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Worked-examples instruction versus Van Hiele teaching phases: A demonstration of students' procedural and conceptual understanding

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Abstract

This study explores the worked-example instruction (WEI) and the van Hiele teaching phases (VHTP) pedagogies to advance students' acquisition of procedural and conceptual understanding of solving simultaneous equations. The quasi-experimental study involved two groups of high school students (age=15): 157 students in total with 72 in one group and 85 in the other. The study followed a pre-, post- and delay tests design and adapted two conceptual frameworks: the structure of the observed learning outcomes (SOLO) model and the Rasch model. It employed Rasch analysis and statistical package for social sciences (SPSS) as data analysis tools. The results indicated that both WEI and VHTP improved students' procedural and conceptual understanding of solving simultaneous equations at the post-test; however, the WEI effects (on both procedural and conceptual understanding) were not sustained after the post-test while the VHTP had a lasting effect on only conceptual understanding. Furthermore, the VHTP group significantly outperformed the WEI group at the post-test and delay test in both conceptual and procedural understanding. These results indicated that the WEI is only beneficial at the initial stage of knowledge acquisition and VHTP is better at the initial and long-term. Practical implications of these results were discussed.

Keywords: Cognitive Load Theory, Simultaneous Equations, Van Hiele Teaching Phases, Van Hiele Theory, Worked-Example Instruction

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Mathematics is globally recognized as an important subject at high school (with students' age range 11–17years) because it enhances the understanding of other subjects and its conscious and unconscious usage in everyday activities. However, students do not seem to have a deep understanding of mathematical ideas (Ngu & Phan, 2021; Omobude, 2014), as demonstrated by their perception that solving mathematical problems is about following a series of disjointed tricks that lead to the answer and are detached from reality (Khalid, 2006; Popovic & Lederman, 2015; Vos, 2018). Specifically, students were reported to be struggling with solving complex algebraic equations with rich knowledge of concepts and procedures that may prompt future use to real-life contexts (Johari & Shahrill, 2020; Ngu et al., 2018). Moreover, several studies identified solving simultaneous equations contextually as one of the difficult topics in high school (Johari & Shahrill, 2020; Kolawole & Ojo, 2019; Ugboduma, 2012). Similarly, Omobude (2014) claims that students may be able to solve a simultaneous equation following automated solution procedures without understanding what equality means, the interconnection of the unknowns, and the conceptual meaning of the variables. Consequently, students may be limited to acquire only formal mathematical knowledge (following procedures) which restrict them from demonstrating competencies of mathematical knowledge and skills outside their classrooms. For example, studies have revealed that despite success recorded in schools and external examinations, students are unable to transfer their mathematical knowledge to further learning and solving real-life problems (Wijaya et al., 2014). This weakness has been attributed to teachers' pedagogical approaches, which are characterized by too much emphasis on passing examinations rather than on life-long learning (Bolstad, 2021; Li & Schoenfeld, 2019; Ngu & Phan, 2021). Hence, many students do not possess the required mathematical skills for work and life.

Quality pedagogy is central to mathematics learning. It requires teachers to consider the position of both conceptual and procedural knowledge when designing learning tasks (Rittle-Johnson, 2019). While procedural knowledge is demonstrated by an ability to select and apply a suitable method (using algorithms) to solve easy and complex problems and produce accurate answers, conceptual knowledge is developed when students explore and use their thinking to build informal strategies to solve varied non-routine problems (Briars, 2016). The superiority of one form of knowledge over the other is still inconclusive among mathematics educators (Cobb & Jackson, 2011; Hurell, 2021). However, most pedagogies adopted by mathematics teachers appears to focus more on developing procedural knowledge than conceptual knowledge. Moreover, the traditional assessment, which focus only on the quantity of students' responses (for example, series of steps leading to the final answer) are still in use in today's mathematics classrooms. These pedagogical and assessment practices continue to impact students' conceptual understanding and their ability to transfer knowledge to solving real-life problems.

Van Hiele Teaching Phases

The van Hiele teaching phases constitute a sequential teaching framework that is based on the van Hiele theory (Van Hiele, 1986) and gradually moves students' thinking from one level to the next. This theory follows the constructivist views, which emphasize students' construction, demonstration, and ownership of mathematical ideas. This theory and the teaching phases were predominantly applicable to the teaching and learning of geometry; however, it has recently been shown to be transferable to other topics in mathematics (Colignatus, 2014). The van Hiele teaching phases are designed for teachers to provide opportunities for students to attain a higher level of thinking.

The five teaching phases are information, directed orientation, explication, free orientation, and integration. These phases begin with teacher-directed activities and gradually move to activities that require students' initiative but remain student-centered through the learning process (Serow et al., 2019). While students move through these phases, they develop 'crisis in thinking' also known as 'cognitive conflict', which allows them to investigate various thinking paths and identify the correct reasoning for the domain of thought (van Hiele, 1986). This strong perception of the structure of the domain leads them to develop insight, experience cognitive growth, and move to the next level of thinking. Hence, crisis of thinking and development of insight must be achieved for students to progress to the next level of thinking. These phases also combine several effective teaching practices such as the gradual introduction and development of technical or formal language, class discussion, student engagement, acknowledging of students' individual needs in progressive understanding, exploration of relevant activities, and problem-solving (Armah et al., 2018; Pegg, 2014; Serow & Inglis, 2010). These attributes and many more contribute to the cognitive development and building of mathematical ideas in students. A summary of the teaching phases according to van Hiele (1986) and Serow et al. (2019) is presented in Table 1.

