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Research Article

Probabilistic thinking in prospective teachers from the use of TinkerPlots for simulation: Hat problem

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Solving real-life problems through mathematical modeling is one of the aims of modern mathematics curricula. For this reason, prospective mathematics teachers need to acquire modeling skills and use these skills in learning environments in terms of creating rich learning environments. With this study, it is aimed to examine the reflections of using a simulation on a problem involving uncertainty on the probabilistic thinking of prospective teachers. The activity includes an experimental review of the famous Hat problem. It was observed that the hat problem, which started as a puzzle, was linked to coding theory and reached the limit of mathematics, statistics, and computer science research. Research findings revealed that the simulation-supported learning environment not only contributes to prospective teachers' probabilistic thinking skills, but also offers the opportunity to experience different methods (such as working with real data, technology assisted learning, modeling) in teaching and learning mathematics. It has been concluded that simulations have a unique potential that other methods do not have in terms of gaining statistical thinking as well as problem solving and modeling skills.

Keywords: Teaching statistics; Statistical thinking; Teaching probability; Mathematical modelling; Simulation

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1. Introduction

In today's society, one of the most common issues that individuals encounter in daily life is probability and statistics. They face concepts such as probability, uncertainty, chance, risk, prize, randomness. It comes face to face with a lot of data in environments and has to decide in situations of uncertainty. They often have to understand data, identify trends, and make decisions. These are indispensable elements of probability and statistics. Probability and statistics are included in the curriculum of many countries because of their benefits in daily life, their roles in other fields of study, and their contribution to the logical inquiry process. It is seen that in the last two decades, more emphasis has been placed on probability and statistics in curricula (Batanero et al., 2014; Inzunza & Rocha, 2021, Koparan, 2022a). In addition, the number of researches on this subject is increasing day by day. However, probability and statistics are at the top of the issues that both teachers have difficulties in teaching and students' learning (Koparan, 2015; Koparan & Kaleli

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Yılmaz, 2015). Although the probability is a socially useful and important branch in mathematics education, there are many difficulties in general, arising from the conflict of intuition with facts (Koparan, 2016a; Koparan, 2019). One of the reasons for these difficulties is that the experiences are generally limited to games of chance (Huerta, 2020). The preparation process that both students and teachers went through while learning probability and statistics was insufficient (Koparan, 2019; Rodríguez-Alveal, Díaz-Levicoy & Vásquez, 2018). Philosophical debates around the meaning of probability, certain features of probabilistic reasoning, students' misconceptions and difficulties, and the growing diversity of technological resources reveal that teachers need special preparation to teach probability (Biehler & Maxara, 2007; Konold, et al., 2007; Rodríguez-Alveal, Díaz-Levicoy & Vásquez, 2018). While textbooks provide some examples, some texts offer a very narrow view of probability concepts or a single approach to probability. Probability applications in textbooks may be limited to games of chance, or the definitions of concepts may be incorrect or incomplete (Cañizares et al., 2002). Practices made by teachers on this subject are limited to textbooks and cannot go beyond some calculations.

Students have great difficulty understanding the basic concepts associated with probability such as randomnes, independence and variation (Batanero et al., 2014). It is a big problem that the concepts remain abstract in this subject, which is mostly explained with traditional methods. These concepts need to be embodied to increase students’ interest and attitude towards the course. Since the visuality required to study some probability problems in traditional environments cannot be achieved, alternative learning environments are needed (Koparan & Kaleli Yılmaz, 2015). The recommendations made in this area are towards raising awareness of students’ probability structure and applications and using technology to develop data analysis and conceptual understanding (Ben-Zvi, 2000; NCTM, 2000; Franklin et al., 2007; Carver et al., 2016; Garfield, delMas & Zieffler, 2012; Biehler, Frischemeier & Podworny, 2017). In probability teaching, various researchers have suggested the use of computers as a way to understand abstract or difficult concepts and to increase students' abilities (Koparan, 2015; Koparan & Taylan Koparan, 2019, Carver et al., 2016).

