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# Exploring the relationship between tacit models and mathematical infinity through history

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**ABSTRACT**

In this article we address the historical and epistemological study of infinity as a mathematical concept, focusing on identifying difficulties, counter-intuitive ideas and paradoxes that constituted implicit, unconscious models faced by mathematicians at different times in history, representing obstacles in the rigorous formalization process of this mathematical concept. It is shown how the active and conscious questioning of these models led to a process of axiomatization of mathematical infinity, which was completed with the works of Cantor (1883) and Robinson (1974). The implemented methodology is supported by a qualitative and argumentative bibliographic research based on content analysis from a meta-ethnography. From this research, information is obtained about the unconscious mathematical structures students are confronted with and the conscious patterns of reasoning they must develop to overcome difficulties and obstacles that these models produce, and thus achieve an adequate understanding of mathematical infinity.

**Keywords:** implicit knowledge, unconscious learning, historical-epistemological study

## INTRODUCTION

Men have been fascinated with infinity since ancient times. Infinity is, beyond dispute, one of the most intriguing and abstract concepts the human mind has created and both mathematicians and philosophers have long debated about its nature throughout history. Full of contradictions, inconsistencies, and controversies, it has also played a central role in defining many areas of mathematics (Weyl, 1949). For this reason, its learning and teaching processes have been widely studied from different theoretical perspectives in mathematics education. Despite this, it is not possible to claim that we grasp the intricate cognitive processes that develop in relation to its understanding. Several works (D'Amore, 2011; Kidron, 2011; Mamolo, 2017; Nasr, 2022) account for the complexity of epistemological obstacles (Bachelard, 2004) in this case.

For example, studies carried out in the past give evidence of the difficulty that arises when students deal with infinite sets of different sizes introduced by Cantor (1883) and the notion that there are infinities larger than others (Núñez, 2005), which is called the *squeezing* of transfinite cardinals (Arrigo & D'Amore, 2004). Previously, in one of his works, Duval (1983) also analyzed the students' difficulties to accept the one-to-one correspondence between the set of natural numbers and its proper subset of square numbers, arguing that this is due to an obstacle that he called *slipping*. Also, in relation to the above, the difficulty known as *inclusion* is reported in the literature (Fishbein, 2001).

Another widespread intuitive conviction among students is that there are more points in a longer segment than in a shorter one (D'Amore, 2011; Sarama & Clements, 2009), which D'Amore (2011) called *dependency* of transfinite cardinals to properties related to geometric measures of segments.

Inspired on Polanyi's (1958) tacit knowledge, Fischbein (2001) claimed that *dependency* as well as *squeezing*, *slipping*, *inclusion* and other similar types of intuitive convictions (that he called *tacit models*), appear when we deal with concepts, which are too abstract or complex. In these cases, we have a natural tendency to think in terms of these simplified mental models at an initial stage of the learning process, that help us represent the original identities with the aim of facilitating and stimulating the task of understanding or resolution, and which later become implicit or tacit, continuing to influence, unconsciously, the interpretations and the solving strategies of the learner.

In general, we are unconscious of how information is being organized by our brains and without resorting to a conscious process, such as intention, to raise questions to perceived information, one can unconsciously ask and seek answers without knowing the underlying factors of these behaviors (Bond & Taylor, 2015; Greenwald & Banaji, 1995; Maxwell et al., 2016; Simons, 2003; Weinberger & Green, 2022). However, the recognition of both questions and answers may need a conscious process. In fact, one of the most challenging problems in educational studies on learning is related to the role of these unconscious and conscious processes (Fukuta & Yamashita, 2021). Evidence indicates that cognitive unconscious processes can lead to unconscious acquisition of knowledge and appears to be structurally and functionally more sophisticated than the conscious ones, freely influencing conscious learning processes (Bargh & Morsella, 2008; Evans, 2008; Fukuta & Yamashita, 2021). Therefore, to achieve a proper understanding of the learning processes related to mathematical infinity, these unconscious learning processes and the corresponding resulting *tacit models* need to be brought to light by conscious intention.

On the other hand, there are three types of obstacles that are distinguished in the learning of any mathematical concept: the ontogenic ones that originate in the characteristics of the student's development; the didactic ones that are a product of teaching; and the epistemological ones that are intrinsically related to the mathematical notion under study. In particular, the origin of an epistemological obstacle faced by students can also be studied, in part, through the difficulties that mathematicians of previous generations faced in situations of a similar nature throughout history. In this case, a historical-epistemological approach could suggest some of the difficulties involved in conceiving mathematical infinity.

Let us recall that in the study of a mathematical concept from the historical and epistemological point of view, not only the constitution and technical functions are analyzed, but also the complicated processes in which theoretical developments are produced, to become elements of mathematical discourses with specific meanings at a given time of history.

Although *tacit models* appearing in students' thought processes have been studied by some authors (for a systematic review see Belmonte & Sierra, 2011), there are no exhaustive reports on the presence of these models in the historical conception and the axiomatization process of mathematical infinity. This type of report would be valuable, considering further that these models have given rise to some of the most remarkable infinity-related paradoxes throughout history that have had a profound influence on mathematics and mathematical thought (Russell, 1992) and are especially puzzling to students. Thus, in this article we analyze the evolution of infinity as a mathematical concept, exploring its relationship with difficulties, counter-intuitive ideas and paradoxes that constituted unconscious, *tacit models* faced by mathematicians at different times in history, representing obstacles in the axiomatization process of this mathematical concept.

