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A theoretical analysis of the validity of the Van Hiele levels of reasoning in graph theory

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Abstract

The need to develop consistent theoretical frameworks for the teaching and learning of discrete mathematics, specifically of graph theory, has attracted the attention of the researchers in mathematics education. Responding to this demand, the scope of the Van Hiele model has been extended to the field of graphs through a proposal of four levels of reasoning whose descriptors need to be validated according to the structure of this model. In this paper, the validity of these descriptors has been approached with a theoretical analysis that is organized by means of the so-called processes of reasoning, which are different mathematics abilities that students activate when solving graph theory problems: recognition, use and formulation of definitions, classification, and proof. The analysis gives support to the internal validity of the levels of reasoning in graph theory as the properties of the Van Hiele levels have been verified: fixed sequence, adjacency, distinction, and separation. Moreover, the external validity of the levels has been supported by providing evidence of their coherence with the levels of geometrical reasoning from which they originally emerge. The results thus point to the suitability of applying the Van Hiele model in the teaching and learning of graph theory.

Keywords: Graph Theory, Levels of Reasoning, Processes of Reasoning, Van Hiele Model

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Discrete mathematics is the part of mathematics devoted to the study of discrete structures such as sets, permutations, relations, graphs, trees, and finite-state machines (Rosen, 2019). Among the benefits of teaching and learning this branch of mathematics, Ouvrier-Buffet (2020) highlights 1) accessibility of problems and concepts, which can be explored by students without an extensive background; 2) richness of discrete concepts, which can usually be defined in multiple ways, so they bring an opportunity to deal with the construction of axiomatic theory; 3) versatility in questioning, proving, and modelling; and 4) involvement of students in mathematical experiences. These benefits have attracted the community of researchers in mathematics education, as shows the recent creation of a specific research group in the teaching and learning of discrete mathematics at the 13th International Congress on Mathematical Education (Hart & Sandefur, 2018), which gives special importance to graph theory. Indeed, there is a wealth of literature on research in graph theory from mathematics education showing several sources of interest: students' learning (Fielder & Dasher, 1968; Hazzan & Hadar, 2005, Medová et al., 2019), design of task sequences (Ferrarello & Mammana, 2018; Lodder, 2014), design of teaching resources (Carbonneaux et al., 1996; Milková, 2014; Geschke et al., 2005), teaching and learning of algorithms (Costa et al., 2014; Khalil et al., 2017; Moala, 2021; Sánchez-Torrubia et al., 2008), and inclusion of graph theory at different educational stages (Blanco & García-Moya, 2021; Ouvrier-Buffet, 2020).

Ouvrier-Buffet et al. (2018) point out the need for the use and development of appropriate theoretical frameworks for research in discrete mathematics education. To approach this issue, one can either create new theories or adopt an already established theoretical framework to extend its scope, being the latter highly recommended by the Educational Studies in Mathematics editors (2002) since it is a way to consolidate theories. In this sense, González et al. (2021) propose to extend the scope of the Van Hiele model to the field of graph theory. This model, mainly applied in geometry, has already been extended to other areas, as suggested by Van Hiele (1986). Indeed, Isoda (1996) applies Van Hiele levels to characterize the development of language about functions, and Nisawa (2018) uses the Van Hiele model to make an experimental study of the comprehension difficulties of high school students about functions. Other extensions, also proposed in the field of mathematical analysis, concern local approximation (Llorens-Fuster & Pérez-Carreras, 1997) and convergence of sequences (Navarro & Pérez-Carreras, 2006).

The idea of extending the scope of the Van Hiele model to graph theory (González et al., 2021) comes naturally since graphs are often represented as points connected by lines, reminding us of the vertices and sides of the geometric figures. Indeed, there is a visual resemblance between the pictorial representations of graphs and 2-dimensional geometric figures. Moreover, each simple polyhedron (i.e., without holes) can be viewed as a planar connected graph just by considering polyhedron vertices as graph vertices and polyhedron edges as graph edges, thus producing a correspondence between polyhedron faces and planar graph faces. Another important relationship between these two mathematical objects is that geometric figures are invariant under rigid movements, while graphs are invariant, furthermore, under any deformation that maintains the original connections between vertices (topological transformations). Therefore, it is expected that some descriptors of the Van Hiele model for geometric figures have their analogs in their extension for graphs and that the properties of graphs that cannot be extrapolated from geometric figures produce different descriptors.

The extension of the Van Hiele model to the field of graphs thus yields a new perspective in mathematics education, and so it should be evaluated from different points of view (Santos-Trigo & Barrera-Mora, 2007). Concretely, exploring the validity of such extension necessarily requires studying how well defined its elements and relationships within the model are, that is, its rigor and specificity (Schoenfeld, 2000). In other words, the objects and relationships in the new extension should be defined unambiguously and fit together in such a way that any researcher in the field could use them without contradicting the original model, bearing in mind that, as we have explained before, geometrical figures are somehow related to graphs. In the context of the present study, this implies that the descriptors of the Van Hiele levels for graphs should not contradict either the characteristics imposed by the theory itself whose scope is extended (internal validity) or the results already obtained in the research (external validity). Indeed, González et al. (2021) first justify the internal validity of their levels by arguing that they have the nature of Van Hiele levels (visual, analytical, informal, and formal), although they leave as pendant work to check that they satisfy the properties of Van Hiele levels (Usiskin, 1982), which differentiate them from levels in other frameworks (Godino et al., 2014; Arnon et al., 2014). Also, these authors give support to the external validity of their results by reviewing the literature on the teaching and learning of graph theory and mention that some descriptors of the Van Hiele model for geometric figures have their analogs in their extension for graphs and that the properties of graphs that cannot be extrapolated from geometric figures produce different descriptors.

