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Learners' Critical Thinking About Learning Mathematics

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ABSTRACT

Developing critical thinking practices among young learners through mathematics education is a topic of attention for mathematics education research community. Learners' critical thinking concerning cognitive and social aspects of their mathematics education have been explored in several research studies. However, learners' critical thinking concerning their personal beliefs about their mathematics learning process have not received much consideration. In this paper, learners' practice with thinking critically *about* and their potential to suggest changes in their mathematics learning process is explored based on their expressed beliefs *about* learning mathematics. Learners of eighth and ninth grade in two Norwegian schools responded to a questionnaire and were interviewed to gather their opinions concerning their mathematics learning process. Data analysis indicates that learners seldom think critically, and hold inconsistent beliefs about mathematics and its learning process. Moreover, they struggle to observe their own beliefs critically, and hesitate in suggesting alternatives to make learning mathematics meaningful for them. Consequently, learners' critical attitude towards their mathematics learning process and their personal beliefs in order to gain a meta-perspective of their learning contexts does not seem to be evolving effectively. However, if learners are encouraged to think critically about mathematics education, their potential of contributing to improve their mathematics learning process becomes visible. We recommend that young learners get training in and are encouraged to think critically *about* their mathematics learning process so that they are equipped to make reflected choices related to learning mathematics in their personal lives.

Keywords: critical thinking, mathematics education, learners' perspectives, learners' beliefs

INTRODUCTION

The word 'critical' can have different meanings depending on the frame of reference of its use (Ernest, 2016). A situation can be 'critical' or at a point of *crisis*, when an action can dramatically improve or deteriorate the conditions. Secondly, *criticism* or *criticizing* can be a form of conveying disapproval, disagreement or negative comments about an argument, situation, decision etc. Additionally, 'critical' can be understood in terms of *critique* (being critical), as contrary to being 'uncritical'. In this sense, being critical includes analyzing the merits, demerits and consequences of any belief, judgement, choice, opinion, product, context etc., be it socio-cultural, political or personal (Ernest, 2016). In this paper, the word 'critical' is used to mean the opposite of being 'uncritical', and refers to the learners' knowledge of evaluating their personal beliefs, inferences, choices, etc. *critically* in order to take informed decisions and action(s) for their personal and societal betterment.

Adopting a critical stance while making choices in life is essential for learners. They should be aware and capable of critically analyzing their viewpoints and situations in cognitive, social and personal spheres of their life to survive and succeed in a complex society (Facione, 1990). In addition, learners are envisioned as future critical citizens of the society, using their critical thinking potential to promote justice and democracy (Skovsmose, 1998). Consequently, teaching and learning of critical thinking practices is often recommended in the education research literature (Bybee & Fuchs, 2006; Saavedra & Opfer, 2012). A number of reports issued by elected commissions (Ludvigsen et al., 2015; Partnership for 21st Century Skills (P21), 2009) and the educational policy documents of some countries also mention development of critical thinking competences as one of the fundamental aims of education. For example, the Norwegian Education Act states that, "[Through education] students and apprentices *must learn to think critically* and, act ethically and environmentally consciously. They *must have co-responsibility* and *the right to co-operation*" (Opplæringsloven, 1998) (translated and added italics). An interpretation of this statement can be that learners should learn to *think critically* through and about their education, take *responsibility* and have the *right to co-operate* in decisions regarding their education. Moreover, learning critical thinking abilities is also mentioned as an educational ideal (Siegel, 1980), and learners' moral right "because in the end students must choose for themselves; there is no escaping this truth" (Norris, 1985, p. 40).

Therefore, the requirement and importance of acquiring critical thinking competence for learners to achieve both individual and societal good is well established.

Mathematics is strongly represented as a fundamental school subject in countries across the globe. Mathematics education, therefore, has a vital role in educating children to become critically thoughtful, responsible and co-operative beings acting in the society. Accordingly, developing critical thinking competence among mathematics learners has been a concern for mathematics education since decades (Jansson, 1986; Kuntze et al., 2017). Evolving "critical thinking skills [through mathematics education] is an implicitly hoped-for outcome of using the NCTM's *Standards*" (Gutstein, 2003, p. 66). However, neither NCTM Standards nor mathematics education research literature explicitly specify or limit the aspect(s) of learning mathematics, concerning which, learners' critical thinking should be developed. Consequently, mathematics education researchers have understood and used the term *critical thinking* diversely in different contexts to discuss development of learners' critical competences.

Learners' *critical thinking* in mathematics education research literature have been discussed mainly as – a set of cognitive skills used to draw logical conclusions and take informed decisions while solving mathematical problems (Aizikovitsh-Udi & Cheng, 2015; Kuntze et al., 2017; O'Daffer & Thornquist, 1993); and secondly as – an attitude to understand and reflect over the role(s) of mathematics and mathematics education in socio-political and cultural contexts to promote justice and democratic concerns in the society (Gutstein et al., 1997; Skovsmose, 1994b, 1998). The first point emphasizes learning critical thinking *in* mathematics to acquire mathematical procedures for problem-solving and finding unbiased logical results. The second point roots critical thinking in the spirit of Critical Mathematics Education (CME) (inspired by critical pedagogy) Freire (1972); (Skovsmose, 1994a) to consider mathematics and mathematics education as objects of reflection and critique in society. This classification highlights the attention directed to imparting critical thinking among mathematics learners regarding the *cognitive* and *social* aspects of their mathematics learning process respectively, whereas the *personal* aspect seems to be missing. Therefore, in this paper we focus on learners' practice with critical thinking regarding their *personal* beliefs *about* their mathematics learning process, and their potential to participate in this process.

Thinking critically in social and cognitive aspects of one's life cannot be complete without, or compensate for being critical in one's *personal* life. Moreover, applying critical thinking cognitively, *in* the process of solving mathematical tasks and understanding social complexities through learning mathematics, cannot be taken to be the same as thinking critically *about* learning mathematics in personal life. Young mathematics learners worldwide need to make a personal choice concerning their mathematics learning early in their educational pathway. In Norway, as in many other countries, 14-15-year-old learners decide if, and what specific direction (vocational or theoretical) of mathematics they want to pursue in their upper secondary school by the end of their compulsory school years (i.e., after tenth grade). Developing learners' critical thinking faculties through mathematics education can be helpful to make a well-reflected choice of this kind. Thus, developing learners' critical abilities through mathematics education is mentioned as a central aim in the recently revised Norwegian mathematics curriculum (specified under '*fagrelevans og sentrale verdier*' [subject relevance and central values]). In the Norwegian mathematics curriculum, it is stated that, "Critical thinking in mathematics involves critical evaluation of reasoning and argumentation, and can equip the learners [with the competence] to make their own choices and to address important questions concerning their own [personal] lives and the society" (Norwegian Directorate of Education and Training, 2020) (translated). Therefore, the scope of learning critical thinking through mathematics education can be seen as directly related to learners' *personal* lives and choices. The Norwegian mathematics curriculum emphasizes that mathematics education should evolve learners' critical attitude towards the decisions they make in their *personal* lives; and gives a personal dimension to learning critical thinking through mathematics education in addition to the cognitive and social dimensions. Consequently, developing a tendency to think critically *about* their personal beliefs concerning mathematics and its learning process may help learners to attain a meta-perspective of learning mathematics and to make informed personal choices about their mathematics education.

Statement of Problem and Research Question(s)

Learners' practice with thinking critically *about* their *personal* beliefs concerning their mathematics learning process are seldom investigated in mathematics education research literature. This research paper addresses this research gap. By practice with thinking critically, we mean learners' tendency to critically observe their *personal* beliefs *about* mathematics and its learning process to gain a *meta-perspective* of it in their own lives. Therefore, in this paper learners' practice with thinking critically and give suggestions *about* their own mathematics learning process is explored based on their expressed beliefs regarding this process. The research question, *"What can learners' expressed mathematics related beliefs reveal regarding their practice with thinking critically about and potential to give suggestions concerning their mathematics learning process?"* is investigated in this paper. The scope of learners' mathematics related beliefs was narrowed down with the help of three sub-questions, namely – *what* subject content do learners find interesting to learn in mathematics?; *why* they learn mathematics in their opinion?; and *how* their mathematics teaching may be improved?