Table 1. The Van Hiele teaching phases

S/No	Teaching phases	Description
1	Phase 1 (Information)	The teacher guides the students to get acquainted with the content of the lesson by introducing a new concept to the students in a familiar context.
2	Phase 2 (Directed orientation)	The teacher directs students to activities related to the new concept, allows the students to explicitly explore the components of the task, and may discuss the activities.
3	Phase 3 (Explication)	After exploration, students are aware of the new ideas, develop acceptable mathematical vocabulary, and verbally express their discoveries using the acquired mathematical terms.
4	Phase 4 (Free orientation)	Students find their way of solving more complex problems, which may lead to crisis in thinking; they demonstrate ownership of knowledge.
5	Phase 5 (Integration)	Students consolidate the whole domain of thought into a coherent whole, which leads to mastery of relationships between ideas and the objectives of the lesson being achieved. No new material is presented at this phase and upon completion of this phase, students' thinking progresses to the next level.

These teaching phases have been demonstrated to be effective in facilitating students' understanding and improving the geometry curriculum (Abdullah & Zakaria, 2013; Alex & Mammen, 2016; Machisi & Feza, 2021; Mayberry, 1983; Usiskin, 1982); however, there are limited studies on the effectiveness of this pedagogy in other aspects of mathematics (Nisawa, 2018; Walsh, 2015). Hence, this study aims to establish if the reported impacts of VHTP on geometry education could be similarly observed in solving complex equations.

Worked-example Instruction

Worked-example instruction (WEI) is one of the instructional designs recommended by the cognitive load theory (CLT) (Sweller, 2011). This pedagogy ensures that students acquire schema with minimal cognitive effort. In WEI, the teacher presents step-by-step solution procedures to students to study and immediately follows it with a similar problem for students to solve. The theoretical tenet of the WEI within the cognitive framework lies in the '*borrowing and reorganizing principle'* of human cognitive architecture, which is an example of a natural information processing system (Sweller et al., 2019). In this context, the information acquired is borrowed from another person through reading and studying, re-organised with existing information, then stored in the long-term memory. This process leads to the construction of new knowledge that can be transferred to solve similar problems (Ngu & Phan, 2021; Sweller et al., 2019). Therefore, when worked examples are presented to students, they study and understand them before transferring their understanding of the worked examples to answer similar practice problems.

Several studies have reported the effectiveness of WEI. These studies emphasize that students construct appropriate schema while solving practice problems because their attention is directed only to activities that are essential for learning, resulting in a low cognitive load (Barbieri et al., 2021; Renkl et al., 2004; Sweller et al., 2019). With the claim of no interference in learning, students maximize the available cognitive resources and effective learning is achieved. Popular studies on worked examples have focused on complete worked examples versus partial worked examples (Richey & Nokes-Malach, 2013; Sweller, 2011; Wittwer & Renkl, 2010) and problem-example pairs versus example-problem pairs versus worked examples only (Alreshidi, 2021; Van Gog et al., 2011). These studies provided evidence for the complete worked examples and example-problem pairs that were followed in this study. Despite several items of empirical evidence that proved the effectiveness of WEI, none of these studies assessed the students' performances according to the quality and quantity of their responses. Also, no study has compared the WEI effect with any other effective pedagogy and related their effectiveness to the forms of mathematical knowledge acquired. Hence, this study aims to investigate these features and also the lasting effects of the pedagogies. This study is significant for the African cultural context, as there are limited studies on the VHTP and no study on WEI in the African context. The results of this study would inform mathematics educators about the pedagogical practices in the classroom and whether cultural context influences the effectiveness of these pedagogies.

Procedural and Conceptual Knowledge

Meaningful mathematics learning requires the acquisition of both conceptual and procedural knowledge. These two forms of mathematical knowledge seem connected and inseparable (Rittle-Johnson & Schneider, 2015). However, there is ongoing debate among mathematics educators on the superiority of one form of knowledge over the other and which form of knowledge should precede the other to facilitate effective and meaningful learning (Baroody et al., 2007; Canobi, 2009; Cobb & Jackson, 2011; Hurrell, 2021). While conceptual knowledge focuses on understanding individual bits of ideas, the connections between the ideas and the thinking progression through several modes and levels that constitute procedural knowledge utilize a series of steps and integrate the rules of symbol representation and algorithms to form solution procedures for mathematical problems. Hence, procedural knowledge has the feature of reaching the desired mathematical goal, while conceptual knowledge is characterized by rich connections between the cognitive constructs related to the mathematical concepts. Many mathematics educators believe that procedural knowledge is always descriptive, not very reliable in developing mathematical reasoning and may be acquired through memorization (Rittle-Johnson & Schneider, 2015). Against this belief, Hurrell (2021) claims that both procedural and conceptual knowledge could either be superficial or deep. This study attempt to indicate the form of mathematical knowledge that is likely to influence the other. Consequently, the current study adopted one of the acceptable and explicit measurement strategies recommended by Rittle-Johnson and Schneider (2015) to assess procedural and conceptual knowledge of students. Specifically, conceptual knowledge was assessed by requesting the students to explain their solution procedures and procedural knowledge was measured by the accuracy of the sequence of procedures leading to the correct answer. Thus, this study empirically builds on existing evidence relating to WEI and VHTP and explore their effectiveness on advancing students' procedural and conceptual understanding. Particularly, this study tests the following hypotheses:

1. Students' procedural understanding in the two groups do not differ after the interventions.
2. There is no significant difference in the conceptual understanding acquired by students in the WEI and VHTP groups after the interventions.
3. The overall learning outcomes of students (in WEI and VHTP groups) do not differ across the time-points.