## MATERIALS AND METHOD

The implemented methodology was supported by qualitative and argumentative bibliographic research. Once the selection of the literature was carried out, the analysis and interpretation of the information extracted from the bibliographic review was made using the content analysis method (Bengtsson, 2016). Content analysis is a research tool used to determine the presence of certain themes, or concepts within some given qualitative data (i.e., texts) and to analyze their meanings and qualitative relationships. It is an interpretive and naturalistic approach. It is both observational and narrative in nature. The selected texts must be coded, or broken down, into manageable code categories for analysis. In this case, the texts were selected and coded according to the observed categories of tacit models and their evolution throughout history was analyzed to reach new insights and conclusions about the axiomatization process of mathematical infinity.

### A Relational Analysis

There are two general types of content analysis: conceptual analysis and relational analysis. Conceptual analysis determines the existence and frequency of concepts in texts. Relational analysis develops the conceptual analysis further by examining the relationships among those concepts. Relational analysis begins like conceptual analysis, where a concept is chosen for examination. However, the analysis involves exploring the relationships between concepts. Individual concepts are viewed as having no inherent meaning and rather the meaning is a product of the relationships among concepts (Krippendorff, 1980).

In our case, to begin the relational content analysis, we first identify all the categories of *tacit models* that could appear through the texts for analysis. Next, texts for analysis are selected carefully by balancing having enough information for a thorough analysis so results are not limited with having information that is too extensive so that the coding process becomes too arduous and heavy to supply meaningful and worthwhile results.

Based on the above, to analyze the evolution of infinity as a mathematical concept, focusing on recognizing and identifying difficulties, counter-intuitive ideas and paradoxes that constituted *tacit models*, the proposed steps or stages of the analysis (Krippendorff, 1980) were followed:

(1) the phenomenon to be studied was decided, identifying texts and materials of interest useful for the analysis (in this case *tacit models* present in the evolution of infinity as a mathematical concept),

(2) it was decided which qualitative studies were relevant, which required determining the scope of the proposed synthesis (one hundred and forty four documents among books on the history of mathematics and advanced mathematics, historical articles and recent editions of original manuscripts, published in the period between 1858 and 2022 were selected, where topics related to the evolution of infinity as a mathematical concept were discussed),

(3) a critical reading of the selected studies was carried out to identify and to code the main concepts and models, and the extraction of interpretive arguments related to these models,

(4) relationship between concepts were explored: once the texts were coded, the relationship between the studies was determined: this stage included the development of lists of key codes and arguments of each study and the analysis of the strength and direction of their relationships with others,

(5) according to the results of the previous step, statements or relationships between concepts of different texts or studies are coded, and

(6) relationships among the identified variables during coding were mapped out and a synthesis of the analysis was made, arriving to new conclusions.

### Synthesis and Interpretation

To carry out the synthesis and interpretation of the information extracted in step (6), the meta-ethnography method (Noblit & Haré, 1988) was used. Meta-ethnography is a thorough qualitative meta-synthesis method in which qualitative studies are selected, analyzed, and interpreted to answer focused questions on a specific topic to come up with new insights and conclusions. Meta-synthesis goes beyond critical literature reports based on a rigorous analysis of the obtained data in qualitative research, which includes the dynamic of constantly identifying similarities and disagreements between collected statements or concepts and its method proposes an approach to the generation of knowledge based on the evidence provided by the results of research (Finfgeld, 2018) extending and fortifying the interpretations that must be obtained based on the critical analysis of literature.

In 1988, Noblit and Haré (1988) determined strategies to perform *meta-ethnography* as an alternative way of approaching meta-synthesis from their experience with ethnographic studies. In this case the product of the synthesis is the "translation" of studies into others, which motivates the researcher to understand and transfer ideas and concepts through different studies (Britten et al., 2002). The "translation" of concepts in *meta-ethnography* is what differentiates it from other traditional methods of literature review, since "translation" does not mean finding the "best truth" but transposing the concepts for a higher understanding. From the translation, comparisons are made between different studies, in which researchers must preserve the structures of the relationships between the concepts. In summary, the interpretation and explanation of the original studies are considered as data and transferred through different studies to produce a synthesis. In this case, for each category of *tacit model* analyzed, the synthesis was carried out among studies that presented the same line of argument, as a whole.

## RESULTS

As early as the 6th century BC pre-Socratic philosophers were already interested in speculations about infinity. Anaximander was one of the first to rigorously consider it. Infinity was for him something neutral, imperishable, without end, unlimited, associated with God (Guthrie, 2000) called the *Apeiron*. His ideas were influenced by the Greek mystical tradition of the archaic period (8th-6th centuries BC) that postulated the *Apeiron* was the primary origin of everything that exists.

Towards the 4th century BC some atomists, such as Leucippus and his disciple Democritus, thought that matter was composed of an infinite number of "indivisibles" and the universe was infinite (Lucrecio, 1985). Parmenides and his disciple Zeno disagreed based on the infinite subdivision of matter principle. There were opposing conceptions about space and time at this point, and consequently, infinity was conceived in two different ways: as *potential infinity* and as *actual infinity*.