The Van Hiele Levels in Geometry

The Van Hiele (1986) model provides a characterization of the reasoning in the field of geometry through five levels, but the last one is omitted here because it refers to the reasoning of professional mathematicians, which is beyond the scope of the present study. The first four levels are described as follows:

- Level 1 (visualization). Students recognize geometric figures by their shape and as a whole. They can describe them in terms of physical characteristics or by comparison with everyday objects, using non-mathematical language.
- Level 2 (analysis). The ability to identify parts and properties of figures allows students at this level to make descriptions of geometric notions in mathematical terms.
- Level 3 (informal deduction). Students can recognize relationships between properties of figures and justify them informally. They have certain handling of propositional logic that allows them to make logical classifications.
- Level 4 (formal deduction). At this level, students can develop formal mathematical proofs. Also, they accept that there might be equivalent definitions for the same geometric concept.

The following properties, which give consistency and coherence to the Van Hiele model, are essential features of its levels (Usiskin, 1982):

- Fixed sequence. A student cannot be at level n without having passed through level n-1.
- Adjacency. Objects that are intrinsic at level n-1 become extrinsic at level n.
- Distinction. Each level has its linguistic symbols and its network of relationships that connect those symbols.
- Separation. Two people who reason at different levels will not be able to understand each other.

As the present work only deals with learning, the description of the other property of the Van Hiele model, attainment, is omitted here because it is related to the *phases*, which are a set of guidelines to be followed by teachers to help students progress through the levels.

Basic Notions of Graph Theory

We first present some background on graph theory (Biggs, 2003) that will be needed to describe the reasoning in this field of mathematics. A *graph* G is defined as a pair (V, E) where V is any given set (called *vertex set*) and E (called *edge set*) is a set of unordered pairs of elements of V. A *subgraph* of a graph $G = (V, E)$ is another graph, say $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$ whenever V' contains all vertices of the edges of E'. For example, the graph $G_1 = \{\{a, b, c, d, e\}, \{ab, bc, cd, de, ad, be\}\}$ contains $G_2 = \{\{a, b, c, d\}, \{ab, bc, cd, ad\}\}$ as a subgraph. Graphs are usually represented by drawing its vertices as points in the plane and edges by (not necessarily straight) segments joining its corresponding vertices (*pictorial representation*), and clearly such a representation is not unique for each graph, as Figure 1 shows. Also, there exist other representation systems based on numerical matrices, sequences of integers, planar representations of geometrical shapes, etc.

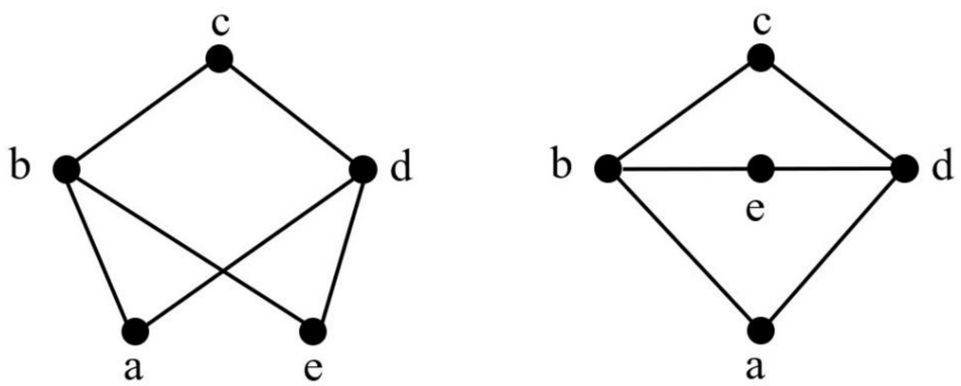

Figure 1. Two pictorial representations of graph G_1

Graph properties can be divided into *global* (i.e., associated with the whole graph) and *local properties* (i.e., associated with parts of the graph). Thus, an example of a local property is the *degree* of a vertex, which is the number of edges that contain it. Examples of global properties include connectivity, Eulerianity, and planarity. Indeed, a graph is said to be 1) *connected* if any pair of vertices can be joined by a sequence of adjacent vertices of the graph; 2) *Eulerian* if there exists a sequence of edges that visits every edge exactly once and that starts and ends at the same vertex; and 3) *planar* if it has a pictorial representation without edge crossings. These basic graph properties allow defining classic families: *cycles*, which are connected graphs having every vertex of degree 2; *paths*, which are connected graphs with two vertices of degree 1 and the rest of the vertices with degree 2; and *trees*, which are connected graphs that do not contain any cycle as a subgraph.

The Van Hiele Levels in Graph Theory

González et al. (2021) provide a detailed characterization of four levels of reasoning in graph theory through the lens of the Van Hiele model. The authors conceived and characterized these levels by means of prior research on students' mathematical thinking, their mathematical understanding of the concept of graph as researchers in mathematics and didactics of mathematics, and their experience as graph theory teachers. The levels are described in general terms as follows:

- Level 1 (GT) (visualization). Students at this level are highly conditioned by a visual type of recognition that restricts them to deal only with known representations of graphs, which are perceived as wholes and described using a non-mathematical language. Also, they can partially handle some global properties and graph notions that do not require specific mathematical knowledge.
- Level 2 (GT) (analysis). These students can handle both global and local properties that enable them to identify graphs independently of their representations. Also, they are able to deal with basic graph notions stated in mathematical terms.
- Level 3 (GT) (informal deduction). Students at this level have logical abilities that allow them to understand and recognize relationships between graph properties. This makes deductive reasoning possible, and so they can make logical classifications of graph families.
- Level 4 (GT) (formal deduction). Students at level 4 regard graphs as abstract mathematical objects, so they can deal with equivalent definitions for the same concept. Also, these students can construct formal proofs of mathematical statements by performing techniques commonly used in graph theory.