METHODS

Research Design

This quantitative research employed a quasi-experiment, followed a pre-test, in-class treatments, post-test, and delay test, and involved two experimental groups; worked-example instruction (WEI) and the van Hiele teaching phases (VHTP). Although quasi-experiments do not randomly assign participants to groups, the groups were randomly assigned to an intervention. This type of experiment is suitable to establish cause and effect relationship where a true experiment cannot be utilized for ethical and practical reasons (Cohen et al. 2018). For this investigation, each student completed three similar tests at three time points across eight weeks. At the initial stage, students completed a pre-test to determine their current knowledge about solving simultaneous equations. The groups were then exposed to eight (40-minute) carefully sequenced lessons, with one group receiving the WEI instruction and the other receiving the VHTP instruction. The students then completed a post-test. Three weeks after the post-test, a delay test was administered to the students to establish the lasting effects of the interventions as recommended by Cohen et al. (2018). The study was conducted in line with the Australian National Statement on Ethical Conduct in Human Research (NHMRC, 2007) and was approved by the Human Research Committee (HREC) of the University of New England, Australia, approval number HE20-224. The students' responses to the three tests were subjected to SOLO coding and Rasch analysis, and hypotheses were tested using inferential statistics (*t*-tests and two-way between subjects repeated measures analysis of variance) on SPSS. The data analysis is discussed in the results section.

Participants

This study involved 157 first-year senior school students (age 14 to 15 years) from two government-owned schools in Nigeria. Multi-phase sampling technique was used to select both students, mathematics teachers and schools that participated in this study. The students in each school formed each experimental group. There were 72 students in one group and 85 in the other. This investigation was carried out at the start of their first term of senior secondary school. All the procedures were carried out in English and all students could demonstrate competency in reading and writing in English. The two schools selected for the study followed the same curriculum and the students had equivalent background knowledge related to solving simultaneous equations. The number of males and females were relatively equal (50.9% male, 49.1% female). The data from the participants were obtained at three time points; all except for one of the students participated at the three time points and the one student participated at two time points. However, Peugh and Enders' (2004) recommended that to avoid biased conclusions, all data, including that from the student with the missing record should be included in the analysis. Due to the health requirements in relation to the COVID-19 virus, two mathematics teachers with equivalent qualifications and teaching experience were selected and trained to implement the interventions. Hence, each experimental group was taught simultaneous equations by their regular mathematics teachers.

Test Instruments

The test instruments were the pre-test, post-test, and delay test. The pre, post and delay tests had identical content and measured the learning growth of students at different time points (pre to post and then to delay testing). They assessed students' conceptual and procedural knowledge of solving simultaneous equations and the application of different methods of solving simultaneous equations. There were nine free-response questions: four questions focused on conceptual understanding and five questions targeted the procedural fluency of solving the simultaneous equations. Students were encouraged to show all relevant solution procedures and state the reasons for their solutions. An example of a contextual question to assess conceptual understanding was "Shade and Ayo are saving money for a holiday to Lagos. Shade has #50 in his money box and Ayo has #120. Shade decides to save #15 a week and Ayo decides to save #10 a week. After how many weeks will they have the same amount of money?" In this question, the researchers expected the students to represent the unknown (number of weeks) by x and the total money each of them would have at the end of x weeks by y. Conceptual knowledge was required to transform the information in the question into equations – $50+15x=y$ and $120+10x=y$ – and decide a suitable method for solving the equations simultaneously. In this instance, students could apply any method of their choice to find the solution to the equations. The success in answering this question demonstrated the students' understanding of the nature of variables, equality, equations, simultaneous solving, associations among quantities, and flexibility in reasoning, which could be more attributed to conceptual understanding. Conversely, procedural knowledge was assessed by students' fluency in step-by-step solution procedures or algorithms leading to the correct answer. In this test, students were required to solve procedural-related questions using a specified method (i.e., solve the pair of simultaneous equations using the elimination method ($2x+y=7$ and $x+y=5$). However, for all the questions, students were to state the reasons for their procedures, which would allow the researchers to understand the student's thinking and classify the responses based on the SOLO taxonomy.

Worked-example Intervention

The researchers developed a lesson plan that was divided into revision and main teaching. The revision addressed all the pre-requisites for solving simultaneous equations, such as linear equations, while the main lesson focused on teaching simultaneous equations using different methods – elimination, substitution, graphical. During the main lesson, the students received an instructional sheet that displayed the meaning of each concept related to simultaneous equations (i.e., pronumeral, terms, expression, and equation), solving equations simultaneously, and the basic steps to follow while solving equations by substitution, elimination, and graphical methods. The instructional sheet also presented a worked example (with explanations) of each method of solving simultaneous equations using the outlined steps and solutions to contextual questions. Students were expected to carefully study the instructional sheet and ensure that they understood the content. They could request assistance from their teachers. The students were then provided with the acquisition problems, which consisted of six worked examples paired with six structurally similar practice problems. It was expected that students would study the worked examples and transfer their understanding to solve similar acquisition problems, post-test and delay test.