Later in the same century these two conceptions of infinity appeared as philosophical categories defined by Aristotle in his *Metaphysics* (Aristotle, 1985), when he distinguished between being "in potential" and being "in act", referring to the *Apeiron* from a different perspective based on the method of exhaustion of Eudoxus. He argued that the "totality" of numbers could not be present in our reasoning, since when generating a list of these, one by one, the complete list could not be generated because there would always be a number that had not been considered before. Therefore, he argued that the *Apeiron* was not something "exhausted" outside of which there is nothing, but something outside of which there is always something, and consequently, it was "inexhaustible". So, it could not be seen as a completed totality, because that which is completed has an end, it has a limit element, therefore it was characterized by its inherent incompleteness, by its existence only as potentiality. Therefore, infinity should be considered as something that had only a "potential existence", and not as something manifested, actual, realized.

From the above, it can be noticed that early philosophers made use of these simplified, unconscious mental models, defined by Fischbein (2001) as *tacit models*, to represent infinity as an original identity from the very beginning of its conceptualization process. In the last case, Aristotle's way of reasoning about infinity is an obvious manifestation of the *tacit model* called *inexhaustible* (infinity was "inexhaustible" because it could not be seen as a completed totality) (Fischbein, 1987).

*Potential infinity* was also considered in Eleatic philosophy on the 5th century BC, in the study of sequences and series through Zeno's paradoxes, who tried to show that the existence of infinity led to apparently contradictory situations. In this case too, these contradictions derive from the appearance of these models. For example, Zeno's paradox Achilles and the tortoise (Aristotle, 1985), shows the *tacit model* called *divergence* (Belmonte & Sierra, 2011) which assumes that result of the infinite sum of finite quantities cannot be finite. The *inexhaustible* model is present in this paradox too because it is considered that the infinite sum cannot be calculated, due to the *undefined* (Belmonte & Sierra, 2011) number of terms, as it is always possible to continue the process of adding terms (*undefined* model). Closely related to the previous models, here is also found the model known as *approximation* (Belmonte & Sierra, 2011) that we have called in a previous study *unreachable*, due to the assumption that the limit of the sequence of the partial sums cannot be reached. It also shows the model called *dependency* (D'Amore, 2011), which identifies the numerical distances separating Achilles and the tortoise at each step, with the segments of the trajectory viewed as geometric spaces. Similarly, due to the same geometric considerations, the model known as *point-mark* (Fischbein, 1987) appears, identifying points with marks in the geometric plane.

Another well-known Zeno's paradox, the *Dichotomy paradox* (Aristotle, 1985), shows the *inexhaustible* model too, as it involves the endless process of repeatedly splitting a path into two parts. Similarly, the *Arrow paradox* (Aristotle, 1985) is an expression of the same *tacit model*, whereas the first two paradoxes divide space without end, this paradox divides time without end (and not into segments, but into points) showing in a similar way the *point-mark* model.

One of the most well-known characteristics of Pythagorean's mathematics was their rejection of irrational numbers (their *incommensurables*) (Stillwell, 2010). Here again we recognize that mathematicians of this early age were faced by the *tacit model* known as *dependency* which represents, in this case, the inability to accept numbers that do not have a common measure, i.e., numbers that cannot be represented as quotients of two integers, which is also due to the *undefined* model mentioned above.

Inspired by these contradictions, Archimedes was very close to the modern concept of mathematical infinity around the 3rd century BC, although the word "infinity" is only named twice at the beginning of his work *The sand reckoner* (Archimedes, 2002). His decision to hide the term "infinity" in his work revealed the demands of the times to avoid it, due to the ideas of Aristotle which exerted great influence on Greek mathematics for many centuries. Despite this fact, in *The sand reckoner* it is shown that he investigated the infinite number of objects considering three pairs of infinite magnitudes, arguing that they were equal in size respectively. This statement suggests that he considered that not all objects infinite in number were equal (O'Connor & Robertson, 2002), that he had notions about the concept of cardinal that Cantor later specified, when reflecting on the existence of *actual infinity* and of infinities larger than others (Gardies, 1997). Thus, he was able to identify the *squeezing* model, which assumes that all infinities are the same size. In the same work, he also questioned the *infinite-unbounded* model (Belmonte & Sierra, 2011) which presumes that an infinite sequence must be unbounded, while setting out to determine an upper bound for the number of grains of sand that fit into the universe.

Also, in *The method* (Archimedes, 2002) it was revealed how Archimedes found his results using arguments related to infinity, and then tested them rigorously by omitting those arguments later (Bagni, 2004). For him, infinity could be used to discover results that were not initially accessible through logical arguments. For example, if you have an infinite sequence of sums $S_1$, $S_2$, $S_3$, ..., where $S_{n+1}$ is obtained from $S_n$ by means of a well-defined rule, and it was known geometrically that the final result of this iterative process must be $S$ (as happens in the application of the exhaustion method to the calculation of the area of a parabolic segment) it was very tempting to think of $S$ as the "completion" of this process, that is, as the limit of $S_n$ when $n$ tends to infinity. So, following this idea, in his work he considered that the sum of an infinite series could be finite (Bagni, 2006) becoming aware of the *divergence* model.