TinkerPlots is a software used by students from primary school to university, providing a dynamic learning environment with data analysis and probability modeling tools (Konold & Miller, 2004). One of the main expectations of today’s mathematics education is to improve students’ problem-solving and modeling skills. Mathematical modeling involves transforming a real-world situation into a mathematical model and then proposing a solution based on that model (Garfield, delMas, & Zieffler, 2012). One type of mathematical modeling is simulation. Simulation is a teaching method in which learners can change their parameters and make experiments one-to-one. Simulations provide opportunities to strengthen the understanding of statistical ideas (delMas, Garfield & Chance, 1999; Konold, Harradine & Kazak, 2007; Koparan, 2016b; Koparan, 2022b) and to support learners' learning processes while working on chance experiments (Maxara & Biehler, 2007; Koparan, 2019). Students can build their knowledge through simulation-based activities (Koparan, 2016a; Koparan, 2022b). Moreover, the simulation promotes active learning and participation. Simulation can be used as a tool in problem-solving to improve conceptual understanding in order to acquire statistical thinking. Thus, students can take a more active role in their learning by searching for various alternatives to answer and solve their questions (Koparan & Taylan Koparan, 2019; Carver et al., 2016). Instead of closed-ended problems, presenting simulation and design activities where the solution is not open can help students to develop some skills required in lifelong learning. Batanero and Díaz (2007) emphasize that students conduct simulations in computer classes in schools that can help solve simple probability problems that are not possible with physical experiments. Simulation is the most appropriate strategy in providing a better focus on concepts and reducing technical calculations (Borovcnik & Kapadia, 2009). Batanero and Godino (2002) suggested that the development of probability ideas should be based on three basic concepts: chance, randomness, and interpretation of probability from the intuitive, classical, frequential, conditional and axiomatic (Batanero, 2005). The use of computer softwares constitutes an important tool to increase the number of samples for random experimentation in the classroom. In addition, modeling skill in mathematics teaching has emerged as an important skill to be acquired by students in recent years. For these reasons, it was decided to include problem solving activities in the simulation context as part of the above-mentioned content in order to increase prospective teachers' awareness of the opportunities that TinkerPlots offers for modeling activities.

Research in this area is scarce and mostly focuses on assessing prospective teachers' probability knowledge (Batanero et al., 2014; Batanero et al., 2016). More research is needed on prospective teachers and applied probability knowledge. One of the major shortcomings in probability education is the design of adequate materials and effective activities to train teachers (Koparan, 2019). Teachers should have both sufficient knowledge of probability and experience in designing research or simulations to work with students (Stohl, 2005). Recent research reveals that many prospective teachers share common biases in probabilistic reasoning with their students (Batanero et al., 2014; Prodromou, 2014, Rodríguez-Alveal et al, 2018). Teaching probability is difficult because the teacher should not only present different concepts and applications of probability, but also be aware of the different meanings of probabilities and the philosophical debates around them (Batanero et al., 2014). Finally, teachers should be familiar with research that reveals students' reasoning and beliefs in uncertain situations, and didactic materials that can help their students develop correct intuitions in this area.

Teachers should be interested in and analyze probability simulations and research. Simulations and experiments are recommended when working with students (Chance, delMas & Garfield, 2004; Batanero, Biehler, Engel, Maxara, & Vogel, 2005; Carver et al., 2016). Teachers need competencies related to this teaching approach. Modeling some probability problems can help teachers increase their mathematical and pedagogical knowledge at the same time (Batanero et al., 2005). Teachers also need experience in planning and analyzing a lesson. When teachers plan and analyze a lesson designed to teach some content, they improve their probabilistic and professional knowledge (Chick & Pierce, 2008). In any classroom, uncertainty arises as a result of dynamic interactions between the teacher, students, and the subject being taught. In the meantime, rich learning environments can be created as classroom discussions, experiences related to mathematical content and content-specific pedagogy can be offered.

1.1. Purpose of the Study

Probabilistic thinking is trying to predict the probability of a situation occurring by using some mathematical and logic tools. It is one of the best tools to have for increasing the accuracy of decisions. This study aimed to assess the inferences of prospective teachers about a problem with uncertainty by considering their contextual thoughts, assumptions, and trigger intuitions, and to present a pedagogical approach to the solution of the problem. For this purpose, the famous hat problem was asked and simulation experiments were created for the problem. The conceptual thinking of prospective teachers when making connections in inferential reasoning and the modeling process for the solution of the problem was presented with a didactic approach.

1.2. The importance of the study

In a universe of possibilities where every moment is infinitely complex, probabilistic thinking helps determine the most likely outcomes. With probabilistic thinking, decisions can be more precise and effective. Therefore, it is important for students and teachers to develop these skills. Intuition is often wrong in probability problems. The simulation provides the desired number of trials and reveals the facts for the solution of the problem (Koparan, 2019; Koparan, 2022b). Hat problem takes place in all countries as a problem that students encounter in some way in high school or university years and does not result in meaningful learning in the memories. What the results obtained from the problem mean and the underlying mathematical facts are not fully understood. This study aimed to present both experimental and theoretical approaches together for a better understanding of the problem. In addition, teachers and prospective teachers need application examples on how to model problems (Çekmez, 2022). Although TinkerPlots is known as a software, it is mostly used by researchers. Its use by teachers is more superficial compared to software such as Cabri, Geogebra, Fathom. Teachers need more practice examples and classroom activities related to simulations the use of TinkerPlots to model non-deterministic phenomena (Koparan, 2019; Koparan, 2022b).