We want to remark that the notation (GT), added to each level of reasoning in graph theory, has been introduced in this paper to facilitate the reading since it allows to better distinguish them from the levels of reasoning in geometry. This work studies the validity of the extension of the Van Hiele model to the field of graph theory with the aim of answering the following research questions:

- Do the levels of reasoning for graphs proposed by González et al. (2021) satisfy the properties of the Van Hiele levels?
- Are those levels coherent with the descriptors of the Van Hiele levels of geometrical reasoning?

METHODS

In general, the research cycle (Jaime & Gutiérrez, 1990) that is intended to be completed in the long term, and which includes the theoretical analysis of the model, consists of the following: the researchers observe a phenomenon in the educational context (in our case the learning of students of Graph Theory); they search for regularities in the behaviour of these students and then they propose an initial model that will serve to describe this learning (in our case the Van Hiele model for graph theory); afterwards, a study of its formal structure is conducted (the objective of this work) and then the model is verified using empirical data (aim of future research), in order, if necessary, to modify or refine some aspects of the initial model. Once the model is validated, it can be used by teachers to improve their teaching and obtain better results. We next perform a theoretical analysis on the validity of the Van Hiele levels of reasoning in graph theory proposed by González et al. (2021), in the same way as other studies that also reflect on the validity of their proposals. Indeed, there are papers that are concerned with the validity regarding the Van Hiele model, such as the work of Llorens-Fuster and Pérez-Carreras (1997), and other frameworks such as the APOS theory, for instance, the theoretical work of Roa-Fuentes and Oktaç (2010). The analysis of this work follows two steps:

1. *Internal validity.* This step verifies that the descriptors of the levels of reasoning proposed for graphs fit the properties of the Van Hiele model concerning levels: fixed sequence, adjacency, distinction, and separation.
2. *External validity.* This step consists of verifying that the indicators of levels of reasoning in graph theory do not contradict those for geometry, thus comparing and contrasting the descriptors for the levels of both fields. Specifically, it is necessary to explore the analogies between graphs and geometric figures and check that the corresponding indicators for both fields agree, and to point out the particular characteristics of graphs not translatable from geometry, which generate extra indicators without an analogue to those of geometric figures.

Carrying out both steps require arranging the abilities involved when learning both fields of mathematics so that they can be compared, as developed below.

A Tool to Organize the Descriptors of Van Hiele Levels: The Processes of Reasoning

Gutie□rrez and Jaime (1998) provide a useful characterization for the Van Hiele levels of geometric reasoning given in terms of the evolution of the processes of reasoning, which refer to students' abilities involved when solving geometrical tasks:

- Recognition consists of the identification of types and families of geometric figures, as well as their parts and properties. This process is based on the physical and global attributes of figures at level

1, while students at level 2 or higher can recognize mathematical properties. Thus, this process does not distinguish between level 2, 3, or 4 students.

- Use of definitions means the handling of geometrical concepts that can be introduced by a book, a teacher, or another student. Level 1 students are unable to use a mathematical notion, whereas level 2 students use definitions with simple logical structure and based on properties they know. Level 3 students can establish logical relationships between properties, which allow them to use any definition. Level 4 students understand the logical structure of mathematics quite deeply, so they accept that the same concept can have several equivalent definitions.
- Formulation of definitions is the ability to describe or characterize geometrical concepts. This process starts at level 1 with students that describe geometric objects through lists of physical properties, with non-mathematical language. In contrast, level 2 students formulate definitions as lists of mathematical properties, but they may contain deficiencies or redundancies. Level 3 students overcome this difficulty, thus formulating definitions as necessary and sufficient sets of properties. This ability is finally improved at level 4 when students can also prove the equivalence of definitions.
- Classification is the assignment of geometric figures or concepts to families or classes. Level 1 students only understand and perform partitional classifications based on physical attributes, so they are not able to place the same object in two different classes. Similarly, those of level 2 only make partitional classifications but based on mathematical attributes. From level 3, students can provide partitional and hierarchical classifications, so this process cannot distinguish between level 3 and 4 students.
- Proof means explaining in some convincing way why a mathematical statement is true or false. Level 1 is not considered for this process since students do not understand the concept of mathematical proof. Level 2 students, when asked to write a proof, check that it is true in one or more examples. Level 3 students can perform informal logic proofs and may use examples to help themselves with reasoning. Finally, level 4 students can understand and write formal proofs.

This characterization of the Van Hiele levels of geometrical reasoning is adapted by González et al. (2021) to the field of graph theory. They provide descriptors for the evolution of the analogous processes along the levels (GT) that they propose, even though there are peculiarities of graphs that cannot be translated from geometric figures:

- *Recognition* entails the identification of specific graphs and families, subgraphs, properties, and relations between properties and representation systems. Level 1 (GT) students recognize graphs according to their visual appearance, which limits them to identify a specific graph, even in different representation systems, only when they are familiar with the shape of the given representation. Also, they can partially recognize global properties, that is, they associate them with particular representations instead of the graph itself. At level 2 (GT), students can recognize local and global properties of graphs, regardless of how they are represented, even in different representation systems. Level 3 (GT) students can recognize relationships between properties, which makes this process more efficient. Also, they improve their understanding of the relationships between different representation systems. Finally, graphs are perceived by level 4 (GT) students with the maximum degree of formality, that is, as formal mathematical objects.
- *Use of definitions* involves the employment of graph theory concepts to solve a task and includes

the understanding of properties or the application of algorithms, among other graph notions. Students at level 1 (GT) can use definitions that do not require any knowledge of graph theory. At level 2 (GT), they can use definitions with simple logical structure, based on known properties related by conjunctions, disjunctions, or negations. Level 3 (GT) students handle any type of definition and finally they accept at level 4 (GT) that many concepts have equivalent definitions.