Van Hiele Teaching Sequence Intervention

This teaching sequence followed the van Hiele teaching phases (information, directed orientation, explication, free orientation, and integration) to revise linear equations and teach simultaneous equations. Similar to the other group, the three methods of solving simultaneous equations were taught, including contextual questions relating to real-life problems. It explicitly detailed the expected activities of both the students and the teacher at every phase of the teaching. This pedagogy involved an iteration of the teaching phases to move students from one thinking level to the next. The sequence allowed the students to explore the conditioned environment created by their teacher and to develop flexible reasoning that could be applied to solve problems. It served as the intervention for the van Hiele group. The content of the van Hiele teaching sequence was similar to the lesson plan designed for the WEI group.

RESULTS AND DISCUSSION

SOLO Model

This study utilized the rubrics of the SOLO taxonomy (Biggs & Collis, 2014) to classify students' responses into increasing levels of thinking. The use of the SOLO taxonomy for assessment is significant because of its qualitative and quantitative consideration of students' responses (Afriyani & Sa'dijah, 2018; Biggs & Collis, 2014). In line with this, both the students' procedural solutions to problems and the reasons for the solution were utilized to judge and place students into SOLO levels. The researchers identified six levels of the SOLO taxonomy from the students' responses to the open-ended questions: pre structural, unistructural, multi structural, relational, formal mode 1, and formal mode 2. While four levels exist in the responses to procedural questions, five levels exist in the responses to conceptual questions. The first four levels fell within the concrete symbolic mode and the last two levels were in the formal mode. Next, the students' responses were scored such that pre structural = 0, unistructural = 1, multi structural = 2, relational = 3, formal mode 1 = 4, and formal mode 2 = 5. To ensure consistency of the scoring process, an intra-rater assessment based on the SOLO model was carried out and yielded 0.93. Table 2 shows the features of the responses in each category.

Table 2. SOLO levels and students' responses to procedural and conceptual questions

Code	SOLO levels	Procedural questions	Conceptual questions
0	Pre structural	No response or response is totally off-track or very irrelevant.	No response or no relevant idea about the question.
1	Unistructural	Response reflects one idea such as labelling the equations correctly	The response indicates a relevant idea such as representing the unknowns with *x* and *y*
2	Multi structural	Response indicates two or more disjointed ideas and missing the correct step(s), getting stuck, and being unable to proceed to the correct answer	Response exhibits two or more ideas or steps towards the solution, such as identifying the unknowns with *x* and *y* but arriving at incorrect simultaneous equation(s)
3	Relational	Two or more connected ideas that lead to the correct answers	
4	Formal mode 1		Two or more connected ideas to solve questions using layman or non-mathematically sophisticated approach; or correctly representing the questions in simultaneous equations but missing the right step(s) that could lead to the correct answers.
5	Formal mode 2		Two or more connected ideas to represent information in simultaneous equations, solving them correctly using acceptable mathematical procedures and proving that the answers are correct

Rasch Analysis Results

The Rasch analysis, which is based on item-response theory, was used to ascertain the degree to which the data (items and persons) fit the model. The Rasch model is suitable because of its significant role in considering both items and persons as connected constructs and the acknowledgement of unequal intervals within the functioning of the items (Bond & Fox, 2013). Hence, the researchers examined the fitness of the items and persons to the Rasch model, and the results are presented in Table 3. When the infit mean square is close to 1, it indicates that the set of items and persons perfectly fit the Rasch model (Bond & Fox, 2013).

Table 3. Rasch summary statistics for Items (I) and Persons (P) estimates

Tests		Separation index (I)	Separation index (P)	Infit (I)	Infit (P)	Outfit (I)	Outfit (P)	Reliability (I)	Reliability (P)
Test1	WEI	7.08	1.31	0.93	0.83	1.70	1.23	0.98	0.63
	VHTP	5.87	1.46	1.01	0.87	0.95	0.95	0.97	0.68
Test2	WEI	5.79	1.52	1.04	0.98	0.99	0.99	0.97	0.70
	VHTP	4.57	1.30	1.02	1.07	0.94	0.94	0.95	0.63
Test3	WEI	4.98	0.92	1.02	1.04	0.95	0.95	0.96	0.46
	VHTP	4.25	1.15	0.97	0.83	0.86	0.86	0.95	0.58

The items with mean reliability indices between 0.98 and 0.95 indicate that a large range of item measures are adequate for stable item estimates, which implies that the sample size can be used to establish a reproducible item difficulty hierarchy. In relation to the persons, most person reliability and separation indices for both groups are greater than 0.5 and 1, respectively. This means that the Rasch model identified more than one level of ability within the participants. And hence, the participants could be classified into low and high ability levels. The infit and outfit of both items and persons range between 0.5 and 1.5, except for the item outfit of WEI, which was 1.70. The outfit measure of 1.70 may be a result of few random responses by the low-performing students. Furthermore, the high item separation indices (> 3) for the two groups indicate that the samples for each group are large enough to identify the item difficulty of the test instrument.

Additionally, the person-item Wright map in Figure 1 indicates the relationship between person abilities and item difficulties before the intervention. The figure shows that the person abilities range between -5 and 3 logits while the item difficulties range between -3 and 1.5 logits. From the figure, the most difficult items are Questions 5, 8, and 9, which are located between 1 and 2 logits, while Question 1, being the easiest question, is located at -3 logits. Since the items difficulties fall within the logits of person abilities, the items of the test instrument are adequate for the targeted students. Thus, it was concluded that the test items fit the Rasch model, have a good range of difficulty, have high reliabilities and are appropriate for the cohort of participants for whom it is targeted. This has the potential for significant productive measurement and results.