CONCEPTUAL FRAMEWORK

Defining Beliefs

Learners' first-hand experiences, opinions and beliefs are used to explore their *critical* orientation while they talk about their mathematics learning process. Learners' beliefs about mathematics and mathematics education play an important role in learning of the subject and are well-researched (Leder, Pehkonen, & Törner, 2002; Maaß & Schlöglmann, 2009). Furinghetti and Pehkonen (2002) mention that it is difficult to point out a single universal definition of beliefs because of different goal-orientations and contexts in which this term is used. However, Leder et al. (2002) cite Bar-Tal (1990), describing *beliefs* to be viewed as, "… units of cognition. They constitute the totality of an individual's knowledge including what people consider as facts, opinions, hypotheses, as well as faith. Accordingly, any content can be the subject of a belief." (p. 12). Beliefs can be formed through a variety of sources, be it direct personal experiences, inferences about a context or information provided by outside sources etc. (Bar-Tal, 1990). Various types, classifications, structures, and assessment methods of beliefs discussed in the literature are beyond the scope of this paper. Therefore, adapting Bar-Tal's definition, learners' mathematics and mathematics education related beliefs are understood as their conscious or unconscious opinions, thoughts, ideas, perceptions or hypotheses about their mathematics learning process which they consider to be true.

Defining Critical Thinking

Analogous to the word *critical, critical thinking* is also defined differently in educational, socio-political and psychological research contexts. Critical thinking has acquired several definitions[1], over its research history. In 1988, the American Philosophical Association (APA) founded a panel of 46 experts (including educationalists, philosophers and psychologists) to develop a consensus *definition* and *fundamental skills* comprising critical thinking for educational instruction and assessment purposes. The panel's consensus presented a total of six *core critical thinking cognitive skills* with 16 sub-skills (**Figure 1**), and *affective dispositions* (not focused in this paper) including habits of mind of a good critical thinker in their conclusive report. The consensus defined critical thinking as, "...[a] purposeful, self-regulatory judgment which results in interpretation, analysis, evaluation, and inference, as well as explanation of the evidential, conceptual, methodological, criteriological, or contextual considerations upon which that judgment is based" Facione (1990, p. 3). Therefore, critical thinking evolved as an ability comprising *core cognitive critical thinking skills* and *affective critical thinking dispositions*.

Interpretation		
Categorization	Decoding Significance	Clarifying Meaning
Analysis		
Examining Ideas	Identifying Arguments	Analyzing Arguments
Evaluation		
Assessing Claims	Assessing Arguments	
Inference		
Querying Evidence	Conjecturing Alternatives	Drawing Conclusions
Explanation		
Stating Results	Justifying Procedures	Presenting Arguments
Self-regulation		
Self-examination	Self-correction	

Figure 1. Cognitive critical thinking skills and sub-skills adapted from (Facione, 1990, p. 12)

Facione (1992) simplified APA's consensus definition of critical thinking as a process to "make a purposeful, reflective judgement about what to *believe* or what to do – precisely the kind of judgement which is the focus of critical thinking" (p.17, italics added). Analogously in this paper, *critical thinking* is understood as an ability to reflect over one's beliefs, circumstances and actions for making purposeful, reflective judgements and choices about what to believe and how to act responsibly for improving one's life, without harming others. In mathematics education context, besides being the ability to solve mathematical problems logically and reflecting over mathematics' role in society, critical thinking can be comprehended as a process tool for learners to consciously reflect upon and gain a meta-perspective *about* their own mathematics teaching-learning process. Therefore, learners' critical thinking *about* their own mathematics learning process was analyzed by using the *self-examination* and *self-correction* sub-skills of the sixth core cognitive critical thinking, *self-regulation* (**Figure 1**). The APA consensus' definitions of sub-skills *self-examination* and *self-correction* were adapted to analyze learners' interview responses.

[1] Mentioning critical thinking as "the propensity and skill to engage in an activity with reflective skepticism" (McPeck, 1981, p. 8) or as "disciplined self-directed thinking which exemplifies the perfections of thinking appropriate to a particular mode or domain of thinking" (Paul, 1993, p. 33), see Beyer (1985) for a literature review on defining critical thinking.

LITERATURE REVIEW

Learners' Beliefs and Mathematics Education Research

In mathematics education, *beliefs* have emerged from being a hidden variable in the mathematics learning process to being a thoroughly investigated topic of research (Goldin et al., 2009; Leder et al., 2002). The research constituting beliefs in mathematics education is rich and diverse, particularly if one also includes the studies using terms like ideas, conceptions, perceptions, attitudes, values etc. in place of the term *beliefs*. Research regarding beliefs about mathematics and mathematics learning process have involved both learners and teachers (both in-service and pre-service) as informants, and may concern either *cognitive* domain (for example, a polygon having three sides is a triangle), or the *affective* domain (for example, mathematics is difficult) (McDonough & Sullivan, 2014). However, most of the studies concerning beliefs are placed under the umbrella of *affective issues* in mathematics education research (Leder & Grootenboer, 2005; Zan et al., 2006). In this study, learners' beliefs about mathematics are explored to get an insight into their practice with thinking critically about and suggesting changes in their mathematics learning process. Consequently, this study takes learners' perspective in focus within the affective domain of enquiry related to their mathematics learning process.

Op't Eynde, De Corte, and Verschaffel (2002) define learners' mathematics-related beliefs as, "… the implicitly or explicitly held subjective conceptions students hold to be true, that influence their mathematical learning and problem solving" (p. 16). Research studies enlightening learners' mathematics-related beliefs conducted within the affective domain may be classified as focusing on either positive or negative affect towards mathematics (Hannula et al., 2016; Zan et al., 2006). Taking the positive affect on one hand, learners' beliefs have paved the way to explore their interest, enjoyment, liking etc. towards mathematics, which is usually seen to be correlated with increased motivation, engagement and achievement (Hannula et al., 2016). Whereas, considering the negative affect on the other hand, learners' beliefs have helped to explore their mathematics anxiety, stress, fear etc. which are seen to be correlated with the feelings of disaffection, failure and loss of interest in learning mathematics (Zan et al., 2006). Therefore, the literature regarding learners' mathematics-related beliefs includes a large spectrum of studies exploring their beliefs about topics such as – the nature of mathematics (Young-Loveridge et al., 2006), attitude towards learning mathematics (Grootenboer & Marshman, 2016), self-efficacy (Kele & Sharma, 2014), mathematics teaching (Mapolelo, 2009), the use-value of mathematics (Kollosche, 2017; Onion, 2004; Pais, 2013), their mathematical identities (Andersson et al., 2015; Bishop, 2012), emotional disaffection (Nardi & Steward, 2003), anxiety (Young et al., 2012) towards mathematics; and more.

Learners' beliefs regarding the nature of mathematics report that they seldom think about this topic (Kloosterman, 2002), they view mathematics as a useful and important (Grootenboer & Marshman, 2016), but also as a difficult and boring subject (Nardi & Steward, 2003). They usually hold a positive attitude towards learning the subject, however, only a few wish to become mathematicians (Grootenboer & Marshman, 2016). When it comes to the specific questions of *what, why* and *how* in mathematics learning process, more studies have examined learners' beliefs concerning the *why* of learning mathematics, rather than *what* to learn in mathematics or *how*. In our literature search, Lindenskov (2010) came up as the only study that took learners' perspective over *what* content learners' themselves are interested to learn in mathematics. The results suggest that if given an opportunity, learners clearly mention what they understand or not while learning mathematics and devote their attention towards related curriculum accordingly. Thus, Lindenskov (2010) suggests that if given the opportunity learners' can critically evaluate their mathematics curriculum and also draws implications of her study for the role of learners in CME research. Concerning the question of *how*, Mapolelo (2009) and Nardi and Steward (2003) reported that learners experience mathematics classes to be tiring, lecture-oriented and the use of mathematical language seemed to be a barrier in learning the subject. However, the learners not only see the problems but, if allowed, can also point out effective instruction strategies to improve the learning outcome of their mathematics classes (Clare & Sue, 2013). Moving to the question of *why*, the investigations Onion (2004), Sealey and Noyes (2010), Pais (2013), and Kollosche (2017) uncover that though learners believe mathematics to be an important subject to learn, yet they struggle to respond when asked about where they use and *why* they need to learn advanced mathematics at school.

These studies indicate that the learners are not used to think over what it takes to learn or why knowing mathematics is important (Kloosterman, 2002), and find it difficult to answer the questions about *what, why* and *how* of their mathematics learning process. In addition, they hold inconsistent beliefs about mathematics and mathematics education, indicating that thinking critically about personal mathematics learning experiences is not a part of a regular mathematics class and despite being potential, their voices are often not encouraged or heard (Clare & Sue, 2013). Though evident, the reasons for this inconsistence in learners' beliefs have not been the focus of research. Learners' beliefs opened a gateway to study their motivation, engagement, affection etc. towards mathematics but their contrary beliefs about mathematics and possible explanation for them are not explored in the research literature. In this paper, learners' beliefs are therefore studied in light of their *critical thinking* skills in order to look for the possible reasons of these logical inconsistencies existing in learners' beliefs about mathematics. The sub-skills *self-examination* and *self-correction* of the sixth critical thinking skill *self-regulation* from critical thinking skills framework (**Figure 1**) are used to analyze learners' expressed beliefs about mathematics (details in *Data Analysis Framework* section).

