, 21, 36. To the right of the table, there are equations: $2n^2 + n = C$ and $2n^2 + n = 10440 = C$. Below the table, there are calculations for differences: $1^2 - 1 = 0$, $2^2 - 1^2 = 3$, $3^2 - 2^2 = 5$, $4^2 - 3^2 = 7$, and $5^2 - 4^2 = 9$. To the right of these, there are more calculations: $2 \cdot 1 = 2$, $2 \cdot 2 = 4$, $2 \cdot 3 = 6$, $2 \cdot 4 = 8$, and $2 \cdot 5 = 10$. At the bottom right, it says $t = 72$.

t	1	2	3	4	...
C	3	10	21	36	

$2n^2 + n = C$
 $2n^2 + n = 10440 = C$
 $t = 72$

$1^2 - 1 = 0$
 $2^2 - 1^2 = 3$
 $3^2 - 2^2 = 5$
 $4^2 - 3^2 = 7$
 $5^2 - 4^2 = 9$

$2 \cdot 1 = 2$
 $2 \cdot 2 = 4$
 $2 \cdot 3 = 6$
 $2 \cdot 4 = 8$
 $2 \cdot 5 = 10$

Figure 2. Resolution of Task 3 by teacher T3 through the study of patterns.

Results

In the teachers' conversation around their procedures used to perform the three tasks, it was found that they reflected on their own teaching practice and knowledge of the functions, as well as on their professional training and the uniqueness of their actions as task solvers (Table 4). In fact, it was identified that the reflections on their actions were the ones that triggered the other three types of reflections, as exemplified in the following excerpts.

T2: It is a new transformation of the previous one! In Task 1 the first differences are 3, 5, 7, ..., in Task 2 the first differences are 4, 6 y 8. Add the first differences and let's go to the differences of Task 3, the first difference is the sum of 3 with 4, which gives 7; adding 5 and 6 gives 11; adding 7 and 8 gives 15, and so on. Then, let's add the second differences which is 2 y 2 (from Task 2) gives 4 and if I add the two functions: t^2 and t^2+2 , wouldn't that give me $2t^2+2$? If we try to relate it, there is a sum of functions there and see something, it is true for the values. If you go to Tasks 1 and 2, adding 1 with 2 gives 3; then 4 with 6 gives 10; 9 and 12 gives 21 and $16+20$ gives 36, do you see?

T6: This is the interesting thing about the Tasks, that if I had taken the time to look at the relationship it would have made Task 3 easier for me.

T4: These tasks are very interesting for students to first review how to establish an algebraic expression and then to recognize the form,

that is, the relationship between the data and establish the sum of functions. I found it interesting how we worked on them.

Regarding the teachers' reflections on their knowledge and practice of teaching functions, it was identified that they were able to recognize areas for improvement in both aspects. For example, they said that this type of experience helps them to improve their knowledge of operations with functions and to have an alternative way of teaching the addition of functions, since they usually teach them by adding algebraic expressions and, in addition, the tasks they use focus more on solving and not on searching for or establishing relationships between what they have learned and the content in each of the exercises (Table 4).

The reflections shared by the teachers on their professional training consisted essentially in realizing that much of what they had studied and learned during their undergraduate studies has been left out of their teaching practice. In particular, they recognize that promoting the development of mathematical thinking in their students, using different ways of representing and treating the same mathematical concept, as well as constantly reflecting on their practice, is a teaching skill acquired in their professional training that they have left out of their practice. In this regard, most teachers agree that time is a factor that leads them to set aside (or avoid) what they have learned in their training, as well as the type of extracurricular demands of their schools. Below are some excerpts that expand on this aspect and can also be seen in Table 4.

T2: The system makes you do what you always do, present concepts and exercises. Sometimes I wondered where T2 left off when I planned so many things and had ideas!

T3: Spaces for reflection such as this one allow us to take up those ideas that we are leaving out of our practice.

T5: This experience made me remember moments from my undergraduate studies, discussing aspects that we should not lose that way of thinking, of reasoning, because that is taken to the classroom and contributes to the student's learning.

In relation to the conceptualization of the function and the procedures for solving the tasks, it was identified that, although the teachers evidenced the phases of internalization and condensation when performing the tasks, this does not happen with the phase of reification, because although they make use of the function concept to obtain the formulas that allow them to calculate the

value requested in the first two tasks, their procedure is focused on working with numerical patterns in isolation and without applying the concept of function, as is evidenced in the case of the third task. The resolution of this last task was unrelated to the use of the functional relations obtained by themselves in the previous tasks.

Finally, the resolutions of the tasks and reflections of the teachers show that, on the one hand, even when the tasks have a low level of complexity, these were key to recognize in them, an absence of the functions as a concept in their knowledge and teaching practice. And, on the other hand, the importance of generating spaces for academic interaction among teachers to promote learning experiences where the conceptual and procedural aspects of the mathematical content they teach are connected, without leaving aside the theoretical foundations of the content.

Discussion

The results provide evidence that the teachers reflected on the need to improve their knowledge and practices of teaching the function seen as a concept and to rely on alternative pedagogical treatments to the exclusively symbolic-algebraic. It is also shown that the conversation helped teachers to reflect on their knowledge and students' learning, which, according to several studies, is complex to make teachers reflect (e.g., [Smith, 2015](#); [Saylor and Johnson, 2014](#)), and to reflect more on learning and not only on teaching (e.g., [Chamoso, Cáceres and Azcárate, 2012](#)).

It is also noted that the teachers' actions on the tasks were essential to trigger in them a more robust didactic-mathematical analysis of the functions and a different way than the one they perform in an algebraic way. This led them to remember and question why they have left out of their teaching work the knowledge and skills acquired in their professional training, for example, the transition between representations, the development of mathematical thinking and the pedagogical objectives of mathematical exercises. Thus, discussing the effectiveness or feasibility of solving the tasks in one way or another helped them to improve their understanding of functions in terms of the relationship between variables and relationships between functions, and led to reflections on how to foster mathematical thinking in their students and the type of tasks to do so. That is, there were reflections on how to teach functions for a better balance in the procedural and conceptual.

Finally, the teachers recognized as part of their conversations and reflections that in their resolution of the tasks they were not able to adequately connect the mathematical content immersed in them. They also commented that this fact is a limitation for the achievement of their teaching practice, thus questioning their knowledge and the reason why they have left out everything they have studied and learned in their professional training as mathematics teachers. After the reflections in the conversation, they agree on the need to reconsider this situation. Thus, these results are in line with what Brodie and Shalem (2011) reported that conversations among teachers favor the development of new professional knowledge and some links between practice and theory, as happened in this study.

Conclusion

This study provides information on the type of reflections that mathematics teachers made to improve their knowledge and teaching practice of functions based on the resolution of tasks and the conversation as the main trigger for the reflections. In this way, the results contribute to deepen in how to make teachers reflect and incorporate a much more structural vision of functions in their teaching practices and, at the same time, incorporate the importance and role of their professional training in their reflections. Thus, we consider that the results have important implications for the learning and professional development of high school mathematics teachers.

Some considerations derived from this study for future research consist in analyzing how this type of experiences are taken to the classroom by teachers and what is their scope in students' learning. This is because the teachers expressed that the conversation space reminded them how little or nothing, they promote mathematical thinking in the classroom, the importance of articulating diverse representations, as well as elaborating exercises for learning rather than for the repetition of procedures.

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