2. Theoretical Framework

The term statistical thinking has traditionally been born in the field of statistics. But recently it has taken on much broader meanings. There are many studies on students' statistical thinking (Ben-Zvi, 2002; Chance, 2002; Garfield & Gal, 1999; Jones, Thornton, Langrall, Mooney, Perry, & Putt, 2000; Mooney, 2002; Rumsey, 2002, Wallman 1993; Wild & Pfannkuch, 1999). Statistical thinking involves understanding how and why statistical research is conducted. This means knowing and understanding the entire research process. According to Chance (2002), statistical thinking is the ability to see the whole process. This process includes understanding the meaning of variables and the relationships between them, having the ability to research data beyond what is described in books, and generating new research questions beyond those asked in basic research. Statistical thinking requires making sense of data. What about making sense out of data? What is happening? What will happen in the future? How can we best understand what the data is telling us? How can we use this information correctly? to find answers to your questions. Wild and Pfannkuch (1999) used empirical data to create a four-dimensional framework of statistical thinking, which included the dimensions of investigative cycle, types of thinking, interrogative cycle, and dispositions. Wild and Pfannkuch (1999) suggests that statistical thinking is more a complex process than a list of four or five broad characteristics. Developing students' statistical thinking has been highlighted as an important learning goal for statistics courses. In general, statistical thinking has been defined as "thinking like an expert applied statistician". However, there is currently no consensus on the characteristics that make up statistical thinking. Also, there is no known assessment that measures the exact nature of statistical thinking (Le, 2017).

According to delMas (2002) if students want to improve their statistical reasoning skills, they should be asked questions about the results they produce in this process (Table 1). For example, the student may be asked to explain this reasoning process with questions such as how the conclusions produced about a probability problem are reached and the connection between the result and the elements explained (delMas, 2002). Likewise, in statistical thinking, applying what they have learned in real-life problems, criticizing and evaluating the purpose and results of studies, or generalizing information obtained from classroom work for new situations supports the development of students in this process (delMas, 2002).

Table 1
Conditions that distinguish reasoning and thinking activities

Reasoning	*Thinking*
How?	Application
Why is that?	Criticism
Description (process)	Evaluation
	Generalization

The problem statement of this study was determined as "How does simulation contribute probabilistic thinking processes of prospective teachers about stochastic processes?" It is aimed to create a didactic learning environment where prospective teachers can experience the problem situation, creating and using the model, evaluation of outputs, extension, generalization, theoretical calculation processes for the thinking types in Table 1. This study focused on the probabilistic thinking skills of prospective teachers before and during model use.

3. Method

The case study method was used in the research. In the case of studies, it is aimed to examine and reflect the special case of a particular phenomenon in-depth (Merriam, 1998). In these types of research, the environment, individuals, and processes in which the research is conducted are investigated in a holistic approach and the relationships and interactions between them are focused. Special case studies enable the use of more than one data collection technique and reach a rich and supportive data diversity (Merriam, 1998).

The research was carried out at Zonguldak Bulent Ecevit University in Turkey. The study group consists of 52 prospective mathematics teachers who take the Probability and Statistics Teaching course. The Probability and Statistics Teaching course is a compulsory and theoretically held 3 hours a week course for the first time in the curriculum in 2020-2021. In the previous year, they took the probability course, which was conducted theoretically for 2 hours per week. The software TinkerPlots was used in this study. Simulation as a method is introduced in parallel to the concept of probability. Situations are modeled mathematically and by simulation and results are compared.

3.1. Teaching and Learning Principles

In this study, the teaching steps adopted in the lessons for prospective teachers are as follows. (1) Problem, (2) Creating and using the model, (3) Extension, (4) Generalization (5) Theoretical Calculation. In the first stage, they were asked to think about the problem and make predictions and assumptions. At this stage, many misconceptions and wrong intuitions emerged. In the next stage, they interacted with the simulation and experienced the processes of deciding whether the model fits the problem, testing what they thought, interpreting the outputs produced by the model, checking the theoretical probability and experimental results and comparing them. Incorrect or insufficient intuitions were reconsidered and evaluated after simulation and theoretical analysis of the problem. Mathematical methods to calculate probabilities were always run in parallel with simulation (law of large number, probability distribution of random variables, the mean, the expected value, fair games, (in)dependence, binomial distribution, Bayes theorem). In lectures and homework, students had to solve problems in a variety of textual contexts, particularly stochastic contexts such as games and the fairness of games.

With the hat problem, it was aimed that prospective teachers experience the reasoning and thinking processes seen in Table 1. Emphasizing the statistical aspects of probability (frequency) during these steps, using real experiments and simulations in lessons to reveal the relationships between data and probability (Konold & Kazak, 2008), including real life situations and real data in probability teaching, learning the modeling process step by step. Emphasis is placed on revealing well-known misconceptions, presenting intuitions and facts together, grounding concepts intuitively and presenting theoretical solutions.

3.2. Data Collection

In this study, the data were collected from the answers given by the prospective teachers to the hat problem. First, the model was not used for the problem, and the prospective teachers were asked to solve the problem with the help of paper and pencil. Then, they were asked to express their views again with the model developed for the problem.