In the 14th century, from the works of Archimedes, progress was made in the development of infinitesimal techniques to calculate areas and volumes of solids. In 15th century Italy, Nicholas of Cusa (1445/2020), inspired by the works of Euclid and Archimedes, considered that a circle was a polygon with an infinite number of sides, so he was implicitly assuming *actual infinity* as the final result of an iterative process, overcoming the *inexhaustible* model that hindered Aristotle's understanding of infinity before. This in turn inspired Kepler in the 17th century to write in his *opera Omnia* (Kepler, 1858-1871) that there was no essential difference between an infinite-sided polygon and a circle, an infinitesimal area and a line, and therefore, between the finite and the infinite, which was contradictory (Tall, 1991). In his work *Nova stereometria doliorum vinariorum* (Kepler, 1613/2018), following in the footsteps of Archimedes, he systematically developed the calculation of areas and volumes. His method consisted of sectioning a solid into a finite number of infinitesimal sections or "indivisible" solids of convenient shapes and sizes, according to each problem. The sum of all these "indivisibles" made it possible to obtain the desired volume. In this work Kepler made the step to the limit, identifying a curve with the infinite sum of infinitely short lines and an area with the infinite sum of infinitely small rectangles (Bell, 2008). Here, again, we see that he was able to deal with the *inexhaustible*, *unreachable* and *divergence* models.

From Kepler's ideas the concepts of *indivisibles* and infinitesimals were introduced in their early stage. Among the first works that stood out in the 17th century, the *indivisibles* of Galileo and Cavalieri can be mentioned. In his work *Discorsi e dimostrazioni matimatiche* (Galilei, 1638/1965), Galileo argued about the continuum when he discussed infinity, starting from the same Aristotelian idea: indefinite divisibility, but starting from a concept opposite to that of Aristotle, which conceived infinity as an endless process. For Galileo *actual infinity* was something finished, realized, completed. Therefore he, as Archimedes and Kepler before, was able to overcome the *inexhaustible* model. From his *indivisibles* he raised certain objections regarding infinity by using paradoxes. For example, based on the famous paradox that bears his name, he pointed out that if the number of natural numbers was not only potentially infinite but also actually infinite then there were as many perfect square numbers as natural numbers, since both sets could be put into one-to-one correspondence. However, he argued, based on Euclid's axiom: "the whole is greater than its parts", that there must be a greater quantity of natural numbers than of perfect square numbers. Upon arriving at this contradiction, he concluded that the infinite quantities were incomparable by stating that: "the attributes 'equal', 'greater,' and 'less,' are not applicable to infinite, but only to finite, quantities" (Galilei, 1638/1965, pp. 32-33). Here we find the *tacit model* known as *inclusion* (Fischbein et al., 1979) which states that a part of infinity must be smaller than the whole of infinity and that has been questioned before by Ibn-Qurra in his book *Questions to thabit Ibn-Qurra* (Lévy, 2001). This model is closely related to the *squeezing* model mentioned above, as well as the *slipping* model (Duval, 1983) which prevents the acceptance of the one-to-one correspondence between the set of natural numbers and its proper subset of square numbers.

On the other hand, Galileo was also able to recognize the *dependency* and the *point-mark* models although he could not find a way to elucidate it, as it is clearly seen from his following statement: "Here a difficulty presents itself which appears to me insoluble. Since we may have one line greater than another, each containing an infinite number of points, we are forced to admit that, within one and the same class, we may have something greater than infinity, because the infinity of points in the long line is greater than the infinity of points in the short line. This assigning to an infinite quantity a value greater than infinity is quite beyond my comprehension" (Galilei, 1638/1965, p. 32).

Following the ideas of Kepler and Galileo, Cavalieri created the geometric method of *indivisibles* for calculating squares, which served as the basis for the infinitesimal methods that would later emerge (Bell, 2013). In 1635, in his work *Geometria indivisibilibus* (Cavalieri, 1635/2010), he calculated the area of a plane figure considering that it was made up of an infinite number of equidistant parallel segments, each of which was interpreted as an infinitely thin rectangle and the volume of a solid considered as an infinite sum of areas of parallel plane surfaces. For him the area was an infinite sum of straight segments (which he called the *indivisible* of area) and the volume was an infinite sum of plane surfaces (which he called *indivisible* of volume). He considered the *indivisibles* as entities of lesser dimension with respect to the continuum of which they were part, affirming that a line was made of points, a plane was made of lines, and a solid was made of flat areas (Bussotti, 2014). Here, once more, we identify the *point-mark* model, which is also called by D'Amore (2011) as the *pearl necklace* model.

Also in the 17th century, Pascal understood that in Cavalieri's method of *indivisibles* the infinitely small appeared implicitly, so he took up the concept of infinitely small of Eudoxus to define his *indivisibles*. However, Pascal's *indivisibles* did not follow some arithmetic rules. By assuming that zero was the indivisible of arithmetic, he was opposed to the fact that a plane was composed of an indefinite number of lines, reflecting the contradiction in his method and revealing, at the same time, the *dependency* and *point-mark* models in Cavalieri's method. He regarded infinite division as something that cannot be represented or understood, and yet, as a geometric principle that cannot be avoided. Still, through his work he was able to calculate limits of sums of an infinitely large number of infinitely small quantities, implicitly handling the *divergence* and *unreachable* models.