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MEASURE    PERSON - MAP - ITEM
               <more>|<rare>
    3             .  +
                     |
                     |
    2                +
                     |S Q5      Q8      Q9
                     |  Q4      Q6
    1             #  +
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    0  .###########  +M
                    S|
           .#######  |
   -1 ############# M+
      .############  |  Q2
        ###########  |S Q3      Q7
   -2      ######## S+
             ######  |
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   -3               T+  Q1
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               <less>|<freq>
EACH "#" IS 2: EACH "." IS 1
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Figure 1. A wright map showing the person abilities and item difficulties

Descriptive and Inferential Analysis

To investigate the extent to which each of the interventions contributes to the acquisition of the procedural and conceptual knowledge of solving simultaneous equations, the person estimates, measured in logits, of the 157 participants (for the three tests) were exported to Statistical Package for Social Sciences (SPSS). The person estimates are raw scores that have been transformed into a genuine Rasch interval scale. The negative values from the pre-, post- and delay tests are because student measures are placed on the same scale as the item measures (which take precedence) and these are constrained to have a mean of zero. Hence, students' ability measures may be below the mean (0) and take the negative values. The descriptive results of the pre-test, post-test, and delay test are presented in Table 4. These initial results measure the changes in the procedural and conceptual knowledge of students across the three tests.

On average, the procedural and conceptual knowledge of students in both groups were relatively low at the initial pre-test stage. Table 4 shows that the students in both the WEI group and VHTP group exhibited an increase in procedural knowledge from the pre-test to post-test, with mean gains of 2.14 and 2.80, respectively. However, the VHTP exhibits a higher influence on improving students' procedural understanding. In relation to lasting effects on procedural knowledge, Table 4 indicates that both experimental groups could not sustain the intervention effects, as there appears to be a slight waning effect. The WEI effect slightly decreases by 0.15 and the effect of the VHTP decreases by 0.40. Although the VHTP group decreases in retention effects, their procedural knowledge is still better than that of their counterparts. This growth pattern (from pre to post then to delay test) of students' procedural knowledge seems to have been impacted by factors, including the interventions.

Table 4. Mean gains of students' conceptual and procedural understanding across the time-points

Knowledge	WEI N=72					VHTP N=85				
	Pre-test	**Post-test**	**Delay test**	**Post-test mean gain**	**Delay test mean gain**	**Pre-test**	**Post-test**	**Delay test**	**Post-test mean gain**	**Delay test mean gain**
Procedural	-0.91	1.23	1.08	2.14	-0.15	-0.23	2.57	2.17	2.80	-0.40
Conceptual	-1.97	-1.79	-2.78	0.18	-0.99	-5.86	0.39	0.76	6.25	0.37

In relation to the acquisition of conceptual knowledge, the WEI group seems to exhibit more conceptual knowledge than the VHTP group at the pre-test. At the post-test, the conceptual knowledge of students in both groups appears to increase, with average gains of 0.18 (WEI) and 6.25 (VHTP). Therefore, the VHTP group achieved more conceptual knowledge than the WEI group. Also, while the conceptual knowledge mean of students in WEI decreases from -1.79 at the post-test to -2.78 at the delay test, the conceptual understanding of students in the VHTP group continues to grow (from 0.39 to 0.76). Thus, the VHTP appears to have both immediate and lasting gains on acquiring conceptual knowledge while the WEI is only beneficial at the initial stage. The summary of the mean changes for procedural and conceptual understanding of the groups is shown in Figure 2.

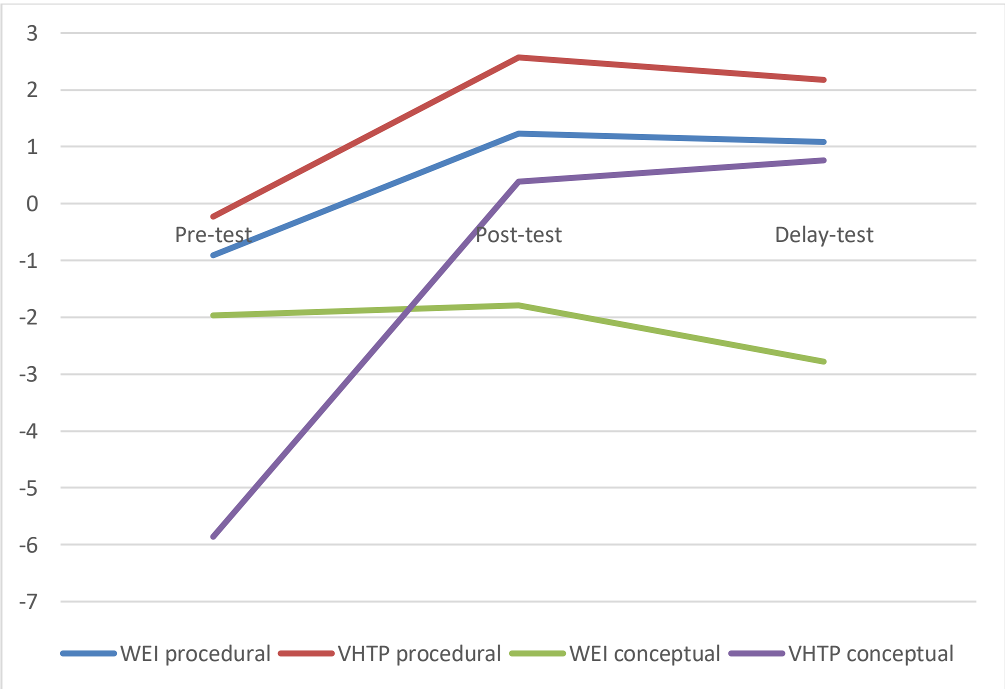

Figure 2. Mean changes of conceptual and procedural knowledge at the three time-points