The Hat Problem: A group of 6 men enter a restaurant and check their hats. The hat-checker is absent-minded, and upon leaving, she redistributes the hats back to the men at random. What is the probability that no man gets the correct hat (given enough tries for example 1000 tries) and how does the probability behave when the number of people changes? The data were collected in the course conducted through distance education and via smart phones. According to Yin (2009), a reliable case study should be based on at least two data sources. Therefore, the present study is based on two data collection tools. They were told that they could make a certain number of attempts at the question using a pair of dice. Thus, they were asked to see how many correct matches were in an experiment. After a small number of trials, they were asked to make predictions about the situations that might be encountered in the event of a large number of experiments.

3.3. Data Analysis

Probability problems are often problems that cannot be answered right away, and intuitions and facts about the problem may differ. The hat problem is one such problem. It was preferred to be used in this study as it is suitable for the emergence of different answers or misconceptions, as well as for simulation modeling. Participants' responses to the questions were analized and classified as misconceptions and errors. Then, the designed learning activities were started. In-class dialogues were analyzed by video recording of the lesson. Prospective teachers' thinking types were presented through direct quotations throughout simulation-supported data-based discussions. The data were read and coded repeatedly. The hat problem is an ambiguous historical problem and is suitable for modelling. It was thought that very different answers could be given by prospective teachers to this problem, and some of these answers might contain misconceptions. It was decided to use the problem with the thought that simulation could be an opportunity to eliminate misconceptions.

4. Results

The frequency and percentage rates of the answers given by the prospective teachers to the problem are presented in Table 2 with various categories (misconceptions and errors).

Table 2
Type of prospective teachers' errors and misconception

Error or misconception category	*Number of errors or misconceptions*	*Percent of errors or misconceptions*
Misinterpreting the problem	9	17%
Equiprobability bias	18	35%
Outcome orientation	2	3%
Representativeness	13	25%
Incorrect use of the proportional model	5	10%
Beliefs	5	10%
Total	52	100%

Some of these may be answers based on intuition and without probability calculations containing probability statements (Fifty/fifty, All results are equally likely, impossible to answer this…) used in everyday language. It was seen that some of the answers were within various misconceptions. Some excerpts from the answers are presented below.

PT2:"The probability of all getting their hat is equal."	**Reasoning-Guess**
PT19:"They all have an equal chance of getting their hat. So it is 1/6."	**Reasoning-Guess**
PT21:"The probability of all of them being correct is 20%."	**Reasoning-Guess**
PT35:"The match situations and probabilities of 0-1-2-3-4-5-6 are equal. The probability of all choosing their hats is 1/6 from here and the probability of choosing their hats for all of them is equal to the product of these. 1/6. 1/6.1/6.1/6.1/6.1/6"	**Reasoning-Guess**
PT34:"Correct match numbers will be equal. So the numbers of 1 correct match, 2 correct matches, 3 correct matches, 4 correct matches, 5 correct matches, and 6 correct matches are equal."	**Reasoning-Guess**
PT5:"Chances are equal. Correct matches may or may not be. So 50%."	**Reasoning-Guess**

The answers above contain an equal probability misconception. The equiprobability bias involves attributing the same probability in a random experiment to different events regardless of their actual chances.

Some prospective teachers focused on making a single "yes or no" decision instead of focusing on the entire storyline. In these answers, the probability of an event occurring or not is treated as a confirmation of certainty rather than a measure of probability. Such misconceptions are included in outcome orientation.

PT12:"No. A person can't choose his hat."	**Reasoning-Guess**
PT41:"Yes. I think they all get their own hat."	**Reasoning-Guess**
PT31:"The probability of each getting their hat is 50%. It either takes or not."	**Reasoning-Guess**

Some answers may be personal answers that are not directly related to the problem. In these answers, it is thought that the final outcome of an event depends on a power beyond their control. Sometimes that power is God or some other power, sometimes luck or wishes.

PT4:"It can be very easy if their sixth feelings are strong."	**Intuition-Guess-Personalized Answers**
PT16:"If they are on their lucky day that day, they can all choose their hat."	**Intuition-Guess-Personalized Answers**
PT43:"We don't know the result. Only God knows."	**Intuition-Guess-Personalized Answers**
PT48:"Depends on people's choice."	**Intuition-Guess-Personalized Answers**

Different from these answers, the following types of answers were obtained.

PT14:"The first person has little chance of getting his hat, while others have more and more chances of choosing their own. 1/6, 1/5,… 1"	**Reasoning-Guess**
PT30:"A person has less chance of getting his hat."	**Reasoning-Guess**
PT23:"There are six people, and if each of them chooses a hat, they will necessarily get their hat."	**Reasoning-Guess**
PT1:"It is unlikely to get one. Others are impossible. There can be only one correct match."	**Reasoning-Guess**
PT29:"Nobody can choose their hat and everyone is equally likely to choose their hat."	**Reasoning-Guess**
PT17:"All of them get their hats. It is a definite event."	**Reasoning-Guess**
PT15:"Two people are more likely to get their hats."	**Reasoning-Guess**

The answers above have been evaluated within the scope of representativeness. This bias implies estimating the likelihood of an event based on how well an outcome represents some aspect of the parent population

In addition, some prospective teachers gave answers that included the incorrect use of the proportional model.