Around this time, the current symbol of infinity (∞) was introduced by John Wallis in his work *Arithmetica Infinitorum* of 1655 (Wallis, 1655/2004), also considering the reciprocal (1/∞) for his infinitesimals and systematically introducing the analysis of infinite series, starting with the geometric progression. His infinitesimals differed from *indivisibles* in that he would decompose geometrical figures into infinitely thin building blocks of the same dimension as the figure. For example, in his works he suggested a thought experiment of adding an infinite number of parallelograms of infinitesimal width to form a finite area. His ideas were the predecessors to the modern methods of integration and showed that he was also able to overcome the *dependency*, *unreachable* and *divergence* models.

Another 17th century mathematician who used the sum of convergent geometric series to calculate areas under a curve and who was able to handle the *tacit models* mentioned above, was Fermat. His technique consisted of considering infinitesimal rectangles inscribed in the figure, for which the dimensions of their bases were in geometric progression. His method, like Wallis's use of infinitesimals, considered the division of an area under a curve into infinitely small elements of area. He approximated the sum of those area elements by the sum of the infinitesimal rectangles and performed something like a limit when the number of elements increased infinitely, while their area became infinitesimally small (Stillwell, 2010).

Based on the work of these mathematicians, Newton and Leibniz developed their Calculus a few years later. They were concerned with manipulating infinitely large and infinitely small quantities and, although their techniques were more rigorous than the previous ones, some contradictions remained in their foundations, due to some of these *tacit models*. For example, they both used infinitesimals involving *actual infinity* but using the *inexhaustible* model related to *potential infinity* to manipulate them. In fact, in his work *Quadrature of curves* of 1676, Newton (1969-1981) expressed his intention to abandon the use of infinitesimals and enunciated his theory of the "first and last ratios of evanescent quantities", anticipating the mathematical concept of limit (the ultimate reasons by which evanescent quantities disappear are limits to which the ratios of continuously decreasing quantities tend to approach, limits that cannot be reached because these quantities diminish ad infinitum). Again, his reasoning in this case also shows the presence of the *tacit model* we called *unreachable*.

On his side, the novelty of Leibniz's work was the investigation of various orders of infinities, considering the infinitely small as a variation (Stillwell, 2010). The existence of infinities of different orders was suggested to him by the type of functions to which he calculated the derivative. However, Leibniz also thought that infinite numbers should be rejected. Through his works the *unreachable* and *inexhaustible* models can be also found. For him, the concept of infinity was logically absurd (which also accounts for the *undefined* model), and if there were no infinitely large numbers, there were no infinitesimals either. In his intellectual testament, written in 1716 shortly before his death, Leibniz said: "As for the calculation of infinitesimals, I am not entirely satisfied ... When I was arguing in France with Abbot Gallois, Father Gouge and others, I told them that I did not believe that there were truly infinite or truly infinitesimal magnitudes: that they were only fictions, but useful fictions to abbreviate and speak universally" (Recalde, 2004, p. 54)

None of these ideas of the 17th and 18th century mathematicians were well received, however, from them, mathematicians of the following centuries developed a rigorous definition of infinity (Bussotti, 2014). It should also be noticed that at this time, to reach agreements and advance the theory, mathematicians resorted to intuition, and considered as valid in the infinite what was valid for the finite, which introduced new *tacit models* in the axiomatization process of infinity as a mathematical concept. For this, considerations of Euclidean geometry and mechanics were generally used, which, for example, accounts for the presence of *tacit models* as *dependency* and *point-mark*.

In the 18th century the problem of infinity had already been raised in the treatment of physical problems such as vibrating string and heat conduction. The vibrating string problem had to do with the search for a mathematical model of the vertical movement of an elastic string, uniformly dense and attached at its ends, to which a small impulse was given. The mathematicians of the 18th century (Euler, Bernoulli, Lagrange, etc.) understood that describing the movement of the rope consisted in determining the function that defined its movement, ideally taken as a curve located in the Cartesian plane. The transition from the physical to the mathematical problem immediately raised the dichotomy between the discrete and the continuous, introducing some of these models. In this case, when Bernoulli assumed the string of length $L$ divided into $n$ masses of length $L/n$, the continuum appeared by making n tend to infinity, making use of the *dependency*, *undefined* and *point-mark* models.

In 1748, Euler published his textbook *Introductio in analysin infinitorum* (Euler, 1748/1988) where an introduction to the treatment of infinities was made and elementary functions developed in infinite series were studied. For the first time functions were developed in infinite products, and to obtain these developments, Euler used infinitely large and infinitely small quantities as a tool. By means of these quantities he put in evidence the internal structure of the functions. In a letter to his friend Goldbach in 1744, Euler wrote: "After having made the plan for a complete treatise on the analysis of the infinities, I realized that many things had to precede it which are not properly included in it nor are hardly found treated nowhere; and from them this work has emerged as a prodrome to the analysis of the infinities" (Durán, 2003, p. 43).

Fourier established the method of separation of variables as a general method to solve partial differential equations in the 19th century, when working on the problem of heat conduction. He concluded that the existence of solutions for a given equation required that the initial condition function could be expanded into an infinite series. The key question, then, was to determine which kind of functions could be expanded in infinite series and to determine its convergence conditions. However, at the beginning of the 19th century, a precise definition of a function had not yet been established, nor of a continuous function, and furthermore, there were no convergence criteria or a rigorous theory of infinite series. In this setting some research problems appeared in which the lack of formal rigor was evident, among which was the treatment of infinities. Thus, at this time in history, the formalization of the concept of infinity and the resolution of the problems and difficulties posed by all these *tacit models*, became increasingly essential for the development of the emerging branches of mathematics.