- *Formulation of definitions* means to describe or characterize graph theory concepts, their properties, families of graphs, etc. Level 1 (GT) students make visual descriptions through comparisons with everyday objects and using non-mathematical language. At level 2 (GT), they can develop definitions consisting of lists of mathematical properties that may contain redundancies or deficiencies. At level 3 (GT), they can write definitions as sets of necessary and sufficient conditions and, at level 4 (GT), they can formulate equivalent definitions for the same concept.
- *Classification* is the organization of graphs into families according to properties. Level 1 (GT) students can make partitional classifications throughout visual recognition, and those of level 2 (GT) can also perform partitional classifications but justifying them with mathematical properties. At level 3 (GT), there is an improvement in logic skills that enables students to provide both partitional and hierarchical classifications. Students at this level (GT) have attained the maximum degree of ability in classification, so this process does not distinguish between level 3 (GT) and level 4 (GT).
- *Proof* of mathematical statements in this field comprises providing justifications to support the validity of classical theorems, equivalence of definitions, validity of algorithms, etc. Level 1 (GT) students usually give visual arguments to convince someone of the veracity of an affirmation on graphs. Also, they can check it in particular graph representations that may be unappropriated for that affirmation. In contrast, level 2 (GT) students use adequate examples of graphs, as they are aware of the variety of their representations, to verify such an affirmation. Level 3 (GT) students know that visual arguments and specific verifications are insufficient to prove a statement. They can write informal proofs that require a few logical steps and, at most, understand the steps of a formal proof, which can only be elaborated by level 4 (GT) students.

To illustrate the evolution of the process of formulation of definitions, we can focus on the notion of cycle. A level 1 (GT) student could say that “it looks like a necklace”, in contrast with a level 2 (GT) answer, which contains mathematical language, such as “it is a graph with all vertices of degree 2”. However, this definition lacks the connectivity property, which might be considered by level 3 (GT) students. Finally, a level 4 (GT) student could provide several equivalent definitions such as “any graph such that the removal of any edge produces a path”. For more examples of the remaining processes, we refer the reader to the work of González et al. (2021).

RESULTS AND DISCUSSION

The descriptions of the processes of reasoning in graph theory (González et al., 2021), seen as a whole, form a characterization of the levels of reasoning in this field. The next four subsections are devoted to verifying that such a characterization fulfills the general properties of the Van Hiele model that concern its levels: fixed sequence, adjacency, distinction, and separation. The fifth subsection studies the coherence with the geometrical case by comparing the processes of both fields of mathematics.

Fixed Sequence

This property indicates that the reasoning skills of a given level require prior acquisition of the skills of the previous level. Indeed, at level 2 (GT), the ability to recognize subgraphs and local properties supports the rest of the skills acquired at this level, not only those of recognition, which concern the independence of the given representation and the ability to make translations between representation systems, but also those involved in the rest of the processes: use of definitions with simple logical structure, formulation of definitions through lists of properties, partitional classifications using properties, and empirical proofs using examples. Thus, the key skill of level 2 (GT) is the recognition of subgraphs and local properties, and this implies having previously visualized the graph as a whole, which occurs at level 1 (GT).

At level 3 (GT), the ability to relate properties, both of graphs and of representation systems, is necessary to achieve most of the skills that arise at this level. On the one hand, the recognition of the relationships among graph properties allows deducing some properties through others. Moreover, this yields a more efficient recognition of such properties and the use and formulation of any definition, the hierarchical classification of graph families, and the elaboration of informal proofs. One cannot think of having acquired this ability to relate properties without having previously gone through level 2 (GT), at which these properties are identified and analyzed separately. On the other hand, level 3 (GT) students have a sufficiently broad vision of the concept of graph that allows them to handle the relationships between different representation systems, which requires previously knowing how to perform translations between them, which is a skill of level 2 (GT).

Level 4 (GT) students see graphs as formal mathematical objects, which supports the handling of the equivalence of definitions. This vision of graphs necessarily includes the understanding of the relationships between the different representation systems that occurs at level 3 (GT). Likewise, at level 4 (GT), students can formally prove propositions about graphs, which implies the ability to relate their properties, as happens at level 3 (GT), since writing a proof consists precisely of correctly linking properties through relationships between them.

Adjacency

We check this property by exhibiting the difficulties that students may face during learning. These difficulties are related to objects that implicitly appear at a given level (i.e., they can be perceived by students but not handled) and become explicit at the next level (Jaime & Gutiérrez, 1990). Indeed, González et al. (2021) include in their description of the evolution of each process along the levels (GT) the difficulties that students can find at each level (GT), as specified below.

At level 1 (GT), all reasoning processes are affected by several difficulties since students do not have the ability to recognize subgraphs and local properties of graphs, which are explicit objects at the following level (GT). For instance, the recognition process is limited to the visual aspect of the graph representations known by students because assimilating the independence of the representation requires a recognition of the parts and properties of graphs that distinguish them from others. Likewise, this lack makes students unable to formulate definitions with mathematical language or classify graphs considering their properties. In the process of proof, students cannot even provide justifications in proper examples of graphs, but in particular known representations.

At level 2 (GT), students can analyze some graph properties and make translations between representation systems. However, difficulties related to the need to recognize the relationships among graph properties or representation systems can arise. For instance, they can find difficulties when trying to identify graph properties that cannot be checked directly but through sufficient conditions. Moreover, these students may have trouble when trying to understand the limitations of a particular representation system, managing complex definitions, classifying hierarchically, or proving statements beyond checking in concrete examples.