PT15:"I've tried a number of times. I made inferences by making proportions with the results I obtained."	**Collection Data Observation** **Proportional Reasoning** **Guess** **Explain**

It was seen that some prospective teachers also tried to connect with pascal's triangle.

PT26:"All of them are more likely to choose their hat than someone is likely to choose their hat, with the highest probability that the 3 and 4 person will choose their hat. Next, come the possibilities for 2 or 5, then 1 or 6 people to choose their hat."	**Reasoning** **Connection** **Guess** **Explain**

4.1. Creating and Using the Model

In the creation of the model, it was accepted that the hats were the same color and model, that the people chose the hats randomly. The person and the hats are randomly matched by the simulation. Figure 1 shows the simulation created for the problem. Let us now have a look at the steps which have to be done in TinkerPlots when conducting the simulation of the hat problem.

Realization in TinkerPlots

Step TP1 Click on the "Sampler" in TinkerPlots.
Step TP2 Add numbers from 1 to 6 with (+)
Step TP3 Enter the repeat value as 6

Figure 1
The simulation created for the problem

Step TP4 Change Attr 1 as Hats in Figure 1.
Step TP5 Run

Figure 2
Defining person and matching variables

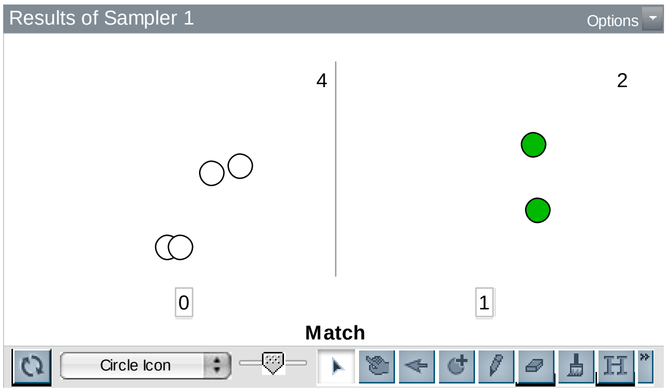
Results of Sampler 1 — Options

	Hats	Persons	Match
1	6	1	0
2	1	2	0
3	3	3	1
4	2	4	0
5	5	5	1
6	4	6	0

Step TP6 Type Persons in the first cell of column 2 in Figure 2
Step TP7 Right-click in the same cell and select edit formula.
Step TP8 Type "caseindex" in the window that opens and then presses OK.
Step TP9 Type "Match" in the first cell of column 3.
Step TP10 Right-click in the same cell and select edit formula.
Step TP11 Type "matchCount" (Hats; Persons) in the window that opens and then press OK
Step TP12 With the Match column selected, click the Plot tab and visualize the match status for each trial.

Figure 3

The matching status of the hats in each trial

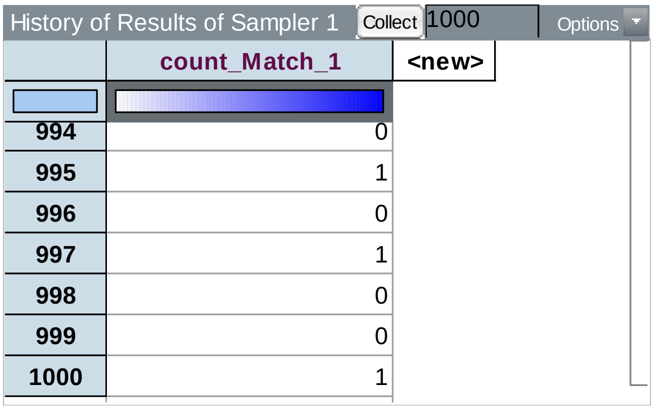


In Figure 3, 0 represents incorrect matches and 1 represents correct matches. From the screenshot in Figure 2, it is understood that 2 people chose their hats, while the other 4 people did not choose their own.

Step TP13 Use "History of Results of Sampler 1" and click the number in the top right in Figure 3.

Figure 4

Tabulation of outputs



History of Results of Sampler 1 | Collect | 1000 | Options

	count_Match_1	<new>
994	0	
995	1	
996	0	
997	1	
998	0	
999	0	
1000	1	

Step TP14 Enter the number of attempts and click collect

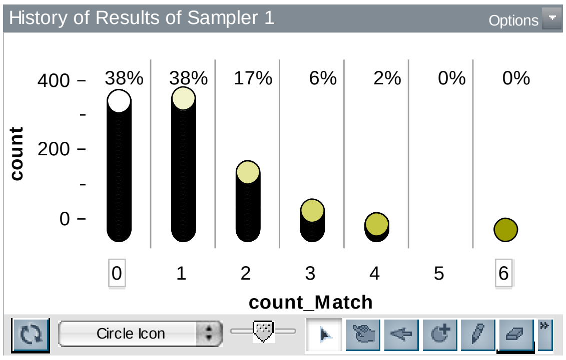
Step TP15 Graph the "History of Results of Sampler 1" with Plot in Figure 4.