Next, to test hypothesis 1, an independent *t*-test was carried out to determine if significant differences exist between the procedural knowledge acquired and sustained by the two groups. The Levene's test statistic was considered at the initial stage of each test to determine the assumption of equal variances. The results presented in Table 5 is not consistent with the assumption of equal variances. Hence, t-test result corresponding to equal variances were considered when Levene's test is not significant. Conversely, when Levene's test is significant, the corresponding non-equal variances were reported. Table 5 indicates that students in the VHTP group had more significant procedural knowledge at the post-test and delay test than the WEI group (*t* (152.38) = -3.84, *p* = 0.00, *d* = 0.60 and *t* (152.10) = -4.92, *p* = 0.00, *d* = 0.78 at α = 0.05 respectively). This means that both interventions improve students' procedural knowledge, however, significant difference exists between the two groups in favour of VHTP.

Table 5. Analysis of students' post-test and delay test scores for procedural and conceptual knowledge

Tests	Groups	Procedural Knowledge					Conceptual Knowledge				
		M	*SD*	*t*	*df*	Sig	*M*	*SD*	*t*	*Df*	Sig
Post-test	WEI	1.23	1.85	-3.84	152.38	0.00*	-1.79	2.39	-7.23	90.70	0.00*
	VHTP	2.57	2.50				0.39	0.97			
Delay test	WEI	1.08	1.19	-4.92	152.10	0.00*	-2.78	1.67	-13.9	154	0.00*
	VHTP	2.16	1.55				0.76	1.51			

*$p < 0.05$

Similar *t*-test analysis for hypothesis 2 indicated significant differences in the WEI and VHTP groups for the conceptual knowledge at the post-test and delay test, in favour of the VHTP group. This means the VHTP significantly outperformed the WEI in assisting students to acquire and retain conceptual knowledge (*t* (90.70) = -7.23, *p* = 0.00, *d* = 1.23 and *t* (154) = -13.90, *p* = 0.00, *d* = 2.23 at α = 0.05 respectively).

Furthermore, the overall effectiveness of the WEI and VHTP at the post-test and delay test was examined. Since proficiency in mathematics is demonstrated by both mathematical concepts and procedures, the overall person measures (in logits) of both conceptual and procedural questions were subjected to an independent *t*-test analysis (Table 6). Again, the Levene's test assumptions informed the choice of the *t*-test values.

Table 6. Independent t-test of students' learning outcomes at post-test and delay test

Tests	Intervention	*N*	*M*	*SD*	*t*	*df*	Sig	Cohen's *d*
Post-test	WEI	72	-0.14	1.19	-6.15	138.63	0.00*	1.01
	VHTP	85	0.95	0.99				
Retention test	WEI	72	-0.31	0.87	-9.76	154	0.00*	1.57
	VHTP	84	1.23	1.06				

*$p < 0.05$

The *t*-test result at the post-test showed that *t* (138.63) = -6.15, p = 0.00, d = 1.01 at a 95% confidence interval. This result indicates a significant difference between the general effectiveness of the WEI and VHTP on students' understanding, with large effect size. The VHTP group significantly outperformed the WEI group at the post-test. At the delay test, there was also a significant difference in the lasting effects of the WEI and VHTP in favour of the VHTP *t* (154) = -9.76, p = 0.00, d = 1.57. Overall, the VHTP is significantly better than the WEI in enhancing students' understanding of solving simultaneous equations both at the immediate and long-term. A two-way repeated measure analysis of variance was used to further test the hypothesis of equal means in the two groups across the three time-points (hypothesis 3). The Mauchly's test confirmed that sphericity is assumed (p = 0.07) and the result indicated that significant difference exists between the WEI and VHTP across the three time points, with a very large effect size [F(2, 308) = 153.30, $p = 0.00$, η_p^2 = 0.50]. A summary of the effectiveness of the two interventions from pre-test to post-test then to delay test is shown in Figure 3.