At level 3 (GT), students' difficulties stem, on the one hand, from a lack of formality in their view of graphs, which is necessary, for example, to admit the equivalence of definitions. On the other hand, these students lack the ability to elaborate chains of interrelated properties, which is necessary to construct formal multistep proofs, an inherent characteristic of level 4 (GT).

Distinction

We verify this property employing the terminology of Jaime and Gutiérrez (1990). These authors represent mental structures as networks of relationships whose nodes are the different concepts and their representations, and whose connections between nodes are given by the established relationships (through memory, observation, or logic) between them. Thus, the progression from a level to a higher level is modeled as the appearance of new concepts and relationships that form a new and more complex network.

The level 1 (GT) network is formed by independent subnetworks corresponding to the different families of graphs, so that each subnetwork is given by the name of the family and the mental images of the representations known by the student, connected by memory and observation. In the subnetworks, there also appear some global properties that students can recognize at this level (GT). These properties are not directly connected to the family name, but to the concrete representations in which students know how to recognize that property (also through memory and observation), since the recognition of global properties at this level (GT) is only partial (Figure 2).

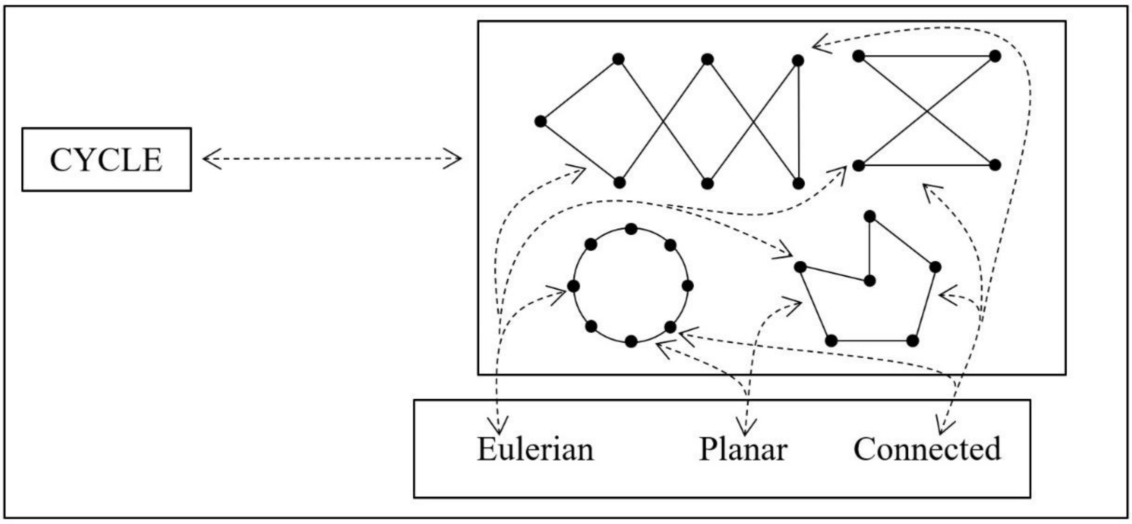

Figure 2. Example of a level 1 (GT) subnetwork related to the family of cycles

During the process of acquisition of level 2 (GT), the structure of level 1 (GT) is enriched by the appearance of local properties in each of the subnetworks, which together with the global properties, allow students to describe families of graphs independently of their representations. This unifies mental images of concrete representations into a single image that encompasses the entire family for each known representation system, giving the possibility to move between these unified images. Thus, in each subnetwork, a direct connection is generated between the family name and its properties, although these connections do not serve to join between different subnetworks (Figure 3). Furthermore, due to the lack of logical skills at this level (GT), the connections between nodes of the subnetworks are still established by memory and observation, as occurs at level 1 (GT).

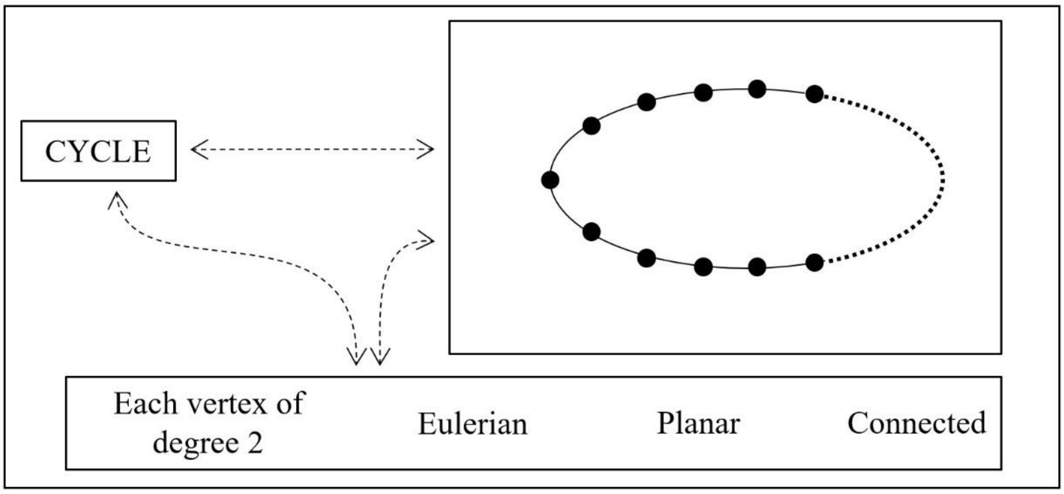

Figure 3. Example of a level 2 (GT) subnetwork related to the family of cycles

At level 3 (GT), new nodes appear that correspond to more complex concepts than those that students can handle at level 2 (GT). In addition to the connections established through memory and observation, logical connections appear for the first time. The relationships between properties enable students to consider minimal sets of necessary and sufficient properties to define graph families. Thus, subnetworks are no longer independent since they are now connected through the properties that characterize them, which generates logical relationships between the corresponding families.