4.2. Evaluation of Outputs

In Figure 3, simulation was used for one experiment. When the simulation is used for 1000 trials and the obtained data is recorded and the graphic feature of the software is used, the graph shown in Figure 5 is obtained.

Figure 5

Graphics of the outputs



In which it is observed that 38% of people do not receive their hat correctly and only one person receives their hat correctly. Some of the inferences made by the prospective teachers regarding the simulation outputs seen in Figure 5 are presented above.

PT22:"It is more likely that out of 6 people not all choose their hat, or only 1 person chooses their hat."	**Visualization-Frequency-Percent-Observation Experimental Probability-Statistical Inference**
PT28:"The chance of a hat to match is gradually decreasing."	**Visualization-Frequency-Percent-Observation Statistical Inference**
PT33:"5 people can't choose their hat. Because if 5 people choose their hat, certainly, the 6th person will also choose their hat. Therefore, there will never be any data in 5 digits."	**Visualization-Frequency-Percent-Observation-Statistical Inference-Generalization**

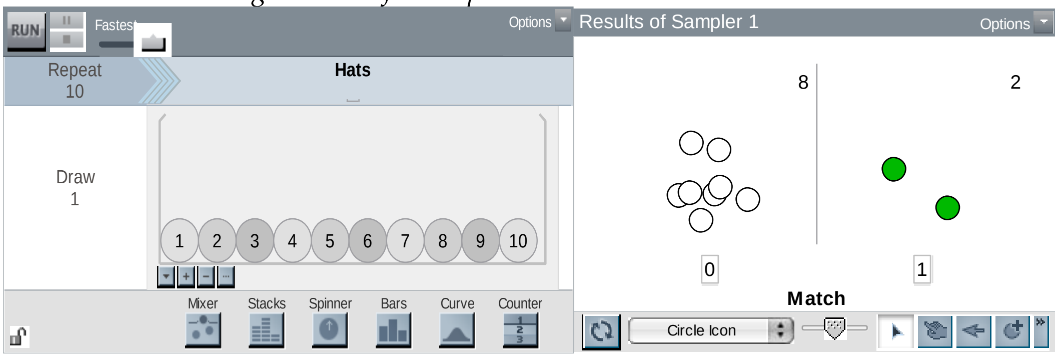
4.3. Extension

If there were 10 people and 10 hats, what results should we expect? If it were, it is a good question in mathematics. Will the result be the same or different in the new situation? If it is the same, it makes us think why it is the same, if it is different, why it is different. So it opens new windows in our minds. Similar to the model created for 6 people, the model is rearranged for 10 people.

PT11:"Can We Experiment With The Simulation By Making The Number Of Hats Less Or More? I Was Wondering About The Result."	**Variation-Using The Model-Testing- Observation**
R:"Sure. How Many Persons And Hats Do You Want?"	**Change of Experimental Variables**
PT11:"10 People And 10 Hats"	**Variation- Recognizing The Need For Data-Using The Model-Testing-Observation**

Figure 6 shows a trial result. Only 2 out of 10 hats matched correctly in this trial.

Figure 6
Simulation and data generation for the problem



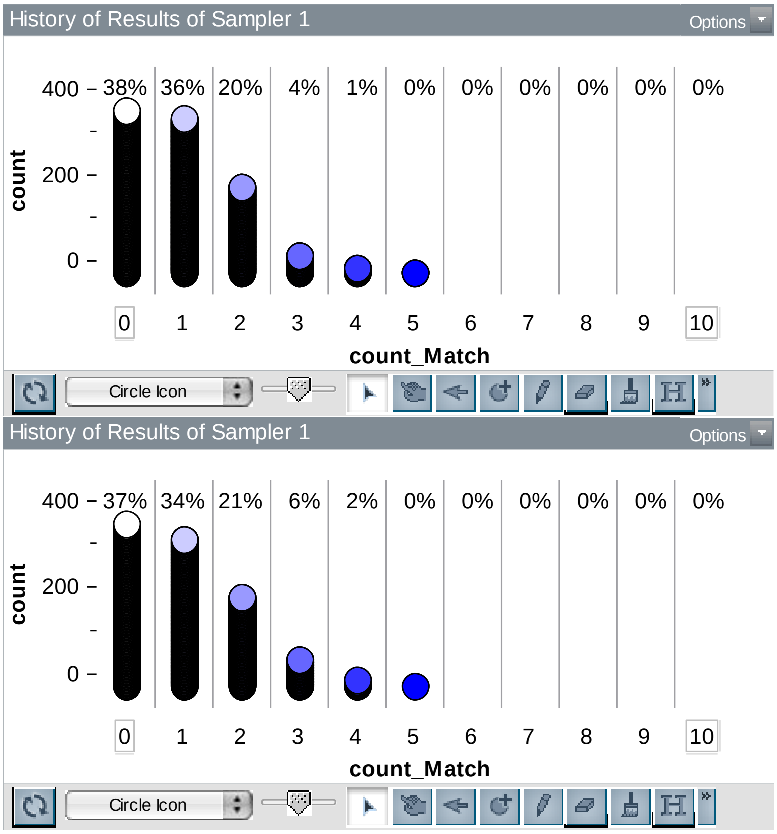
PT11: "We can see how many of them matched in each trial."	**Using the Model- Visualization-Speed-Observation-Evaluation**