Learners' Critical Thinking and Mathematics Education Research

Analogous to the research concerning beliefs in mathematics education, the research related to learners' critical thinking is also wide and diverse. Critical thinking is discussed from being a cognitive toolkit in order to solve mathematical problems in a logical and deductive manner (Jablonka, 2014; O'Daffer & Thornquist, 1993), to an attitude which learners as future citizens of mathematical society should be able to adopt in order to strive for a just and equal social structure (Gutstein, 2003; Skovsmose, 1998). This section elaborates these variations concerning critical thinking research in relation to mathematics education and suggests the connection between learners' beliefs and critical thinking in light of the research question asked in this study.

Critical thinking for mathematical problem-solving and its limitations

The “first wave” (McLaren, 1994, p. ix) of psychological and educational research concerning critical thinking presented it as the cognitive processes resulting in analyses of information objectively, argumentation, drawing conclusions, deductive and logical reasoning etc. (Ennis, 1964). The requirement of being objective, logical and rational while thinking critically had similarities with the *absolutist* view of mathematical knowledge (Ernest, 1985, 1991). This view of mathematics education in which being objective was essential for the knowledge to be seen as mathematical, provided convenient grounds for the partnership of critical thinking and mathematical problem-solving. Gutstein et al. (1997) also highlight this correlation by drawing on NCTM Standard (National Council of Teachers of Mathematics [NCTM], 1989) using *mathematics as reasoning* by citing “… students understanding and applying reasoning processes, creating and judging mathematical arguments, and validating their own thinking and answers” (p. 712).

This partnership leads to testing learners’ critical thinking skills quantitatively (using mathematical tasks as test items), and usually in experimental settings (pre-and post-tests, control and treatment groups). Standardized tests, such as, Ennis-Weir Critical Thinking Essay Test (E-W), California Critical Thinking Skills Test (CCTST) and Cornell Level X Critical Thinking Test (CL-X)[2] got developed for the purpose. Likewise, CCTST-N[3] and James Madison University’s Quantitative Reasoning (QR) Test[4] were developed to assess learners’ critical thinking skills (named as QR, numeracy or quantitative literacy skills), specifically in mathematics. Since these tests measure cognitive critical thinking skills such as, proposing hypotheses, analysis, evaluation, drawing inferences and conclusions, most of the questions (usually multiple-choice) posed included graphs, tables, numbers and data to reach a decision. Mathematics education researchers also employed these tests (Aizikovitsh-Udi & Cheng, 2015) and contextual mathematical problems (Firdaus et al., 2015; Palinussa, 2013) to assess learners’ critical thinking skills.

Using critical thinking to solve tailored (contextual) mathematical problems (in classroom/critical thinking tests) can indicate learners’ successful implementation of critical thinking within a *cognitive* domain, usually divorced from their *social, political, cultural* contexts and *personal* life-choices. Jablonka (2014, p. 122) mentions that, even *cognitive* critical thinking “does not automatically emerge as a by-product of any specific mathematics curriculum …”. Hence, learners’ critical thinking ability may not transfer from a particular area of the discipline to social and personal spheres of their lives by itself. To gain a meta-perspective of their mathematics learning process, understand and participate in improving it, learners should be acquainted with applying critical thinking to reflect the role of mathematics and mathematics education in the *social, political, cultural* aspects of their lives, and in making their *personal-life* choices.

Critical thinking for understanding mathematics and mathematics education in socio-political and cultural contexts and its limitations

By introducing the “second wave” of *critical thinking*, critical pedagogy researchers reminded that all *critical thinking* is carried out by *someone* in *some social context* (McLaren, 1994, p. x). They insisted that if one is to use critical thinking in their lives, a complete de-subjectification and de-contextualization of any daily life situation cannot be possible. Hence, achieving full objectivity while thinking critically by discarding thinker’s social, political, cultural contexts, and personal beliefs and/or prejudices is illusionary. Moreover, creativity and imagination may seem like opposites to *critical thinking,* and any creative solutions to a problem would be inacceptable if critical thinking should only propose logically deducible solutions.

Critical pedagogy, through education, aims to make learners conscious of *thinking critically* to reflect over their socio-political and cultural settings, and their personal beliefs to understand and rectify hidden hegemonic ideas, power, privilege, injustices and inequalities in society (Freire, 1972). Thus, ‘second wave’ critical thinkers propose an alliance of critical thinking and critical pedagogy, in which *critical thinking* should enable learners to critically observe, reflect, understand and analyze their social contexts and *personal* beliefs from a distance; and take planned actions to promote social justice, equity and peace in the society. This alliance also influenced mathematics education research.

CME (Skovsmose, 1994a) and the socio-political turn (Gutiérrez, 2013) in mathematics education make it clear that imparting *critical thinking competence* among mathematics learners is important to promote social justice, equity and critical citizenship through mathematics education (Gutstein et al., 1997). The need of making mathematics learners critically reflect and critique mathematics in the spirit of critical pedagogy is argued, though not particularly using the term *critical thinking*. Gutstein (2003, 2006) and Sriraman and Knott (2009) provide examples of using mathematics education with critical pedagogy to promote learners’ socio-political and cultural consciousness about mathematics and mathematics education’s role in society; and how it can help promoting social justice, critical citizenship and equity. This way learners can be made aware of the role mathematics and mathematics education play in their socio-political and cultural surroundings and how it indeed shapes their thoughts, lives and society. Also, Skovsmose (1994b), seemingly supports the ‘second wave’ of critical thinking, rejecting the possibility to “… reduce ‘reflection’ and ‘critical thinking’ to ‘logical awareness’ ... informal logic and to criticism ‘inside the disciplines’ …” (p. 217). He recommends CME to develop *mathemacy* in learners – a competence parallel to *literacy* (Freire, 1972) gained by critical education, as advocated in critical pedagogy. Evolving learners’ critical reflective skills for attaining mathematical, technological and *reflective* knowledge[5] are favored to develop *mathemacy* (Skovsmose, 1994b). Learners are expected to critically reflect the

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[2] See Hatcher (2011) and Liu, Frankel, and Roohr (2014, pp. 5-7) for detailed overview of available Critical Thinking tests.

[3] California Critical Thinking Skills Test with Numeracy (CCTST-N).

[4] See https://www.madisonassessment.com/view-demo/ for sample test items and Grawe (2011) for a detailed review of QR assessment tests.

[5] *Mathematical* knowledge includes learning mathematical content – numbers, algebra, methods, theorems, proofs etc. used in mathematical modelling and problem-solving. *Technological* knowledge enables learners to model a given situation mathematically and applying mathematical knowledge to analyze or solve the situation. *Reflective* knowledge, however uses the *critical reflection* component to reflect over the legitimacy and role of applying mathematical and technological knowledge to a situation and the influence mathematics exerts on those contexts.

use of mathematics in society using their *reflective* knowledge about mathematics gained through CME. Similar to *literacy*, the aim of *mathemacy* is to empower learners to be critical citizens through mathematics education.

However, critical socio-political awareness alone is not sufficient to bring social justice and equality, since to act consciously for bringing change in personal lives or society, people need *self-awareness* prior to socio-political awareness (Freire, 1972). In mathematics education context, Gutstein (2003) says that being critical about use of mathematics in socio-political contexts such as statistical data and results can help learners only *read* the world using mathematics. This reading can promote equity and justice *theoretically*, whereas to *write* (to *act* for changing) the world using mathematics, learners need to be active carriers of that change. Therefore, though learners may understand the role of mathematics and mathematics education in their socio-political and cultural contexts using critical thinking, it is not obvious that they can think critically about their own mathematics learning process for taking reflected decisions in their *personal* lives.

Critical thinking for gaining a meta-perspective of learning mathematics in personal life and its potential

Previous sections describe the *first* and *second* waves of research concerning *critical thinking.* McLaren (1994, p. xii), however, describes a potential "third wave" of *critical thinking* aimed at making critical thinkers aware of the importance of being critical to their *personal beliefs*. Such critical thinkers observe their own beliefs, choices, decisions and actions in their personal lives and societal circumstances critically to reflect on how they themselves influence and are affected by inequalities and injustice prevalent in society. Simplified, the 'third wave' of *critical thinking* can be seen as being a *critical thinker* in *personal* sphere of one's life.

