

Organizing the mathematical proof process with the help of basic components in teaching proof: Abstract algebra example

Aysun Yeşilyurt Çetin¹ and Ramazan Dikici²

¹ Atatürk University, Erzurum, Turkey

² Mersin University, Turkey

The aim of this study is to identify the basic components of the mathematical proof process in abstract algebra and to organize the proof process into phases with the help of these basic components. A basic component form was prepared by arranging a draft basic component form, which was created as a result of a document analysis in accordance with the opinions of three academicians, who were experts in algebra. The data obtained as a result of both document analysis and expert examination were analyzed by the descriptive analysis method and are explained here in a detailed manner. It is believed that this basic component form will facilitate the step-by-step addressing of such a complex process as proofs in a non-compulsory and non-hierarchical order.

Keywords: teaching abstract algebra, basic components, mathematical proof process, teaching proof

This article is based on the doctoral dissertation of the author Aysun YEŞİLYURT ÇETİN: Yeşilyurt Çetin, A. (2017). *The processes of pre-service mathematics teachers' ability to write pre-determined key ideas in mathematical proof* (Publication No. 480352) [Doctoral dissertation, Atatürk University]. National Thesis Center of the Council of Higher Education.

ARTICLE DETAILS

LUMAT General Issue
Vol 9 No 1 (2021), 235–255

Received 26 January 2020
Accepted 12 April 2021
Published 22 April 2021

Pages: 21
References: 38

Correspondence:
aysun.yesilyurt@atauni.edu.tr

[https://doi.org/10.31129/
LUMAT.9.1.1497](https://doi.org/10.31129/LUMAT.9.1.1497)

1 Introduction

The aims of mathematics are to understand the rules on which numbers, algebra, and geometry are based; to find new and non-routine ways in these systems; and to explain new situations that are encountered. The world of mathematics is a world of ideas, insights, and discoveries, which will not be realized by people who are only interested in how to separate a function, and it is necessary to use mathematical proof and abstraction to enter this world (Goldberg, 2002). A mathematical proof, which is far more than just one or several examples supporting a mathematical statement (Derek, 2011), is a logical explanation of why a mathematical statement is true (Altıparmak & Öziş, 2005). The best proof is the one that helps understand the proved theorem by showing not only that it is true but also why it is true (Hanna, 2000).



1.1 The purpose of the study

Proofs constitute the basis of mathematics (CadwalladerOlsker, 2011; Sari, 2011). Therefore, many studies have been conducted on mathematical proofs. However, the mathematical proof process was not presented step by step in any study as in this study.

Abstract Algebra is one of the most important course of mathematics and mathematics education at the university level. According to Agustyaningrum, Husna, Hanggara, Abadi and Mahmudi (2020), abstract algebra is full of definitions and theorems which all require proof. Therefore, the students need to understand every definition and theorem they learn and be able to organize the concepts needed to proving theorems.

For that to be possible, students should have the ability to organize the required information for proving theorems in abstract algebra. This requires them to have an idea about the structure and stages of the proofs.

Due to these reasons, the mathematical proof process in abstract algebra should be examined and presented step by step. The aim of this study is to classify the mathematical proof process in abstract algebra with the help of basic components. Furthermore, it was also aimed to determine the knowledge that would help students facing problems and would also increase their understanding of mathematical proof. The following two research questions are posed for this purpose:

1. What are the difficulties students face during the mathematical proof process?
2. What are the stages of the mathematical proof process in abstract algebra?

The answer to the first research question was sought by reviewing the literature addressing the difficulties experienced in the mathematical proof process and analyzing lecturers and students' notes in order to see how the mathematical proof process in abstract algebra was experienced by students and teacher candidates and what kinds of difficulties they faced in this process. Similarly, the second research question was replied based on literature and the opinions of experts. Thus, the proof process was staged as basic components and then exemplified.

Unlike previous studies (Boero, 1999; Leron, 1983) in which the structure and stages of proofs were revealed, this study is inspired by the studies that deal with the difficulties students experience in the process of proof and the existing literature. In studies investigating the difficulties of proof, the content of the proof is also addressed, albeit not directly, and this gives us an idea of the nature of the proof.