At level 4 (GT), the corresponding network experiences few changes in terms of the number of nodes that appear. However, the abilities acquired at this level (GT) allow students to create new connections between nodes, such as, for example, those between sets of properties that equivalently define the same concept. Another change in the network concerns the nature of the level 4 (GT) connections, which are now formal and abstract, in contrast to those of level 3 (GT), mainly informal and based on manipulation.

Separation

This property refers to the fact that students at a certain level cannot frequently understand students at higher levels in relation to a common object of reasoning. Indeed, level 1 (GT) students do not understand level 2 (GT) students when the latter provide reasons based on local properties. Likewise, level 2 (GT) students cannot understand level 3 (GT) students who relate properties to justify that one family of graphs is included within another. Similarly, a level 3 (GT) student cannot follow a proof that a level 4 (GT) student could give to demonstrate the equivalence between two definitions, as level 3 (GT) students usually do not accept multiple definitions for the same concept. The following statements about graphs clearly show these differences between the forms of reasoning at each level (GT):

1. "This graph is a path because it reminds me of a rope with knots".
2. "A tree is a connected graph without cycles".
3. "Since paths are connected and they have no cycles, then they are also trees".
4. "We will use the induction method to prove that trees can also be defined as connected graphs that have one edge less than vertices".

These statements are also clear indicators of the close relationship between language and levels (Jaime & Gutiérrez, 1990). Indeed, the language of level 1 (GT) uses a non-mathematical vocabulary, based on visual attributes, and employs comparisons of graphs with objects familiar to students, in contrast to the language of level 2 (GT), whose words belong to the mathematical vocabulary (local properties, types of subgraphs, etc.). Level 3 (GT) features a language that links the words of level 2 (GT) by means of terms that come from propositional logic: “implies”, “then”, “if and only if”, “necessarily”, etc. Finally, level 4 (GT) students use the vocabulary of formal mathematics to organize the relationships learned at the previous level (GT), thus handling terms such as “induction”, “proof by contradiction”, etc.

A Comparison between the Van Hiele Levels of Reasoning in Graph Theory and in Geometry

The recognition process in graph theory is quite different from the recognition process in geometry, since the former has indicators generated by some characteristics of graphs that cannot be inferred from characteristics of geometric figures. Indeed, although at level 1 both objects are perceived as a whole, numerous global properties of graphs can be identified, even if only partially due to the limitations of this level in graph theory. This generates an additional indicator for this process at level 1 for graphs that does not occur in geometric figures. Likewise, although a level 2 indicator for both cases is the recognition of their components and mathematical properties, it is remarkable that students at this level for graphs must, in addition to performing translations between different representation systems, recognize them independently of the chosen representation. This is a crucial moment in the learning of graph theory because, unlike the geometric case, there is a great variety of transformations that leave graphs invariant in addition to rigid movements.

The acquisition of levels 3 and 4 (GT) requires an improvement in recognition skills that again implies another notable difference with respect to geometric recognition, whose maximum degree of acquisition is at level 2. Indeed, the complexity of certain reasonings on graphs requires a more efficient level 3 (GT) recognition than that of the preceding level, through which students can recognize the relationships that exist between properties. Likewise, it is necessary at level 3 (GT) to understand the relationships that exist between different systems of representation, a skill necessary to perceive graphs with the depth that level 4 (GT) students do, that is, as formal mathematical objects.

In contrast to the recognition process, the formulation of definitions for graphs unfolds in the same way as in the geometric model: visual descriptions that highlight the physical characteristics of the objects using a language that is not necessarily mathematical (level 1), lists of mathematical properties (level 2), sets of necessary and sufficient conditions (level 3), and equivalence of definitions (level 4). Note that Gutie□rrez and Jaime (1998) mention that, in this process, level 4 students can prove the equivalence of definitions, which rather concerns the proof process. Thus, it can be inferred that this descriptor necessarily involves the formulation of equivalent definitions for the same geometrical concept, which agrees with its analogue for graphs proposed by González et al. (2021).

The same is true for the classification process, which begins with partitional classifications at levels 1 and 2, based, respectively, on visual recognition and mathematical properties, and ends at level 3 when students admit partitional and hierarchical classifications.

An intermediate case is the use of definitions, which is similar for graphs and for geometric figures but only from level 2 since this process does not make sense at level 1 for the geometric case. In the case of graph theory, there are notions that can be handled at level 1 (GT) without the need for knowledge of graphs beyond the mere distinction between vertices and edges. In the remaining levels, for both mathematical objects, students handle definitions with a simple logical structure at level 2, any definition at level 3, and equivalent definitions at level 4. The other intermediate case is the process of proof, which is analogous for both mathematical objects from level 2 since it is also meaningless at level 1 for geometry because there is not yet a recognition of properties. Instead, level 1 (GT) students recognize some global properties that they can try to justify by visual arguments or by checking them on specific representations of graphs. At level 2 (GT), students use real examples of graphs to try to prove properties, just as geometry students do with particular examples of figures. Similarly, at the following levels, the process of proof is analogous for both mathematical objects: informal proofs at level 3 and formal proofs at level 4.