In mathematics education, research regarding *critical thinking* is not precise. Different perspectives highlight learning and applying *critical thinking* in *cognitive* and *social* spheres of one's life through mathematics education (see *Introduction*). However, learners' potential of *critical thinking* to make reflected choices and take informed, well-reasoned decisions about learning mathematics in their *personal* lives does not seem to be carefully researched. This focus on researching learners' application of critical thinking skills only in the cognitive and social spheres of their life while learning mathematics has failed to consider the opportunity to find out if and how they adopt a critical attitude towards learning mathematics in their *personal* lives. Consequently, the potential and practice of learners in applying critical thinking skills to their *personal* mathematics learning process remains undiscovered. Learners are expected to make choices in mathematics education, especially at the transition points (at the end of their compulsory school years) of their educational pathway where they take important decisions regarding education for their future lives. Such choices should not be made uncritically. Therefore, McLaren's third wave of critical thinking applies to mathematics education as well. Learners should use critical thinking to gain a *meta-perspective* of learning mathematics in their personal life, so that they not only decide what to *believe*, but also can figure out how to *act* to improve their mathematics learning process. These *meta-perspectives* about their mathematics learning process can be different. A call for learners to achieve such *meta-perspectives* is also scattered around in the literature concerning CME. We outline three types of non-exhaustive and overlapping *meta-perspectives* below.

Firstly, learners' *critical thinking* about personal *beliefs, choices* and *decisions* concerning mathematics learning in their lives may involve *critically* understanding *what* content they are interested/not interested in learning, *how* mathematics is being taught to them, *why* they are/are *not* learning mathematics, and the purpose it serves in their present and future lives. Learners choose if they want to study mathematics or *not* and eventually the direction(s) they wish to pursue in further mathematics education. Such decisions involve *critical thinking* about personal reasons, desires and ways of learning mathematics or *not*. This *meta-perspective* would allow them to not only "grasp a meaning but also to have the possibility of *negotiating* a meaning about the content of their [mathematics] education" (Skovsmose, 1994b, p. 93) (parentheses and italics added). Acquiring such a meta-perspective can give learners the possibility to co-operate with teachers to influence and evaluate their teaching strategies and take decisions to make mathematics classes meaningful for themselves. This way learner cans develop a meta-language to have a say, take *charge*, start *acting* and *participating* (Skovsmose, 1994b, 2001) in their mathematics classrooms. Similar concerns are found in previous studies encouraging to extend the focus of CME to involve students more profoundly (Clare & Sue, 2013; Powell & Brantlinger, 2008), give them freedom to decide their own curriculum in mathematics (Lindenskov, 2010) and know its relevance in their *personal* lives (Kollosche, 2017).

The second type of *meta-perspective,* involves learners' *thinking critically* and understanding their strategies of learning mathematics and the ways to improve them. For instance, while doing mathematics, they can *critically reflect* upon their patterns of study, learning strategies, and regulate their personal practicing methods, studying structures, schedules etc. that work for them. By gaining such a *meta-perspective*, learners can improve their *learning* strategies leading to feel success in learning mathematics and enhance their mathematics learning experience on their own (Malmivuori, 2006).

In the third type of *meta-perspective*, learners become aware of *critically* understanding their personal beliefs about mathematics and mathematics education and the reasons for holding them. First and second *meta-perspectives* allow learners to understand how they can influence their mathematics learning, while the third *meta-perspective* encourages learners to *think critically* about how learning mathematics influences them and the society. Several papers forward concerns about learners acquiring an inferiority complex while struggling to learn and succeed in academic mathematics under CME research. Greer and Mukhopadhyay (2012) raise the issue of hegemony of mathematics education in society and "intellectual violence" (p. 236) exerted on learners by the dominant societal status and teaching structure of mathematics. D'Ambrosio (2010) also pointed out the responsibility of mathematics and mathematics educators to address the problem of survival with dignity for all individuals in the society. The need to *critically* consider the interplay between their personal beliefs about mathematics, and how these beliefs in-turn influence and reflect in the dominant structures of society through mathematics education is addressed for both struggling

	Strongly disagree	Disagree	Agree	Strongly Agree	Do not know
To learn mathematics and science is important for me because it will improve my career opportunities	☐	☐	☐	☐	☐
I am interested in what I learn in mathematics and science	☐	☐	☐	☐	☐
Mathematics and science are important subjects for me because I need them for my future studies	☐	☐	☐	☐	☐

Figure 2. An excerpt of statements from the questionnaire

and successful learners of academic mathematics. This self-awareness can make learners *conscious to reflect* over their role[6] in hegemonic structures and *able to act* for changing the status quo of dominant ideologies and traditions of mathematics education in society in and respect people holding different views.

Gaining these meta-perspectives through critical thinking can provide learners with a *potential* to improve their mathematics learning process for the betterment of their personal lives and the society. They can figure out better ways to learn mathematics, co-operate with their teachers to improve their mathematics teaching, and prioritize their choices while learning mathematics in their present and future lives. In addition, aim of the third wave of *critical thinking* to make the thinker aware of his/her own ideological situatedness in the society and work for emancipation (not adaption) can be realized through this *potential*. However, in this study, learners' meta-perspective of *first* type is explored since their expressed beliefs regarding *what, why* and *how* of their mathematics learning process are analyzed to identify their practice with *thinking critically* about this process.

METHOD

A qualitative approach is adopted to investigate learners' *personal* beliefs and reasons for learning mathematics (Creswell, 2014). Presented data was collected as part of the research project *Local Culture for Understanding Mathematics and Science* (LOCUMS, 2016); aiming to explore the role of *practical tasks rooted in students' own interests and local culture* to learn mathematics and science. Here, we analyze parts of the data collected under sub-project of LOCUMS carried out in middle Norway. Two schools which included learners from diverse cultural backgrounds were chosen as the sites of data collection in accordance with the research aims of LOCUMS. Four classroom interventions were planned which took place in 8th (two interventions) and 9th (two interventions) grades of two schools located in middle-Norway. Each intervention included three steps of data collection. In first step, learners responded to a paper-pen questionnaire, second step included learners working to solve practical group tasks (4-5 learners in each group) and the final step included face-to-face individual interviews with selected learners. This paper is based upon the data collected in questionnaires and interviews. In total, 74 learners from these two schools participated in the questionnaires and interventions, and 20 learners were selected for conducting semi-structured interviews. Both the questionnaires and interviews were conducted in Norwegian, audio-recorded, and later transcribed for analyses.

The questionnaire was designed by deriving the inspiration from the ROSE survey (Schreiner & Sjøberg, 2004). Different parts of the questionnaire were devoted to different themes which LOCUMS focused on. The questionnaire started for example with asking for the introduction of the learner and moving ahead to their personal life interests, leisure time activities, thoughts about culture, their multicultural classrooms, connection with mathematics and science, future education and job perspectives, school environment, social participation and a particular section for the immigrant learners. Each of these parts included both five-point Likert-scale statements and some open-ended questions where learners could write down their opinions freely. Since LOCUMS was directed to both mathematics and science subjects, the statements in the questionnaire include both these subjects. This paper discusses the results from learners' responses to the section "your connection with mathematics and science" in the questionnaire part. This section included 24 Likert-scale statements concerning learner's relation with mathematics and science subjects. Out of these 24 statements, 12 statements were about both mathematics and science, eight concerned only science, and four only mathematics. Learners had to respond to the extent they agreed with each given statement on a scale from *strongly disagree* to – *strongly agree* (see **Figure 2**).

Due to the focus of LOCUMS on both mathematics and science education, the statements in this section of questionnaire were about both these subjects. In addition, the statements carried both positive and negative connotations, for example, some statements dealing with mathematics were, *'I like to learn mathematics'; 'I think what I learn in mathematics is a waste of time'; 'I am simply not good at doing mathematics'* etc. Since statements dealing with both mathematics and science made it difficult to separate learners' beliefs about mathematics from those about science, their questionnaire responses were used as a starting point to design the interviews. Any incoherence noticed in their questionnaire responses laid the foundations for preparing interview questions directly related to their beliefs regarding mathematics, and its learning process. Consequently, the

[6] Either by contributing to authoritative ideologies, by adapting and adjusting to these hierarchical structures or by contributing to injustice and inequality in any other sense.

questionnaire responses are presented here only as a precursor to the interview analyses, which is the main source of information for this study.