Among the many mathematicians who undertook the task of giving the necessary rigor to infinity as a mathematical concept at this time, we have Cauchy, Weierstrass, and Bolzano. Cauchy was the first to understand that to build a solid foundation to support infinite structures, it was necessary to establish a clear and precise definition of limit. In 1821 in his *Analyse algébrique* (Cauchy, 1821/1989) he defined this concept incorporating infinitely small and infinitely large quantities as variable quantities. Infinitesimals were for him variable quantities whose limit was zero, while infinity was the limit of "numerical values of the same variable that grow more and more so that they exceed any given number" (Stillwell, 2010). Therefore, he did not accept *actual infinity* for an infinitely large quantity, since his reasoning was making use of the *inexhaustible* and *divergence* models. On the other hand, with respect to the continuum, he defined "continuous variation" referring to "the values that a variable takes successively". This, together with the use of infinitesimals, showed two conceptions of the line: to conceive or not, a line made up of points, which explicitly exposed the *dependency* and the *point-mark* models.

The issue of infinity could be better understood through Kant's works of 1790, although he also rejected *actual infinity* (Sanhueza, 2015) exhibiting some of the *tacit models* mentioned above in his reasoning. Like his Greek colleagues, Kant and the mathematicians of this time were opposed to accept the existence of infinity because of the many paradoxes and difficulties that it caused. Based on his works, Bolzano was the first to formally address the problem of *actual infinity* and conceptually explored the properties of infinite sets in his book *Paradoxes of infinity* (Bolzano, 1851/1950) in the 19th century. In this work he expressed: "Certainly most of the paradoxical statements encountered in the mathematical domain... are propositions which either immediately contain the idea of the infinite, or at least in some way or other depend upon that idea for their attempted proof" (Bolzano, 1851/1950, p. 75).

In this book he also addressed the comparison of infinite sets, taking up the work of Galileo, who had established one-to-one correspondences between infinite sets and their own subsets before. Following the ideas of Galileo, Bolzano reasoned: "in inferring the equality of the two sets, in the event of their being infinite, with respect to the multiplicity of their members–that is, when we abstract from all individual differences... two sets can still stand in a relation of inequality, in the sense that the one is found to be a whole and the other a part of that whole" (Bolzano, 1851/1950, p.98), thus showing that he, like Galileo before him, failed to handle the *inclusion* model. In this work he further stated: "Although every quantity in A or B allows of coupling with one and only one in B or A, yet the set of quantities in B is other and greater than in A, since the distance between the two quantities in B is other and greater than the distance between the corresponding quantities in A" (Bolzano, 1851/1950, p. 100), showing in this way the *dependency* model.

Later, Dedekind (1901), following in the same direction as Bolzano and the Greek mathematicians of the Pythagorean school, understood that he had to rigorously characterize irrational numbers by means of cuts (the well-known Dedekind cuts), starting from the fact that the linear continuum was not completely "filled" with rational numbers. From this, and after many centuries of contradictions, Dedekind achieved, for the first time, a precise definition of infinite sets (Eves & Carroll, 1966). His definition, which states that "a set A is infinite if there is a proper subset B of A such that B has the same number of elements as A" shows that Dedekind was finally able to consciously overcome the *inclusion* model, by actively questioning Euclid's axiom: "the whole is greater than its part" valid for finite sets (Ferreirós, 2016).

However, it was Cantor in 1883 who, developing the ideas of Bolzano and Dedekind, declared that "whenever two sets - finite or infinite–can be paired by a one-to-one correspondence, they have the same number of elements" (Maor, 1991, p. 57). He concluded that there were as many even numbers as there were natural numbers, because such a correspondence could be established between the natural and even numbers, resolving in this way Galileo's paradox and explicitly unraveling the *inclusion* model. It became clear to him, following Dedekind's definition, that this simply showed a property of infinite sets. Furthermore, for him, thinking in these terms meant accepting *actual infinity* which also required awareness of the *undefined*, *divergence*, *unreachable* and *inexhaustible* models.

However, due to the persistence of these models in ways of reasoning about infinity throughout history, these properties of infinite sets were so puzzling to mathematicians of the day, that many of them were reluctant to accept Cantor's ideas. Cantor himself admitted that certain conclusions drawn from his work were strange and counterintuitive. In his letter of June 25, 1877, Cantor asked Dedekind: "Can a continuous manifold of $p$ dimensions, with $p$>1, be uniquely related to a continuous manifold of one dimension such that at a point of one corresponds one and only one point of the other?" questioning in this way the *dependency* and *point-mark* models. Just four days later, in another letter dated June 29, he wrote to Dedekind about the affirmative answer he had found for his question: "As long as you don't approve it, I can only say: I see it, but I don't believe it" (Arrigo & D'Amore, 1999). Thus, although he doubted his own results, he was finally able to overcome the two *tacit models* mentioned above.