CONCLUSION

According to the scheme of creation and use of an educational model (Jaime & Gutiérrez, 1990), this work provides a theoretical analysis of the formal structure of an initial model for the learning of graph theory. Concretely, the present study deepens the validity of the extension of the Van Hiele model to the field of graphs introduced by González et al. (2021), who provide descriptors of four levels of reasoning in this area. To do this, the analysis carried out has been structured through the processes of reasoning proposed by Gutie�rrez and Jaime (1998): recognition, use and formulation of definitions, classification, and proof.

The internal validity of the levels of reasoning in graph theory, which provides an affirmative answer to the first research question of this work, has been supported by verifying the properties of the Van Hiele levels (Usiskin, 1982). Indeed, they have shown hierarchical structure (fixed sequence) and both intrinsic and extrinsic objects of reasoning (adjacency). Also, these levels (GT) are characterized by the use of a series of symbols and relationships among them (distinction) that may impede communication between students at different levels (GT) (separation). Thus, a future line of research could focus on the fifth property of the Van Hiele model, attainment, which concerns prescriptive aspects. Specifically, the model proposes the design of task sequences according to the levels and the different learning phases that allow students to progress in the acquisition of the Van Hiele levels. It is necessary to explore these features of the model, since they could be used by teachers to help their students advance in the learning of graph theory.

The external validity, which gives an affirmative answer to the second research question, has been supported by the analysis of the coherence of the Van Hiele levels for graph theory with the levels of geometrical reasoning. Indeed, the comparison between the indicators of both types of reasoning has allowed verifying that they are coherent despite the existence of indicators for the reasoning in graph theory without an analogue in the geometrical context. This fact is due to the nature of graphs, which are mathematical objects that sometimes require more abstraction than geometric figures. Thus, the descriptors of the levels of geometrical reasoning can be regarded in some way contained in those for the reasoning in graph theory. This allows detaching the Van Hiele model from the geometrical context, which makes interesting the study of connections among the existing particularizations of this model to other mathematical objects. Such an approach would go towards a new formulation of the Van Hiele model independently of the underlying mathematical object, which follows the spirit of the community of researchers in mathematics education that encourage to go from middle-range theories to a grand theory (Silver & Herbst, 2007).

Therefore, the results obtained in this paper support a priori validation of the levels of reasoning in graph theory under the lens of the Van Hiele model, which suggests the possibility to apply this model also in the teaching of graph theory, similarly as it has been done in the field of geometry by means of the instructional component of the Van Hiele model. However, a limitation of this work is precisely that a posteriori validation should be explored. Indeed, it is necessary to collect empirical data for future works aiming to either confirm the adequacy of the initial model or introduce modifications that allow us to better describe the phenomena under study, that is, the understanding of graph theory concepts. This would lead to the refinement of the initial proposal of González et al. (2021) with more precise descriptors and thus to the completion of a research cycle (Jaime & Gutiérrez, 1990).

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REFERENCES

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. Springer. https://doi.org/10.1007/978-1-4614-7966-6

Biggs, N. L. (2003). *Discrete mathematics* (2nd ed.). Oxford University Press.

Blanco, R., & García-Moya, M. (2021). Graph theory for primary school students with high skills in mathematics. *Mathematics, 9*(13), 1567. https://doi.org/10.3390/math9131567

Carbonneaux, Y., Laborde, J. M., & Madani, R. M. (1996). CABRI-Graph: A tool for research and teaching in graph theory. In F. J. Brandenburg (Ed.), *Graph Drawing. GD 1995. Lecture Notes in Computer Science* (Vol. 1027, pp. 123-126). Springer. https://doi.org/10.1007/BFb0021796

Costa, G., D’Ambrosio, C., & Martello, S. (2014). Graphsj 3: A modern didactic application for graph algorithms. *Journal of Computer Science, 10*(7), 1115- 1119. https://doi.org/10.3844/jcssp.2014.1115.1119

Educational Studies in Mathematics Editors (2002). Reflection on educational studies in mathematics: The rise of research in mathematics education. *Educational Studies in Mathematics, 50*(3), 251-257. https://doi.org/10.1023/A:1021259630296

Ferrarello, D., & Mammana, M. F. (2018). Graph theory in primary, middle, and high school. In E. W. Hart & J. Sandefur (Eds.), *Teaching and learning discrete mathematics worldwide: Curriculum* *and research. ICME-13 Monographs* (pp. 183-200). Springer. https://doi.org/10.1007/978-3-319-70308-4_12

Fielder, D. C., & Dasher, B. J. (1968). Some Classroom Uses of Ambits in Teaching Graph Theory. *IEEE Transactions on Education, 11*(2), 104-107. https://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=4320356

Geschke, A., Kortenkamp, U., Lutz-Westphal, B., & Materlik, D. (2005). Visage – visualization of algorithms in discrete mathematics. *ZDM Mathematics Education, 37*(5), 395-401. https://doi.org/10.1007/s11858-005-0027-z

Godino, J. D., Aké, L. P., Gonzato, M., & Wilhelmi, M. R. (2014). Niveles de algebrización de la actividad matemática escolar. Implicaciones para la formación de maestros [Algebrization levels of school mathematics activity. Implication for primary school teacher education]. *Enseñanza de las Ciencias 32*(1), 199-219. https://doi.org/10.5565/rev/ensciencias.965

González, A., Gallego-Sánchez, I., Gavilán-Izquierdo, J. M., & Puertas, M. L. (2021). Characterizing levels of reasoning in graph theory. *EURASIA Journal of Mathematics, Science and Technology Education*, *17*(8), article em1990. https://doi.org/10.29333/ejmste/11020

Gutie￾rrez, A., & Jaime, A. (1998). On the assessment of the van Hiele levels of reasoning. *Focus on Learning Problems in Mathematics, 20*(2/3), 27-46. https://www.uv.es/Angel.Gutierrez/archivos1/textospdf/GutJai98.pdf