An interview guide was developed in accordance with selected learner's responses on the questionnaire statements about mathematics and science, and their performance and engagement levels in the intervention. Each interview lasted for about one to one and a half hour, where the questions were divided into different sections. These sections started with an individual introduction and moved to enquire into the learner's general views about education, school environment, culture and cultural differences[7]. Further, learners' beliefs in-depth about their learning process in mathematics and science subjects separately were explored; and the concluding section enquired their experiences with the interventions. Since the research question deals with learners' practice with thinking critically *about* and give suggestions concerning their personal mathematics learning experience, the interview questions focused on their choices, decisions and personal experiences in mathematics classroom are presented here. Specific issues such as, *what* they are/are not interested in learning, *why* they learn and *how* they are taught mathematics, were discussed with learners. In this way, learners were made to *self-examine* their opinions and suggest ways, which can enrich their mathematics learning experience.

Our informants include learners studying in 8th and 9th grades (13-14 years old) who will soon reach a transition stage in their educational pathway (i.e., in 10th grade) where they prioritize and take important decisions concerning their further mathematics education. The whole data set consists of 74 responded paper-pen surveys, video recordings of learners solving group tasks and audio recordings from 20 individual (10 boys, 10 girls) interviews with selected learners. All the 74 learners got and answered the questionnaires (no sampling), whereas the 20 informants for in-depth semi-structured interviews were selected by following the principle of maximum variation. Based on a learner's questionnaire responses and his/her performance and participation level in the classroom intervention, a representative selection of five learners (one from each group in a classroom) after each intervention was made. The selected learners included students who liked/did not like, were highly, moderately or not much interested in learning mathematics, and were highly, moderately or not so enthusiastic while participating in the intervention. Attention was also paid to interview the learners from different cultural backgrounds. Majority (15) of the interviewed learners were Norwegian but some of them had different mother tongue (Sámi (2), Eritrean (1), Arabic (1) and Pashto (1)) and diverse cultural backgrounds. All except one (interviewed in Pashto) of them had been in Norway for more than three years and understood Norwegian well during the interviews despite having a different mother tongue. However, to investigate their critical thinking in-depth, this study focuses on learners' responses, not the variation in their culture and language.

DATA ANALYSIS FRAMEWORK

Both questionnaire and interview data were analyzed to explore learners' *critical thinking* about their personal choices and decisions concerning learning mathematics in their lives. Preliminary analysis of questionnaires highlighted learners' inconsistent responses to statements concerning their mathematics learning. For instance, a learner *strongly agreeing* that she/he likes learning mathematics but strongly disagreeing that she/he is interested in what she/he learns in mathematics was interpreted as an incoherent response. These conflicting responses were followed up in the interviews. Both questionnaire statements revealing learners' incoherent responses and selected interview excerpts are presented here. Questionnaire responses are analyzed in MS Excel, and interview excerpts are analyzed descriptively using the sub-skills, *self-examination* and *self-correction* of the sixth core cognitive critical thinking, *self-regulation* (**Figure 1**).

While holding certain beliefs, making specific choices or decisions in lives, people often have personal reasons and justifications for doing so while being unconscious of these reasons. Therefore, when asked to explain their beliefs, choices and decisions, the process of clarifying can make them conscious of those underlying reasons, assumptions, bias etc. *Critical thinking*, as defined, presumes existence of beliefs, hypotheses or opinions, but being *critical* involves questioning those beliefs and ideas logically. Beliefs are the objects of reflection for critical thinking and *self-regulation* critical thinking skill provides the opportunity to analyze our personal beliefs rationally (Facione, 1990). *Self-regulation* focuses on the process of self-consciously questioning and evaluating reasons for the judgements and decisions made by oneself (i.e., *self-examination*), and correcting one's reasoning or beliefs in case it reveals any bias or erroneous assumptions (i.e., *self-correction*). Since the incoherence of learners' beliefs concerning their mathematics learning process is focused in this study, the skills of *self-examination* and *self-correction* are employed to analyze learners' interview responses. Learners' beliefs are enquired along with the reasons and justifications for holding those beliefs to explore if this inconsistence may be related to their practice with *thinking critically*. The definitions of *self-examination* and *self-correction* were operationalized as follows:

- *self-examination* is understood as "… an objective and thoughtful meta-cognitive self-assessment of one's opinions and reasons for holding them'; 'judging the extent to which one's thinking is influenced by deficiencies in one's knowledge, or by stereotypes, prejudices, emotions or any other factors constraining one's objectivity or rationality." (Facione, 1990, p. 19), and
- *self-correction*, is understood as, " where self-examination reveals errors or deficiencies, to design reasonable procedures to remedy or correct, if possible, those mistakes in one's opinions and their causes" (ibid.).

[7] The questions concerning learners' views about education, mathematics, science, culture and cultural differences in their own class etc. intended to meet the objectives of LOCUMS.

Our aim in these interviews was to explore learners' practice with critical thinking *about* learning mathematics. These definitions allowed us to explore learners' beliefs, assumptions, thoughts, opinions etc. presented in the interview conversations to highlight their practice with critical thinking skills of *self-examination* and *self-correction* about learning mathematics.

Ethical Considerations

This project was reported to the Norwegian Centre for Research Data (*Norsk senter for forskningsdata*, NSD) and permission to collect data was obtained with a clearance number 50556/3/AMS. Both young learners and their parents and/or guardians were informed in written about the project and their written consent was obtained to collect data involving their children. They were made aware that all data will be treated confidentially. It was also conveyed that their participation is voluntary and they can withdraw their consent whenever they wish without giving any reason.

DATA INTERPRETATION AND DISCUSSION

Questionnaire Analyses

This section presents analyses of learners' responses to the statements of the section "your connection with mathematics and science" in the questionnaire. There were 24 statements in this section, out of which eight statements dealt only with science and 16 with both mathematics and science. In 11 of these 16 statements learners were asked about their achievement levels in mathematics and science, and the remaining five statements enquired about their interest, liking and importance of learning these subjects for further studies and career opportunities. Since learners' achievement level is not the priority of this paper, some analyses of learners' responses to only these five statements are presented here. These statements being – 'I like to learn mathematics'; 'I am interested in what I learn in mathematics and science'; 'To learn mathematics and science is important for me because it will improve my career opportunities', 'Mathematics and science are important subjects for me because I need them when I will study further'; and 'I think that what I learn in mathematics is waste of time'.

On comparing learners' responses to these statements, we observed an inconsistence in their answers. A comparison analysis of opinions of several learners appeared to be incoherent and unsure. For example, if a learner strongly agreed/agreed to both the statements, *'I like to learn mathematics'* and, *'I think what I learn in mathematics is waste of time'* or vice-versa[8], it was interpreted as an incoherent response. Another example is a learner strongly agreeing/agreeing that, *'I like to learn mathematics'*, but strongly disagreeing/disagreeing that, *'I am interested in what I learn in mathematics and science'*; or the opposite[9]. Therefore, we collected learners' responses to these five statements from all 74 questionnaires and compared them with each other to find out the proportion of incoherent answers for each pair of the statements. **Table 1** shows these statement pairs and the number of incoherent responses from the learners. This analysis promoted our interest in enquiring if practice with thinking critically can be a possible explanation of the 'mismatch' between learners' responses, which formed the bases of our research question for this paper. **Table 1** is presented as a forerunner leading towards the interview process and research question.

As depicted in **Table 1**, 18% (13 out of 74) learners stated that they *like* to learn mathematics, but are *not interested* in learning the same, or vice-versa. 32% hold divergent views about *liking* to learn mathematics, and the *importance* of learning mathematics and science when it comes to future career opportunities. Simultaneously, responses of 31% of learners are incoherent when it comes to *liking* to learn mathematics and the *importance* of learning mathematics and science for further studies. 28% responded that what they learn in mathematics is *waste of time* despite they *like* to learn mathematics, or they do not think that learning mathematics is a waste of time, yet they do not *like* it. Further, 22% give disconnected responses to being *interested* in learning mathematics and science in relation to its *importance* for future careers. 23% have paradoxical views regarding their *interest* in learning mathematics and science, and the *need* for learning these subjects for further studies, whereas, 20% reported that though they are *interested*, what they learn in mathematics seems to be *waste of time*. The results for the last three pairs of statements in **Table 1** seem to be more intelligible. Only 5% of students have diverging views regarding *importance* of learning mathematics and science for *future careers*, and the need of these disciplines for *further studies*. Just 11% state that learning mathematics is *waste of time,* which seems to be correlated with that learning mathematics and science is *important* for their *future career*. Learning mathematics seemed to be a *waste of time* for only 5% of learners as they report that they need to learn mathematics and science to *study further.*

This preliminary analysis of questionnaire responses brought the confusion apparent in learners' beliefs to our notice which provoked our curiosity to know if learners consciously and critically examine and correct their personal beliefs about learning mathematics. Though majority of the learners (over 50% for all the statement pairs) provide coherent answers, but the lack of harmony and indecisiveness seems evident for about 30-40% of learners. The questionnaire responses, however, did not help us to understand why learners were indecisive or held divergent opinions. Therefore, semi-structured interviews were conducted to delve deeper into learners' application of critical thinking while interacting *about* their mathematics learning process. In interviews, learners were asked to follow-up and justify any incoherent answers in-depth. Their responses were then analyzed using *self-examination* and *self-correction* to explore their use of critical thinking for making choices and decisions about learning mathematics in personal lives.