4.4. Generalization

Trials are made with the new model created. The results obtained are interpreted. Figure 7 shows the results obtained for 1000 trials.

Figure 7

Two estimations of probability based on 1000 trials



The use of simulation in this problem offers opportunities to observe by changing the number of trials, comparing the number of matches, and understanding that the probability of mismatch by changing the number of hats is independent of the number of hats.

PT8:" For 10 people and 10 hats, 38% did not match on the first try. In the second attempt, 37% did not match. In other trials, I observed more 37%."	**Observation-Randomness-Variation-Statistical Inference-Generalization**
PT8:" With the simulation it is very easy to change the number of objects (Hats) or the number of experiments and see the results immediately."	**Using the Model-Change of Experimental Variables-Visualization-Observation-Evaluation**
PT8:"*The probability that the hats will not match, regardless of the number of hats, is 37%.*"	**Visualization-Observation-Experimental Probability-Statistical Inference-Generalization**

4.5. Theoretical Calculation

This step is aimed to make theoretical calculations and compare them with experimental results.

R:"Anyone have any ideas for a theoretical solution?	*(They think for 3-4 minutes)*
PT14: "I can offer a way for 4 people and 4 hats."	**Simplify the Problem**
R:"How? In how many ways is this possible? "Suppose people A, B, C, and D, and not each of them get their hat. In how many ways is this possible?	
PT14:"All possible situations are 24." **ABCD**, **AB**DC, **A**CB**D**, **A**CDB, **A**DBC, **A**DCB, BA**CD**, **BADC**, BCA**D**, **BCDA**, **BDAC**, BDCA, CAB**D**, **CADB**, CBAD, CBDA, **CDAB**, **CDBA**, **DABC**, DACB, DBAC, DBCA, **DCAB**, **DCBA**.	**Connection-Different Representations– Permutation**

PT14:"There are only 9 irregularities (shown in red italics) where not each one gets their hat. In every other permutation of this set, at least one person gets their hat (shown in dark blue). The probability that each person will not choose their hat $\frac{9}{24} = 37.5\%$"	**Permutation-Counting-Probability**
R:"In such problems, the number of mismatch of a set of n elements is calculated by calculated by $!n \approx \frac{n!}{e}$ $(e = 2{,}718\ldots)$. This problem was first considered by Pierre Raymond de Montmort (de Montmort, 1708) in 1708 and he solved it in 1713, as did Nicholas Bernoulli at about the same time."	**Explain**

$$!n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!} \quad (1)$$

$$e^x = \sum_{i=0}^{\infty}\frac{x^i}{i!} \quad (2)$$

R:"If the value of x is taken as -1 in the above equation (2) and replaced in the above equation (1), the following result is obtained."	**Explain**

$$\lim_{n\to\infty}\frac{!n}{n!} = \frac{1}{e} \approx 0.36789\ldots$$

R:"This is the limit of the probability that a randomly selected permutation of a large number of objects is an irregularity. The probability converges to this limit extremely quickly as n increases."	**Explain**
PT19:"Let's calculate for 6 hats now. The number of all matches is $n! = 6!$. The number of times the hats did not match correctly $!6 \approx \frac{6!}{e}$ $(!n \approx \frac{n!}{e})$ Accordingly, the probability that the hats do not match correctly is obtained as	**Testing-Verification**

$$P_{Mismatch} = \frac{!n}{n!} \approx \frac{6!/e}{6!} \approx \frac{1}{e} \approx 0{,}37$$

The probability that at least one of the hats is matched correctly is obtained as	**Testing-Verification**

$$P_{Match} = 1 - \frac{1}{e} \approx 0{,}63$$

PT5:"It is seen that the simulation produces results compatible with the theoretical probability."	**Comparison with Model Results**

The experiences in the learning environment designed are presented under the themes in Table 3.