Hart, E. W., & Sandefur, J. (Eds.) (2018). *Teaching and learning discrete mathematics worldwide: Curriculum and research*. Springer. https://doi.org/10.1007/978-3-319-70308-4

Hazzan, O., & Hadar, I. (2005). Reducing abstraction when learning graph theory. *Journal of Computers in Mathematics and Science Teaching, 24*(3), 255-272. http://citeseerx.ist.psu.edu/viewdoc/download? doi=10.1.1.96.5275&rep=rep1&type=pdf

Isoda, M. (1996). The development of language about function: An application of van Hiele levels. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 105-112). University of Valencia. https://math-info.criced.tsukuba.ac.jp/isoda_lab/isoda_pdf/unk.0000.00.07.pdf

Jaime, A., & Gutiérrez, A. (1990). Una propuesta de fundamentación para la enseñanza de la geometría: El modelo de van Hiele [A proposal for the foundation for the teaching of geometry: The van Hiele model]. In S. Llinares & V. M. Sánchez (Eds.), *Teoría y práctica en educación matemática* (pp. 295-384). Alfar.

Khalil, E., Dai, H., Zhang, Y., Dilkina, B., & Song, L. (2017). Learning combinatorial optimization algorithms over graphs. In U. von Luxburg & I. Guyon (Eds.), *Advances in Neural Information Processing Systems*, (pp. 6351–6361). https://dl.acm.org/doi/pdf/10.5555/3295222.3295382

Lodder, J. (2014) Networks and spanning trees: The juxtaposition of Prüfer and Borůvka, *PRIMUS, 24* (8), 737-752. https://doi.org/10.1080/10511970.2014.896835

Llorens-Fuster, J. L., & Pe￾rez-Carreras, P. (1997). An extension of van Hiele's model to the study of local approximation. *International Journal of Mathematical Education in Science and Technology, 28*(5), 713-726. https://doi.org/10.1080/0020739970280508

Medova□, J., Pa□leni□kova□, K., Rybansky□, L̆., & Nas̆ticka□, Z. (2019). Undergraduate students' solutions of modeling problems in algorithmic graph theory. *Mathematics*, *7*(7), 572. https://doi.org/10.3390/math7070572

Milkova□, E. (2014). Puzzles as excellent tool supporting graph problems understanding. *Procedia- Social and Behavioral Sciences 131*, 177-181. https://doi.org/ 10.1016/j.sbspro.2014.04.100

Moala, J. G. (2021). Creating algorithms by accounting for features of the solution: The case of pursuing maximum happiness. *Mathematics Education Research Journal, 33*(2), 263-284. https://doi.org/10.1007/s13394-019-00288-9

Navarro, M. A., & Pe□rez-Carreras, P. (2006). Constructing a concept image of convergence of sequences in the van Hiele framework. *CBMS Issues in Mathematics Education, 13,* 61-98. https://doi.org /10.1090/cbmath/013

Nisawa, Y. (2018). Applying van Hiele's levels to basic research on the difficulty factors behind understanding functions. *International Electronic Journal of Mathematics Education, 13*(2), 61-65. https://doi.org/10.12973/iejme/2696

Ouvrier-Buffet C. (2020) Discrete mathematics teaching and learning. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 227-233). Springer. https://doi.org/10.1007/978-3-030-15789-0_51

Ouvrier-Buffet, C., Meyer, A., & Modeste, S. (2018). Discrete mathematics at university level. Interfacing mathematics, computer science and arithmetic. In V. Durand Guerrier, R. Hochmuth, S. Goodchild, & N. M. Hogstad (Eds.), *Proceedings of INDRUM 2018 – second conference of the international network for didactic research in university mathematics* (pp. 255-264). University of Agder. https://hal.archives-ouvertes.fr/hal-01849537/document

Roa-Fuentes, S., & Oktaç, A. (2010). Construcción de una descomposición genética: Análisis teórico del concepto transformación lineal [Constructing a genetic decomposition: Theoretical analysis of the linear transformation concept]. *Revista Latinoamericana de Investigación en Matemática Educativa*, *13*(1), 89-112. http://www.scielo.org.mx/pdf/relime/v13n1/v13n1a5.pdf

Rosen, K. H. (2019). *Discrete Mathematics and its Applications* (8th ed.). McGraw Hill.

Sánchez-Torrubia, G., Torres-Blanc, C., & Giménez-Martínez, V. (2008). An eMathTeacher tool for active learning Fleury's algorithm. *International Journal Information Technologies and Knowledge*, *2*, 437-442. http://sci-gems.math.bas.bg/jspui/bitstream/10525/213/1/ijitk02-5-p07.pdf

Santos-Trigo, M., & Barrera-Mora, F. (2007). Contrasting and looking into some mathematics education frameworks. *The Mathematics Educator, 10*(1), 81-106. https://www.uaeh.edu.mx/investigacion/icbi/LI_EconomiaFinanzasMat/Barrera_Mora/Santos-Barrera-2007.pdf

Schoenfeld, A. H. (2000). Purposes and methods of research in mathematics education. *Notices of the AMS, 47*(6), 641-649. https://www.ams.org/notices/200006/fea-schoenfeld.pdf

Silver, E., & Herbst, P. (2007). The role of theory in mathematics education scholarship. In F. Lester (Ed.), *Second handbook of research in mathematics teaching and learning* (pp. 39-67). Information Age.

Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry.* CDASSG Project. https://files.eric.ed.gov/fulltext/ED220288.pdf

Van Hiele, P. M. (1986). *Structure and insight. A theory of mathematics education*. Academic Press.