[8] That is, strongly disagreeing/disagreeing to both the statements, *'I think what I learn in mathematics is waste of time'* and, *'I like to learn mathematics'*.

[9] That is, strongly disagreeing/disagreeing that, *'I like to learn mathematics'*, but strongly agreeing/agreeing that, *'I am interested in what I learn in mathematics and science'*.

Table 1. Overview over learners' incoherent responses to pairs of the five statements in questionnaire

Pair number	Statement number 1	Statement number 2	No. of learners (out of 74) giving incoherent responses to statement number 1 and 2
1	I like to learn mathematics	I am interested in what I learn in mathematics and science	Agree or disagree to both the statements (Coherent responses): 46 Agree to statement no. 1 and disagree to statement no. 2 or vice-versa (Incoherent responses): 13 Do not know/other answers: 15
2	I like to learn mathematics	To learn mathematics and science is important for me because it will improve my career opportunities	Agree or disagree to both the statements (Coherent responses): 41 Agree to statement no. 1 and disagree to statement no. 2 or vice-versa (Incoherent responses): 24 Do not know/other answers: 9
3	I like to learn mathematics	Mathematics and science are important subjects for me because I need them when I will study further	Agree or disagree to both the statements (Coherent responses): 40 Agree to statement no. 1 and disagree to statement no. 2 or vice-versa (Incoherent responses): 23 Do not know/other answers: 11
4	I like to learn mathematics	I think that what I learn in mathematics is waste of time	Agree to statement no. 1 and disagree to statement no. 2 or vice-versa (Coherent responses): 43 Agree or disagree to both the statements (Incoherent responses): 21 Do not know/other answers: 10
5	I am interested in what I learn in mathematics and science	To learn mathematics and science is important for me because it will improve my career opportunities	Agree or disagree to both the statements (Coherent responses): 43 Agree to statement no. 1 and disagree to statement no. 2 or vice-versa (Incoherent responses): 16 Do not know/other answers: 15
6	I am interested in what I learn in mathematics and science	Mathematics and science are important subjects for me because I need them when I will study further	Agree or disagree to both the statements (Coherent responses): 41 Agree to statement no. 1 and disagree to statement no. 2 or vice-versa (Incoherent responses): 17 Do not know/other answers: 16
7	I am interested in what I learn in mathematics and science	I think that what I learn in mathematics is waste of time	Agree to statement no. 1 and disagree to statement no. 2 or vice-versa (Coherent responses): 41 Agree or disagree to both the statements (Incoherent responses): 15 Do not know/other answers: 18
8	To learn mathematics and science is important for me because it will improve my career opportunities	Mathematics and science are important subjects for me as I need them for further studies	Agree or disagree to both the statements (Coherent responses): 60 Agree to statement no. 1 and disagree to statement no. 2 or vice-versa (Incoherent responses): 4 Do not know/other answers: 10
9	To learn mathematics and science is important for me because it will improve my career opportunities	I think that what I learn in mathematics is waste of time	Agree to statement no. 1 and disagree to statement no. 2 or vice-versa (Coherent responses): 54 Agree or disagree to both the statements (Incoherent responses): 8 Do not know/other answers: 12
10	Mathematics and science are important subjects for me because I need them when I will study further	I think that what I learn in mathematics is waste of time	Agree to statement no. 1 and disagree to statement no. 2 or vice-versa (Coherent responses): 56 Agree or disagree to both the statements (Incoherent responses): 4 Do not know/other answers: 14

Learners' Interview Responses about What, Why and How of Learning Mathematics

This section presents selected interview excerpts in which learners were asked about *what* they think is interesting to learn in mathematics, *why* they think it is important to learn mathematics and *how* to improve mathematics teaching. Being semi-structured, the interviews were conducted like informal conversations and the questions asked from different learners were similar but not identical. In order to explore learners critical thinking in terms of self-examination and self-correction, they were often asked to provide reasons for their responses. In the following text, different learner's response (indicated by using a number as a subscript under letter 'L' in the transcripts below) to same question is used as the unit of analysis. A descriptive analysis of learners' interview responses is presented below. Learners' responses starting with the phrases *"I" think*, *view*, *believe, use* etc. have been chosen to evaluate their application of self-examination and self-correction skills. First, representative interview responses of learners concerning *what* is interesting and *what* seems to be waste of time for them to learn in mathematics are presented and discussed. Later, learners' responses concerning *why* they learn and *how* they suggest to improve mathematics teaching are explored. An interview excerpt concerning the interest in learning mathematics follows:

I (Interviewer): … is there something in mathematics you are especially interested in learning?

L_4 (Learner$_{Number}$): I like to calculate the areas and … construct

I: is there something in mathematics which you think is useless to learn?

L_4: no…

I: but do you think then that you need to learn everything in mathematics that you learn in the class?

L_4: maybe not everything but …it is smart to be able to do that …

Here, the learner quickly expresses what content of mathematics she likes to learn; however, it seems difficult to answer if there is any content in mathematics which seems useless for her to learn. This was common for most of the interviewees. 16 out of 20 learners promptly mentioned what specific content in mathematics they are interested in learning. Moreover, when justifying their choices 12 out of 20 were able to provide reasons for their interest in learning that specific content, may it be just for fun, further studies or future job. These numbers give an impression coherent to Lindenskov (2010) interpretation that leaners usually *self-examine* and can justify their personal interests in learning specific mathematics content. However, learners seemed to be confused and unsure when asked to name the topics which may not be useful for them. In the interview excerpt presented above, the learner quickly concludes that she does not find any topic in mathematics to be useless to learn. The paradox, however, is that though the learner thinks that not everything is useful to learn, yet she could not mention any topic in mathematics that is useless to learn. This unawareness of critically evaluating both perspectives indicates a negligence of self-examining one's personal opinions in-depth. This dilemma was also clear in other interviews, for instance:

I: … is it something in mathematics which you think is especially interesting?

L_6: I think it's like the fractions and the forms [figures] and such things… I think that it's fun to do that…

I: is there something in mathematics you think is waste of time?

L_6: eh… no…it's well maybe… or I don't know actually

I: but eh… maybe there is something that you feel that you have no need to learn or … like you cannot use it for something? Or is it just that … like okay it's nice to know about everything then?

L_6: yea… it's maybe like this or like… I don't know actually. I can't think of any words [topics] like which could be felt like waste of time…

I: yea but it's no hurry like you don't have to answer quickly… you can take your time to think and if you think it would help?

L_6: it's maybe the same [histograms, as mentioned earlier in the interview] …

…

I: will you say that it's difficult to learn or you just can't find any use of it?

L_6: I don't just find any use of it… I don't know…

In this extract, learner easily picks up what is especially interesting for her to learn in mathematics, but when asked about the content she does not find useful, the words *'I don't know'* appear frequently. Only on being urged and given time and to reflect, she came up with a topic which did not seem useful for herself. Only eight out of 20 could name topics they think are waste of time to learn for them and just seven could give a reason for why learning that specific topic in mathematics is in vain for them.

Learners' responses and frequent use of the words *'I don't know'* indicate their self-doubt, unfamiliarity and lack of *critically self-examining* their personal interests in learning mathematics. Even if learning everything they are taught does not appear to be correct, possible or useful, but not trying to do so is experienced to be negative since mastering mathematics is experienced to signify being clever and smart. Therefore, an unconscious resistance to gain an overall critical meta-perspective of one's own mathematics learning also surfaces. These contrasting responses reveal that learners are not used to think critically about the questions related to the mathematical content they *do not* think is interesting to learn. However, when asked to be critical and clarify their own opinions mentioned earlier in the interview, they do mention the content which does not seem personally beneficial for them. Considering both, *what* content *is* and *is not* interesting for them, is important for learners to make reflected personal choice regarding what field of mathematics they want to pursue in the future, and opt out of the content which is boring for them.