**Table 1.** Tacit models in the axiomatization process of mathematical infinity

| Tacit model | First recorded appearance | Conscious resolution-overcoming |
|---|---|---|
| Undefined | Zeno's paradoxes, 5th century BC | Archimedes, 3rd century BC; Robinson, 20th century |
| Divergence | Zeno's paradoxes, 5th century BC | Archimedes, 3rd century BC; Robinson, 20th century |
| Unreachable | Zeno's paradoxes, 5th century BC | Archimedes, 3rd century BC; Robinson, 20th century |
| Dependency | Zeno's paradoxes, 5th century BC | Cantor, 19th century |
| Point-mark | Zeno's paradoxes, 5th century BC | Cantor, 19th century |
| Inexhaustible | Metaphysics-Aristotle, 4th century BC | Archimedes, 3rd century BC; Robinson, 20th century |
| Squeezing | The sand reckoner-Archimedes, 3rd century BC | Cantor, 19th century; Robinson, 20th century |
| Infinite-unbounded | The sand reckoner-Archimedes, 3rd century BC | Cantor, 19th century; Robinson, 20th century |
| Bounded-finite | The sand reckoner-Archimedes, 3rd century BC | Cantor, 19th century; Robinson, 20th century |
| Inclusion | Questions-Ibn-Qurra, 9th century | Dedekind, 19th century |
| Slipping | Discorsi-Galileo, 17th century | Dedekind, 19th century |
| Bounded-unbounded | On a property of the collection of all real algebraic numbers-Cantor, 19th century | Cantor, 19th century |

In his investigations of infinite sets, he also understood that it was necessary to analyze the nature of the continuum and of infinite sets of points. In this regard, to further illustrate the contradictions that arise in reasoning about infinity, Cantor introduced a bounded nonempty subset of the real interval [0, 1], which nevertheless has an infinite number of elements, and which is known as the Cantor set (Cantor, 1883). By introducing this set, he was also able to throw light upon the *bounded-finite* model (Belmonte & Sierra, 2011) which assumes that a bounded set cannot have an infinite number of elements, and which is closely connected to the *infinite-unbounded* model earlier questioned by Archimedes. Similarly, by using his diagonal argument he was able to show that the set of real numbers could not be put into a one-to-one correspondence with the set of natural numbers, and therefore that there were infinities larger than others, clarifying in this way the *squeezing* model. On the other hand, arguing the one-to-one correspondence, he proved that the real intervals [0, 1] and [0, +∞) have the same size, shedding light on the *bounded-unbounded* model (Belmonte & Sierra, 2011) which assumes that an unbounded set must be larger than a bounded set.

Cantor's ingenious work and its application to infinite sets was an extraordinary conceptual achievement in mathematics. During the process he created infinity as new formal mathematical concept, and with it, new mathematics that could not have been created without a conscious intention and purposeful effort to overcome all those implicit, *tacit models*.

However, despite his works, Cantor rejected infinitesimals in the actual sense because they seemed to contradict its very conception, due to the presence of all these *tacit models* for the infinitely small. It was Robinson (1974), based on the work of Skolem (1934) and following Leibniz' ideas and a rigorous and axiomatic approach, who was able to introduce infinitesimals in their actual sense, thus overcoming those *tacit models* (*undefined*, *unreachable*, *divergence*, *squeezing*, *infinite-unbounded* and *bounded-finite*) which were so persistent over time. By doing so, he created non-standard analysis and introduced non-standard infinity, using the theory of models as the foundation of his work. It is worth noticing in the 19th century, Weierstrass, following the work of Cauchy, had established the formal definition of the limit in the epsilon-delta language (Weierstrass, 1895/2013), thus giving a rigorous definition of infinitesimal in the potential sense and thus paving the way to the development of the work of Skolem and Robinson.

**Table 1** summarizes the analysis done by showing the dates of the first recorded appearance and of the resolution of the conflict or obstacle posed by each of the *tacit models* considered, as well as the mathematicians involved in these processes.

## DISCUSSIONS

In the previous section we analyzed the evolution of infinity as a mathematical concept from a historical-epistemological perspective, focusing on recognizing difficulties, counter-intuitive ideas and paradoxes that constituted *tacit models* faced by mathematicians throughout history and prevented the acceptance of the existence of mathematical infinity for long periods of our history. It was found that *tacit models* appeared in the axiomatization process of infinity as a mathematical concept from the very birth of mathematics as a science in Ancient Greece.

The historical-epistemological reconstruction of this concept under the gaze of these models shows us some of the same difficulties students face when trying to comprehend topics related to mathematical infinity. Thus, problems faced by students in the comprehension of this mathematical concept can also be studied, in part, through the obstacles that mathematicians of previous generations faced throughout history while trying to understand infinity, because they are of a similar nature.

Furthermore, the analysis performed allows us to identify mathematical structures that students must develop to overcome difficulties and obstacles posed by these models, and thus to achieve an adequate understanding of mathematical infinity, through the analysis of the conscious efforts and the new mechanisms and ways of reasoning developed by mathematicians of the past, to overcome these obstacles.