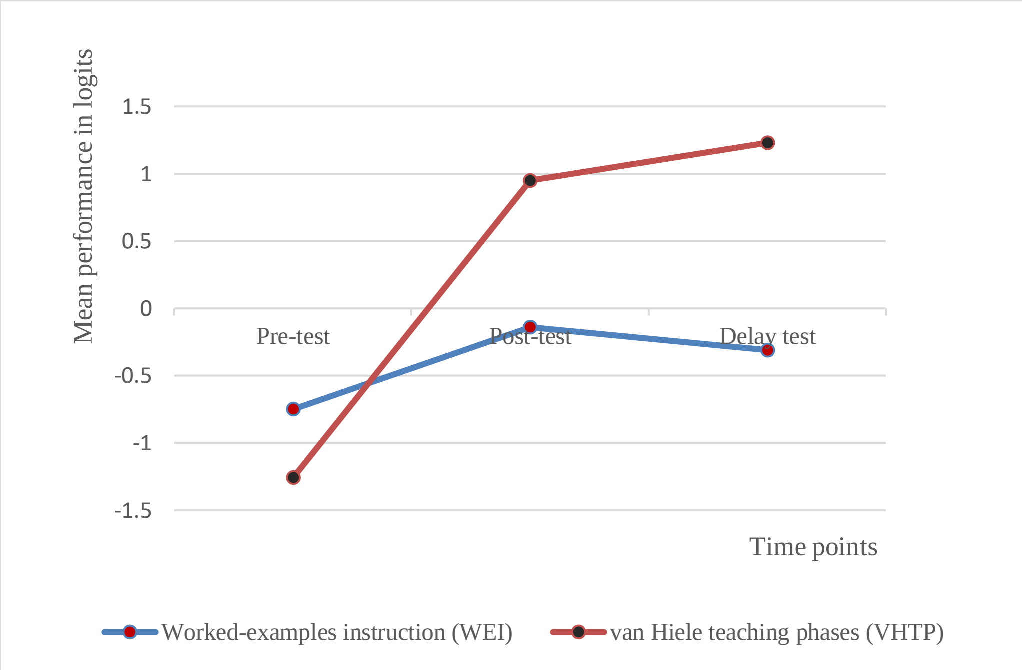

Figure 3. Line graph showing a summary of students' overall performance at each time point

This study explores the effectiveness of the VHTP and WEI for students' acquisition of conceptual and procedural knowledge. The findings from this investigation indicate that both the WEI and VHTP facilitate students' acquisition of conceptual and procedural knowledge at the time of learning. However, while neither group retained all the procedural knowledge over time, the VHTP group's conceptual knowledge continued to grow after the post-test. This suggests that the WEI effect on procedural and conceptual knowledge was observed only at the post-test while the VHTP had an immediate effect on both procedural and conceptual knowledge and a lasting effect on conceptual knowledge only. This pattern of effects was not expected and may be attributed to many factors, including the nature of the instruction and forgetfulness.

Particularly, WEI facilitates more procedural knowledge than conceptual knowledge. This result, which is consistent with the findings and claims of several studies (Baroody et al., 2007; Booth et al., 2013; Renkl, 2017; Rittle-Johnson & Schneider, 2015), may be because the worked-example instruction provides a step-by-step guide to solving a problem, and thus is closely related to procedural knowledge. Moreover, conscious efforts were made to encourage students to follow a sequence of actions to arrive at the correct answers via building on their prior knowledge and transferring their understanding of the worked-example solution procedure. Hence, students in the WEI group were able to demonstrate constructs that reflected the mastering of patterns and mathematical rules for only familiar problems. Similarly, the findings from this study were supported by the claims that self-explanation of the procedural steps facilitate students' conceptual understanding (Barbieri et al., 2021; Booth et al., 2013; Renkl, 2017). However, none of these studies considered the use of the SOLO taxonomy nor did they focus on complex algebraic equations. Furthermore, the researchers expected that the students in the WEI group would acquire limited but relevant schema, facilitated by resources within their cognitive capacity. This result seems to support our assumption. The decrease in students' performance in the retention test seems to reflect the "redundancy effect" and suggests that the practice of teaching students with WEI is necessary but may not be sufficient to build lasting procedural and conceptual knowledge.

For the VHTP group, students understanding of concepts and procedures improved at the post-test, suggesting that the students acquired beneficial schema during the intervention. However, the students could not maintain the acquired procedural knowledge after the post-test, resulting in a waning effect. The researchers expected that students would retain or increase their procedural and conceptual knowledge from the post-test to the delay test. This assumption was only valid for conceptual knowledge, which continued to increase after the post-test. More importantly, the use of VHTP developed more conceptual knowledge than procedural knowledge. This development of conceptual knowledge may be a result of the strong insight students developed from the crisis of thinking experienced during the teaching phases. This finding is similar to the effectiveness of VHTP in geometry (Luneta, 2014; Machisi & Feza, 2021) and trigonometry (Walsh, 2015) but inconsistent with findings on function (Nisawa, 2018). Given this result, it appears that students meaningfully acquire conceptual knowledge from the pedagogy introduced to them and were able to richly connect the concepts of solving simultaneous equations. The researchers made this assumption based on the principle that the VHTP allows students to explore, discover, and find their own way of solving a problem out of several possible ways. The students might have experienced an increase in the procedural knowledge at the post-test as a result of the report by several mathematics educators that sound conceptual understanding aids procedural understanding more so than the reverse (Hecht & Vagi, 2010; Rittle-Johnson & Schneider, 2015). These findings appear to indicate that the acquisition of conceptual knowledge aids retention and application of the knowledge to both familiar and unfamiliar contexts, including real-life problems.

The next question of interest is what underlying mechanism facilitates students' mathematical understanding when exposed to either the WEI or VHTP. For the worked examples, students were provided with fully guided instructions, which help to direct their attention to activities that are only essential for learning, resulting in no distractions and maximally utilizing the available working memory resources (Sweller et al., 2019). Hence, the WEI seems to manage students' cognitive load, and the practice problems help students to remember the procedural steps of familiar problems. Also, the worked examples are designed using sequencing, where students first study simple questions before difficult questions and then build on their prior knowledge. All of these strategies contributed to the growth in their procedural and conceptual knowledge. Moreover, students gradually move from being a dependent learner to an independent learner as they transfer their understanding from the worked examples to practicing other similar problems. As reported in the analysis, students' conceptual understanding may be facilitated through their exposure to the instructional sheet that contained the meaning of different concepts relating to simultaneous equations and the prompts for self-explanation of the procedures (Booth et al., 2013; Renkl, 2017). However, the decline in the conceptual and procedural knowledge observed at the delay test may imply that worked examples are only beneficial at the initial stage of knowledge acquisition.