Being able to use and apply acquired knowledge in practical lives and professions legitimates to a great extent *why* that knowledge is gained. Therefore, learners were asked to justify *why* they learn mathematics by providing examples of using mathematics in their lives. While mentioning these uses, 17 of 20 learners named only the application of elementary mathematics in real-life (i.e. basic arithmetic calculations while shopping, cooking etc.), whereas only three of them presented examples of using mathematics they learn in their current grade. For instance:

I: but like in general? …, is it something then you can use mathematics for?

L_8: eh… not for a lot of things but maybe … or at least little bit… if you have to calculate something so…eh…mass density or you should find out … or if you have to make a bit like makeshift or MacGyver [a TV hero]… it's like to make something out of the things you find and then it becomes something…you can calculate how things work out like if you had to for example make an electric cycle from only the things you find in a garage… so you need to calculate things and know how the things work out … such that it could work…

L_8's argument of being able to do machine makeovers represents a situation where one may need 8th and 9th grade mathematics in a real-life situation, however, the learner imagines a fictional character (i.e., MacGyver), not himself using mathematics in own real-life. Two more learners provided such examples including: to read geometrical and numerical data using maps and a compass; and applying statistics to any data collected, though specifying that he uses statistics only at school. Though specifying that they have not used mathematics in this way yet in their own everyday life, these three learners display some potential to *self-examine* their personal beliefs about *why* they learn mathematics. Uses of elementary mathematics provided by the remaining 17 learners include examples such as:

> *I: but can you give examples on where you use mathematics? Like the mathematics you learn at the school? Do you use it anytime in your daily life or…?*
>
> *L_6: yes… I use it like when I am at a shop so I use to pay by cash and to find out how much I'm going to get back and different things like this and then I use it when I make food …then one needs to know like liters and deciliters and such things then…*

L_6's response associates "the relevance of mathematics with learning skills *in elementary mathematics*" (Kollosche, 2017, p. 637) (italics in original), without considering why and where this learner needs advanced mathematics such as algebra or geometry in his/her life. Estimation of price using basic arithmetic operations is curriculum of primary school, not what they learn in 8th and 9th grade mathematics. Lack of *self-examining* and *self-correcting* one's beliefs about *why* one learns mathematics is also visible in the following response:

> *I: yes… I am listening to what you are saying but actually it is a bit interesting that… algebra is a very… ok it's right what you are saying that it's [algebra is] just setting in a letter instead of a number but why are you going to need it…like? You are saying that it is useful but can you give an example about where you will need it?*
>
> *L_5: I've just heard that one needs it…*
>
> *I: heard…? Where?*
>
> *L_5: from teachers and other adults…*
>
> *I: ok, but have you sometime asked them where they use it?*
>
> *L_5: no…*
>
> *I: no… have you sometime asked them 'where can I use an equation?'*
>
> *L_5: no … I haven't asked him [the teacher] no…*
>
> *I: has he [the teacher] said anything about where you will use an equation?*
>
> *L_5: no…*
>
> *I: ok… well …but you still think that though you don't know that where one needs or can use it …so you think that it is useful to learn?*
>
> *L_5: yes… because we learn it…*
>
> *I: and important?*
>
> *L_5: it's not the most important thing but …like… eh… it is no…I just mean that people should know about it and be able to do it but I don't know why …we learn it at the school so one thinks that one will use it…*

L_5 is evidently influenced by the widespread belief that mathematics is a useful subject to learn and everyone is going to need it later in their lives. The phrase '*we learn it at the school so one thinks that one will use it…*' illustrates learner's trust in mathematics curriculum adopted in their classroom to an extent that she does not feel the need to critically examine the information received by the teachers or other adults. Consequently, she is not conscious of demanding reasons, justifications and real-life applications of mathematics learnt in school. Moreover, though she could not mention any uses of advanced mathematics in her life, yet she denied that learning it may not be so useful for her. This instance not only illustrates the lack of *self-examination* of one's personal beliefs about learning mathematics, but also the denial to *self-correct* those beliefs if found unjustified, which again can hinder her in developing a meta-perspective of first type about learning mathematics. Similar seems to be the case for the other 17 learners who could only provide elementary uses of mathematics despite being in 8th or 9th grade, but still believed that learning advanced mathematics is and will be useful for them. This unconsciousness to *self-examine* and *self-correct* (if needed) their beliefs can inhibit evolving *self-regulatory critical thinking* skills among learners and sustain the incoherence of their beliefs regarding their own mathematics learning process.

Finally, the interview questions focused on *how* learners are taught and what changes they suggest improvement in their mathematics lessons. Several learners mentioned that mathematics lessons at times are either difficult, boring or obligatory for them. This general expression of learner dissatisfaction encouraged questioning the learners about their personal experiences of and suggest changes in mathematics teaching to improve it. Learners were not acquainted with answering such questions and it was not easy for them to spontaneously suggest changes in their mathematics lessons. However, when persuaded, 15 out of 20 learners proposed changes in their mathematics teaching. For example:

I: hmm …. have you sometime thought that there is something I would like to be changed in [mathematics] teaching?

L_{12}: no … I have not thought about it …

I: are you satisfied with the way teachers teach you?

L_4: yes…

I: why do you think that this way to be taught is alright?

L_4: because… eh… I don't know maybe I'm used to having it like this so…I have done it all these years so, I think it's a fine way to learn…

I: but if you had a chance…would you change the teaching in mathematics and science?

L_4: no…

I: nothing?

L_4: maybe a bit more [practical] activities in mathematics but otherwise I don't think so of anything…

Learner L_{12} replied that she has *never thought* about bringing any changes in their mathematics teaching, symbolizing unawareness to think critically and *self-examining* their mathematics teaching. Similarly, L_4 gave an expression of being *'used to having taught like this'*, thus thinking some other approach than the *usual* way seems to be difficult for her. However, when asked to reconsider, she could recommend more practical mathematics lessons since much sitting and getting instructions from the teachers was mentioned as being boring (Nardi & Steward, 2003). Changes proposed by these 15 learners ranged from including more practical activities to teachers using more real-life based tasks, alternative explanations and giving more time for slow learners to catch-up in the lessons. However, lack of practical activities in mathematics lessons was a recurring concern and also frustrating for some learners as illustrated:

I: had you involved other activities like more practical activities… [in mathematics lessons]?

L_8: eh… I would have done it because what we've done in the whole 8th grade is just to write… write and write and solve the tasks and then it becomes quite boring and you lose the interest and you sit there just to write and when the class is over so you think 'oh yes, finally finished…'

Here, L_8 suggests including more practical activities in order to get over the monotony of mathematics lessons. The changes mentioned above include general suggestions, but one of the learners even managed to sketch a complete lesson plan for teaching a topic she suggested in the interview. She expressed that learning to make budget for a family in an Excel worksheet can be useful instead of using one to two weeks to cram formulas for calculating volumes and areas of geometrical figures, which are easily accessible on their smartphones. Her reply follows:

I: hmmm…if you have a suggestion about that you could have learnt to set up a budget…have you also thought of how would you have liked to learn it? In what way?

L_5: eh… if we could have got a realistic situation… and then we could have got a task about it so it would have… for example set up budget for a whole family for a month and you get different expenses and the teacher and you get the income and you have to pay the tax and you have to pay different and you should have a bit sum as saving if you sometimes get into a trouble and such things … like which are important to learn like you don't need to take loan and you don't have a lot of debt because you just got into a trouble which you never expected in your budget… so we learn how one should use his money because I mean that mathematics…we use a lot of mathematics in society like money and much is controlled by money and money gives power so… because people should not use up…because of people should have better … because like now we have got confirmed… like I don't have any concept of… like I don't know what 1 million kroner is… it's a lot and lot but you can't manage to buy a house for one million kroners… so we should learn more about the value like how much 1000 kroner [NOK] is worth … and such…

L_5's reply presents an apt instance of using critical thinking to evaluate and examine her beliefs and in both personal and social domains of learning mathematics in her life. This critical outlook indicates learner's potential to acquire a meta-perspective of *first type* about her own mathematics learning not only to make choices, decisions and suggest improvements in her mathematics learning experiences, but also to critically observe mathematics' role in the society. These extracts show that though learners possess the capability to think critically and suggest changes in mathematics lessons, but their potential is hidden and they are unsure about doing so due to lack of training and experience. Further, it seems unlikely that learners themselves will initiate any such discussion and communicate their ideas to mathematics teachers unless asked. These examples indicate that learners are likely to gain a meta-perspective regarding *what, why* and *how* to improve their mathematics lessons by evaluating and suggesting improvements in these lessons, provided their *self-examination* and *self-correction* critical thinking skills are encouraged and facilitated. Our data however does not indicate that learners' views are different or appear to be influenced by variations in their cultural backgrounds.