As an example, let us take the *tacit model* known as *inclusion*, reported by Ibn-Qurra in the IX century and by Galileo in the 17th century and that was deeply rooted on the unconscious assumption "the whole is greater than its part" based on the experience we have with finite collections. Let us recall that Dedekind in the 19th century was able to overcome the obstacle posed by this *tacit model*, by precisely questioning this unconscious preconception. He noted that, to be able to characterize the notion of size (cardinal number) for infinite sets, it was essential to develop a new mechanism for the comparison of the number of elements (that is, cardinality) of infinite sets. To do that, Dedekind first realized that if two finite sets have the same numbers of elements,

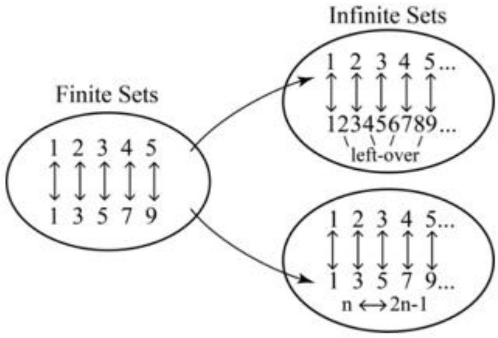


**Figure 1.** Different extensions of "same number as" & "pairable" for infinite collections (Diaz-Chang & Arredondo, 2022)

then we can put their elements in one-to-one correspondence, which also means that these two sets are "pairable". He further realized that if we consider that a finite set A has the "same number" of elements as a finite set B, it means that for each element of A, there is a corresponding element of B that can be removed, and that no element of B remains "left-over".

However, when he tried to extend these ideas to infinite sets, he realized that "pairable" and "same number as" were two different ideas that had the same meaning for finite collections but were cognitively different and did not have the same extension for infinite collections (Lakoff & Núñez, 2000), as it is shown in **Figure 1**.

By realizing this, Dedekind was able to choose "pairability" instead of our everyday notion of "same number as" to define his notion of "cardinality" for infinite sets, overcoming in this way the *inclusion* model and enabling the comparison of the number of elements of infinite sets. Note that in order to do that, he had to consciously ignore the "left-over" clause that is unconsciously implicit in our ordinary notion of "more than" (Lakoff & Núñez, 2000).

In an analogous way, each of the mechanisms that allowed to overcome each of the models mentioned above can be analyzed, to show students how those new mathematical structures must be developed in order to achieve a proper understanding of this mathematical concept (see, for example, Diaz-Chang & Arredondo, 2022).

## CONCLUSIONS

The analysis made revealed that the historical-epistemological reconstruction of this concept under the gaze of these models can show students how mathematical infinity evolved throughout the successive periods of its conception, exposing obstacles and difficulties that mathematicians of the past faced, showing to students, in this way, that they aren't the first or the only ones to make mistakes when dealing with this abstract concept. From this point of view, it is more understandable and natural for them to make mistakes, helping them realize that many times these mistakes are rooted in our unconscious perceptions, intuitions, and preconceptions about infinity (Fischbein, 2001). In Fischbein's opinion, the influence of such tacit, elementary, intuitive models on the course of mathematical reasoning, is much more important than is usually acknowledged, and he hypothesize that this influence is not limited to the pre-formal stages of intellectual development. His claim is that even after the student becomes capable of formal reasoning, elementary intuitive models continue to influence his ways of reasoning.

The relationships between the concrete and the formal in the reasoning process are much more complex than it is generally supposed. The idea of a tacit influence of intuitive, primitive, unconscious models on a formal reasoning process does not seem to have attracted Piaget's attention. In fact, our cognitive mechanisms are not controlled only by logical structures but, simultaneously, by a world of intuitive models acting tacitly and imposing their own constraints (Fischbein, 2001; Fukuta & Yamashita, 2021).

On a more general note, this analysis also helps students to reflect on the relativity of mathematical infinity as a concept throughout history, highlighting the importance of conveying that mathematical concepts are not something immutable, but rather are the fruits of their dialectical evolution as knowledge embodied in its socio-cultural context and historical development. As Mamolo (2017) points out, the history of mathematical practices and ideals related to infinity do seem to suggest that views of this mathematical concept depended largely on perspective. Aristotle's view that actual infinity cannot exist because it would take eternity to enumerate such a set of numbers, Galileo's conclusion of the impossibility of comparing infinite sets, Bolzano's notions about "distance" between elements, Cantor's sets, and Robinson's infinitesimals all are plausible within its contexts.

Moreover, this type of study shows the connection between the rigorous formalization process of infinity as a mathematical concept through history and the active conscious overcoming of these unconscious patterns of reasoning about infinity. Therefore, from our perspective, this type of reflection allows us to improve our teaching practice by designing activities aimed at achieving the development of conscious cognitive mechanisms relevant in each case, for each of these models, that could significantly affect the comprehension of this concept at the university classroom. Thus, further implementation of experiences incorporating the results of this study into class activities is recommended, and we argue that this implementation could be an element to be considered in the research agendas in mathematics education.

On the other hand, this study could also have a positive effect on the mathematical training of teachers and students, helping to broaden and enrich the cultural horizon from which the study of this mathematical concept is approached (González, 2004), even more if we take into account that mathematical infinity is a fundamental concept from the epistemological point of view, and its study is highly relevant for the history of the emergence of various disciplines of modern mathematics and for the understanding of numerous mathematical concepts at the university level (Weyl, 1949). This could also help students to grasp the complexity and relevance of mathematical infinity, and its unique place in "the paradise which Cantor has created for us" (Moore, 2002).

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