The van Hiele teaching phases present information in an organized structure. At the initial stage, there is a focus on developing students' concepts and vocabulary related to a task. The students are encouraged to use their own words to express meaning before advancing to formal language (Serow et al., 2019). Also, students' reasoning develops in sequence while they are actively engaged in verbalizing their understanding and using deductive thinking, which helps in the construction of mathematical ideas at each stage of their development (Serow & Inglis, 2010). Moreover, during VHTP instruction, students build strong and rich connections between concepts as they are challenged with more complex tasks while they find their own way of solving both familiar and unfamiliar problems (Luneta, 2014). All these strategies seem to contribute more to conceptual knowledge development than procedural skills. In the same vein, the results of this study indicate that conceptual knowledge aids procedural knowledge, which could explain the increment in both conceptual and procedural knowledge at the post-test. The strong conceptual understanding continues to grow while the procedural understanding slightly decreases.

It should be acknowledged that a number of factors might have interfered with these results. The investigation was carried out in Nigeria after almost seven months of total lockdown and no learning due to the COVID-19 pandemic. To reduce this impact, students had a revision class, where the pre-requisites of the simultaneous equation were revised. Also, students' prior conceptual and procedural knowledge are likely to determine how they grow in these domains. At the pre-test, the WEI group performed better in conceptual items while the VHTP group demonstrated more procedural understanding. Against our expectation, the VHTP group demonstrated more growth in conceptual knowledge than procedural knowledge, while the WEI group demonstrated more growth in procedural knowledge. These results could inform practices in the mathematics classroom.

The results of this study extend the knowledge obtained from previous studies on the effectiveness of WEI and VHTP in mathematics teaching and learning. The current findings provide much-needed evidence to support the effectiveness of pedagogies that advocate discourse, active participation, sequential development of students' reasoning, and exploration of learning materials. The interventions in the natural classroom setting, judging students' responses based on quality and quantity (as prescribed by the SOLO model), and the use of multiple post-tests provides stronger reliability for the results of this exploration. In general terms, the VHTP is more effective than the WEI in learning to solve simultaneous equations. The only benefit of the WEI was observed at the immediate post-test.

The implications for mathematics teaching and learning are that the WEI should be improved from strongly focusing on developing procedural knowledge to focusing explicitly on the two forms of knowledge required to demonstrate mathematical proficiency. A few possibilities are for teachers to integrate oral discourse that targets rich connections of concepts into the use of WEI, emphasize that students provide self-explanation of the worked examples presented to them before transferring the understanding to solving familiar problems, encourage group work while using the WEI, and deemphasize the ultimate need to arrive at the correct answer without in-depth and practical understanding. This may help students taught with WEI to solve unfamiliar problems and apply their mathematical understanding to solve real-life problems.

The VHTP seems more beneficial to mathematics learning and students learn unconsciously through the activities. However, teachers need to carefully sequence and construct activities that will fall within the capacity of students' working memory, otherwise students may be distracted by activities that may not facilitate their learning and transfer. As perceived in this investigation, activities that help students to build conceptual understanding take a longer time, and for this reason, teachers who use this pedagogical approach may not complete the teaching syllabus required of them in an academic term. Hence, the use of VHTP requires carefully selected and precise activities that will highlight important and meaningful concepts and procedures for students to learn. At the same time, teachers need to be sufficiently familiar with the particular aspect of students' thinking that the activities are targeting. Moreover, this pedagogy may become very stressful to use when students' levels of thinking vary.

The study acknowledges the unavoidable limitation imposed by noise that results from conducting the study in the students' natural setting. Nevertheless, the results of this investigation are similar to studies undertaken in controlled conditions. Further research on WEI and VHTP could target other topics in mathematics and also utilize the SOLO model for assessing students' responses. Another line of investigation that can be undertaken in the future is to examine which pedagogy (WEI or VHTP) is more beneficial for the students' range of ability levels, gender or learning styles and their acquisition of conceptual and procedural knowledge. Thus, there are further research areas to address to compare the effectiveness of WEI and VHTP and demonstrate their mathematical proficiency for life-long learning.

CONCLUSION

This study found that the van Hiele teaching phases (VHTP) improved students' knowledge of concepts and procedures both immediately and in the long-term more than the worked-example instruction (WEI). Nevertheless, the WEI is beneficial only at the initial stage and facilitates more procedural knowledge than conceptual knowledge. While the VHTP enhances conceptual knowledge both at the initial and long-term, the procedural knowledge acquired through the VHTP intervention could not be sustained after the post-test. Hence, the VHTP facilitates more conceptual knowledge than procedural knowledge. Furthermore, the overall effects of the VHTP on students' learning was significantly better than the WEI. These results add to the growing evidence that procedural knowledge emphasized by the WEI is not absolutely sufficient to demonstrate mathematical proficiency and may not be dependable to solve real-life problems. The evidence from this study also indicates the attributes of effective pedagogy that can contribute to the development and sustenance of conceptual knowledge and procedural knowledge. Moreover, the acquisition of conceptual knowledge (as demonstrated by VHTP group) was shown to be more reliable in solving real-life problems and facilitating long-lasting learning than the reverse. This study has contributed to the empirical validation of effective teaching practices for life-long learning in mathematics education.

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Declarations

Author Contribution	:	All authors contributed to the study conception and design. The material preparation, data collection and analysis were carried out by Saidat Adeniji, under the supervision of Penelope Baker. Similarly, the first draft of the article was written by Saidat Adeniji, while Penelope Baker commented on the previous versions of the manuscript. All authors read and approved the final manuscript.
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