Lack of adopting such a critical attitude towards learning mathematics in their personal lives makes it difficult for learners to gain the meta-perspective of *first type* so that they can make personal life choices and decisions concerning their mathematics education for further studies or career. This inexperience in decision-making indicates learners' *uncritical* obedience to the pre-decided mathematics curriculum and the traditional or dominant ways of teaching and learning mathematics at school. These speculations about learners' unconsciousness can be grounded in their contrasting questionnaire responses, combined with interview answers such as, *'I don't know'* and *'I'm used to having it like this so…I have done it all these years so, I think it's a fine way to learn…'*. However, though being inexperienced in thinking critically, the learners do not lack the potential to do so. When given time, encouragement to clarify their responses, and asked to reflect over their choices, they seem willing to explain their thoughts in depth and can also investigate and understand the reasons of their beliefs. Some of them have an advanced potential to contribute in improving their mathematics teaching-learning process, but they are not likely to initiate or communicate these thoughts to their teachers if not encouraged to do so.

Learners investigated in this study are young so it is not surprising that they are unsure and *'don't know'* their choices exactly, but it can be problematic if they have *'never thought about it'* until now. They will soon reach a transition stage in their educational pathway (i.e., starting higher secondary after 10th grade), where choosing the right direction (theoretical or vocational) for further mathematics learning will be important for their lives. Our study suggests that these learners lack the practice in thinking critically *about* their learning process of mathematics and struggle to mention their personal interests, reasons and favorable strategies to learn mathematics. These findings indicate a mismatch between what is expected of the learners at this age in terms of deciding the direction of their future mathematics education, and what prerequisites they have acquired in terms of critical thinking to assess the options available to make a well informed and reasoned choice concerning mathematics education in their personal life. Regardless of their achievement levels in mathematics, and whether or not they decide to study mathematics further, mathematics education should be experienced as meaningful, not causing inferiority complex or a feeling of unworthiness among these learners. Therefore, more awareness and attention needs to be diverted for inspiring them to be critical towards their mathematics related beliefs so that they can suspect the widespread claims prevalent in the society about mathematics instead of just accepting or leaving them unquestioned.

TRUSTWORTHINESS AND LIMITATIONS

The trustworthiness of this study is addressed using the criteria *credibility*, *transferability*, *dependability* and *confirmability* (Bryman, 2016). The *credibility* is achieved by triangulating the two research methods – questionnaires and semi-structured interviews. The paradox in learners' responses is evident in both questionnaires and the interviews. Moreover, the findings of the study are in coherence with the previous studies reviewed in the paper. The criterion of *transferability* is taken care by providing a thick description of the context of this study, the details of informants and an elaborated account of data analyses procedure. *Dependability* is assured by keeping the log and regular meetings of the research team through all phases of the research process. A language and quality check of questionnaire, intervention design and semi-structure interview guide was done by fellow researcher having experience in conducting similar studies. In addition, the analyses of questionnaire and interview responses were also discussed with the research team to review the findings. Furthermore, though complete objectification cannot be obtained but *conformability* of the study is ensured by keeping an objective outlook during the phases of data collection and analyses. The researchers neither had any involvement in learners' mathematics and science lessons on daily bases, nor any control over their achievement in these disciplines, before or after this project. In this way, it was avoided to exert any influence on the learners to provide falsified or have hidden any information while answering the questionnaire and interview questions. Researchers' subjective values or preferences therefore, can be assumed to have little influence on the conduct of the research and the findings derived from it.

Despite adopting the measures to establish the trustworthiness of the findings, this study has limitations such as, lack of similar previous research, and the constraints on time and resources to well-design instruments for exploring other possible reasons of the incoherence in learners' beliefs. Employing the lens of critical thinking faculties, *self-examination* and *self-correction* does not seem to be a familiar approach to visualize the incoherence observed in learners' mathematics related beliefs. Therefore, scarcity of similar previous research can be considered as a limitation for the way this study was designed and conducted. In addition, the overarching focus of LOCUMS on both mathematics and science subjects restricted the amount of time and resources available. Consequently, it became difficult to develop professionally advised questionnaires to access learners' self-examination and self-correction critical thinking skills, specifically for their mathematics learning experiences. Though investigating mathematics and science beliefs simultaneously can be considered as a multidisciplinary approach, the study's accountability and reliability within mathematics education might be increased by adopting such measures. Nevertheless, these limitations open up the possibilities for further research on developing such instruments and interview guides to advance the concerning field of investigation.

SUMMARY AND CONCLUSION

Learners' mathematics related beliefs are investigated under the domain of affect research in mathematics education. Most studies concerning learners' beliefs explore their motivation, engagement, anxiety, stress etc. towards learning mathematics through their beliefs, but a few studies are found enquiring their beliefs regarding the *what, why* and *how* of their mathematics learning process. These investigations illustrate that the learners' hold incoherent beliefs about mathematics and mathematics education, however, this incoherence has not been the center of attention for further research in mathematics education. We take this incoherence in learners' mathematics related beliefs as a starting point to explore their practice with thinking critically about and their potential to suggest changes in their mathematics learning process. Specifically, the critical thinking skills of *self-examination* and *self-correction* are analyzed. Therefore, along with their beliefs learners were also asked to provide the justifications and reasons for holding them.

Our literature review highlighted that research concerning critical thinking in mathematics education mainly discusses it as a tool to learn *cognitive* practices of inference, drawing conclusions etc. within a discipline; and sometimes to highlight *socio-political* power imbalances. However, APA's consensus definition of critical thinking[10], critical pedagogy research's argument for *second* and *third wave* of critical thinking; and recent mathematics education research challenge the limited view of critical thinking. The importance of developing and exploring learners' application of critical thinking in *personal* domain of their lives, along with the *cognitive* and *socio-political* is recognized in this paper. We propose that learners should acquire habit of critical thinking to gain a meta-perspective *about* their *personal* mathematics learning process so that they can make choices and decisions about learning mathematics in their future lives. In this way, they may become aware to participate and take actions for making mathematics learning process meaningful for them in accordance with the central values of the Norwegian mathematics curriculum.

Consequently, in this study learners' practice with thinking critically and their potential to suggest changes in their mathematics learning process is explored by analyzing their expressed mathematics related beliefs. The incoherence evident in the questionnaire responses of 8th and 9th grade learners is presented and *self-examination* and *self-correction* (**Figure 1**) skills are used to interpret their interview responses to explore their application of critical thinking with respect to their *personal* mathematics learning process. Interview excerpts including incidents where learners apply, and do not apply *self-examination* and *self-correction* skills to their mathematics related beliefs are represented. Based on our interpretations of the questionnaire and interview data, we conclude that the inconsistence evident in learners' mathematics related beliefs can be related to the lack of practice in thinking critically *about* their mathematics learning process. Both questionnaire and interview responses indicate that these learners have limited or no experience in thinking critically about learning mathematics, and specifically in applying the skills of *self-examining* and *self-correcting* on their mathematics related beliefs. This leaves an impression that learners' critical thinking faculties *in* learning mathematics seems to be common in classrooms whereas, critical thinking *about* learning mathematics does not seem to get much attention. Moreover, learners do not seem aware of their right to observe their mathematics learning process critically in order to take a co-responsibility and co-operate in improving their own mathematics learning experience as per the Norwegian Act of Education. Thus, learners' potential to cooperate with their teachers to influence their mathematics education is intimidated, hence, being hardly visible and utilized.

The focus on critical thinking in mathematics concerning just the *cognitive* and *social* aspects do not provide a holistic picture of interaction between mathematics education, the learner, and the society. Therefore, it is significant to make mathematics learners aware from young age of being critical to their *personal* beliefs *about* mathematics, their mathematics learning process, and the role mathematics education plays in the society to make them reflect and consciously take decisions regarding mathematics in their personal and social lives. Therefore, we encourage that learners should learn to argue for and justify the legitimacy of their personal choices about their own mathematics learning process in mathematics classrooms while learning mathematics. Such critical thinking can stop learners from *uncritically* feeling obliged to follow a pre-decided mathematics curriculum, and use their potential to influence their mathematics learning process for the best of themselves and the society. We recommend that young mathematics learners are encouraged to and get training in thinking critically *about* their mathematics learning process along with gaining critical thinking skills for problem-solving in mathematics. Such training can equip them with a meta-perspective of their mathematics education in order to gain a holistic view of it and make reflected choices concerning mathematics in their personal lives. More research on this area is welcome and needed.

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[10] That is, making a reflective judgement about what to believe and what to do in situations one faces in their real-life.

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