Table 3
The learning experiences of prosectice teachers in the designed learning environment

Technological Experiences	*Pedagogical Experiences*	*Field Experiences*
-Creating a model	-Class attendance	-Permutation
-Testing the model	-Discussion	-Combination
-Using the model	-Critical Thinking	-Theorical probability
-Collection Data	-Sharing experiences	-Reasoning
-Visualization	-Sharing observations	-Proportional reasoning
-Different representations	-Cooperation	-Guess
-Lots of experiments	-Peer learning	-Intuition
-Speed	-Learning experience	-Personalized answers
-Table and chart	-Teaching experience	-Observation (die)
-Frequency		-Explain
-Percent		-Testing
-Observation (software)		-Verification
-Experimental probability		-Refutation
-Change of experimental variables		-Simplify the problem

Table 3 continued

Technological Experiences	*Pedagogical Experiences*	*Field Experiences*
-Connection		-Proof
-Evaluation		-Statistical inference
-Comparison model results with other methods		-Generalization
		-Problem Solving
-Randomness		-Mathematical communication
-Variation		-Mathematical connection
-Recognizing the need for data		-Different representations

In the Table 3, the first part (problem) presents the thinking before the simulation is used, the other parts present the thinking types in the simulation environment. As can be seen from Table 3, the experiences in the learning environment designed by evaluating the codes and researcher observations are presented under three themes: technological, pedagogy, and field (Mathematics) experiences. Some sections of the views of the prospective teachers regarding their above-mentioned experiences about the learning environment are presented below.

Technological experiences

PT11: TinkerPlots simulation offers experimentation and observation with real data.

PT2: It was quite effective to conduct various experiments using a problem-oriented simulation model and compare the experimental and theoretical results.

PT30:With TinkerPlots, it is very easy to experiment with random processes, experiment variables can be changed, and test outputs can be visualized with graphics using frequencies and percentages.

Pedagogical experiences

PT16: I really liked the simulation-based probability course. We shared our observations, collaborated to uncover what was going on about the problem. We discussed about some results.

PT25: It was a different learning experience for me. When I become a teacher, I also want to use such software in teaching.

Field experiences

PT5: In this problem, I realized how our intuition misled us. Most of us gave personal answers unrelated to the problem.

PT13:We made predictions on a probability problem. We used concrete material and simulation. We experimented, observed, tried to explain and generalize the results. Finally, we made theoretical calculations.

As can be seen from the direct quotations, prospective teachers state that simulation provides the opportunity to work with real data, to conduct many experiments in a short time, to visualize and observe the experimental outputs in a concrete way, and to establish a relationship between experimental and theoretical probability. Prospective teachers also stated that simulation provides a different learning environment and serves as a bridge to confront intuitions about probability and to transition from experimental probability to theoretical probability.

5. Concluding Remarks

Since the visuality required for studying experimental probability problems in traditional environments cannot be achieved, alternative learning environments are needed. It is realized that TinkerPlots is an important and useful tool in developing ideas about experimental probability. It was seen that it offers the opportunity to create simulations and make observations, especially for real-life problems. The data obtained from the trial results are transformed into a dynamic and visual working environment with tables and graphics. Thus, learners develop an understanding of probabilistic concepts through the activities they perform and gain opportunities to verify or change their intuition about probability and randomness. In this sense, the different features of dynamic statistics software have made me believe that they create suitable learning environments for teachers and students, and offer opportunities to make data-based discussions and inferences that are not possible in data analysis activities that they can do with pen and notebook. One of the problems with the probabilistic model approach is that students are not introduced to the more nuanced ideas of model building and evaluation. This is a topic not often covered in probability and statistics courses (Zieffler, Justice, delMas, & Huberty, 2021). From my experience of conducting this activity in the classroom, I have witnessed that students benefit from both their cognitive and affective perspectives. Cognitively, linking theoretical knowledge to real-life situations contributed to the creation of a rich learning environment where learners can test, change, validate and discuss their mathematical ideas during the activities. There was some positive feedback from prospective teachers that they had a different, interesting teaching experience and concrete approaches to how technology could be integrated into probability teaching. Since an education that focuses only on technical skills will not be enough to overcome the false beliefs of prospective teachers, it is necessary to help them build a bridge between conceptualization and pedagogy, as well as to try different ways such as the use of simulation in probability teaching. I gave information about an experimental approach that connects ancient knowledge of the basic concepts of probability theory. Results of this research successfully demonstrate that there is room for such an approach to education. It has been emphasized that computer-based simulations will form a strong mathematical foundation for future teachers, and will provide opportunities to meet all important demands of modern mathematics teaching such as analyzing and representing real situations, problem-solving, and making decisions based on mathematical reasoning (NCTM, 2000). In summary, I believe that these types of activities will be effective in raising awareness in prospective mathematics teachers about the potential of simulations to design modeling activities.

Probability education should focus on affecting prospective teachers' internal dialogue in higher education; the various correct and incorrect approaches used by prospective teachers to solve probability problems should be addressed in probability education. The straightforward link between response and misconception has to be questioned on probabilistic reasoning. Future research should broaden the research scope by further focusing on learners' reasoning processes and in comparing these results with similar populations of pedagogy students from other national and international training institutions.

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