2 Literature Review

The proof process is a complicated and hierarchical process in which more than one thinking process and stage are included, and mathematical information is used in an intertwined manner. Students come to a dead end when they do not know what to do when constructing proofs (Selden & Selden, 2008; Weber, 2001). In the proof process, students mentally go through steps of identifying the problem, making an assumption, testing the truth of the assumption by outlining the proof, conclusions, re-checking the proof that has been obtained, and making justification (Faizah, Nusantara, Sudirman, & Rahardi, 2020). Researchers state that pre-service mathematics teachers who encounter a mathematical proof for the first time face problems such as not knowing how or where to start, not understanding the logic of the proof, not being able to decide on what kind of method and conceptual information to use, and not being able to conclude the proof (Güler, 2013; Güler & Dikici, 2014; Moore, 1990; Moore, 1994; Polat & Akgün, 2016; Selden, Selden, & Benkhalti, 2018; Weber, 2001; Yeşilyurt Çetin & Dikici, 2020).

To understand the abstract and conceptual structure of mathematics, it is very important for students to understand the concept of proofs, what a proof is, why proofs are constructed, and the proof process (Sarı, 2011). If students do not know how to construct a proof, they try informal approaches, such as using examples or looking at a graph (Raman, 2002). At the undergraduate level, when mathematics students are expected to construct a proof, sufficient time is not allocated to helping students learn how to do so, and therefore the difficulties that students face cause many of them to quit studying mathematics (Selden & Selden, 2008). Pre-service teachers have difficulty in constructing long, acceptance-based proofs that they encounter for the first time (Polat & Akgün, 2016). Furthermore, most pre-service teachers do not know how to start a proof (Güler & Dikici, 2014; Polat & Akgün, 2016).

According to Weber (2001), the primary cause of failure in constructing proofs is the lack of strategic knowledge and it must therefore be ensured that students gain effective strategic knowledge. Pre-service teachers have difficulty in determining a proper proof method and strategy when constructing proofs (Doruk & Kaplan, 2015; Güler, 2013). According to Karakuş and Dikici (2017), students of secondary school mathematics teaching have difficulty in using proof methods effectively, although they think that proof methods play a significant role in the proof process. Demiray (2013) determined that pre-service teachers are highly successful in refutation and proof by contradiction, but they give incorrect answers to contrapositive proof

questions since they have difficulty in realizing and understanding the equivalence of contrapositive expressions, and pre-service teachers further fail to see the difference between an empirical argument and a valid proof. According to Ceylan (2012), the use of examples in the proof process by pre-service teachers may mean that they do not have sufficient logical inferences. Furthermore, according to Güler, Özdemir, and Dikici (2012), pre-service teachers failed to understand the relationship between mathematical induction steps completely and perceived this proof method as a procedure to be followed. The results of that study showed that pre-service teachers understand mathematical induction but fail to generate the induction step by using the induction hypothesis. In another study, Jones (2000) pointed out that pre-service mathematics teachers do not have a sufficient level of skills to construct proofs; that they do not have the mathematical knowledge necessary for effective mathematics teaching, including those in advanced grades; and that they graduate this way. It was also shown that pre-service teachers in advanced grades can construct proofs more smoothly in technical terms, but this does not provide them any benefit in terms of their associated mathematical knowledge in conceptual terms.

Karaoğlu (2010) stated that sufficient conceptual knowledge is needed to complete a proof and to understand how to use such knowledge in the proof of a theorem. Pre-service teachers have difficulty with proofs that can be done by using definitions even at the most basic level (Şahin, 2016). A pre-service mathematics teacher's experience with proofs, sufficient conceptual understanding, and skill in following different methods are all important for directing towards conceptual images when encountering a mathematical problem and having the skills to begin a proof (Bayazit, 2009). When pre-service teachers do not have conceptual knowledge about the theorem they want to prove, they cannot start to construct the proof, and so they begin with the help of the concepts obtained by investigating all the concepts related to the theorem in their mind and try to find the one that works. However, if their available knowledge is insufficient, they cannot conclude the proof (Karaoğlu, 2010). Another difficulty encountered in the proof process is incomplete or incorrect preliminary knowledge (Polat & Akgün, 2016; Yeşilyurt Çetin, & Dikici, 2020). Even if the pre-service teachers know which property and definition to use in the proof, they have difficulty using them while constructing the proof (Güler, 2013; Yeşilyurt Çetin, & Dikici, 2020).

By using the relevant literature, Moore (1990) addressed the difficulties experienced by undergraduate mathematics students in understanding proofs and the construction of proofs within seven categories. Accordingly, these students:

- do not know or cannot express definitions;
- have an inadequate intuitive understanding of the concepts;
- are not competent in proving conceptual images;
- are unsuccessful or unwilling to create and use their own examples;
- do not know how to use definitions to create the whole structure of a proof;
- are unsuccessful in understanding and using mathematical language and notation; and
- do not know how to start the proof process.

According to Moore (1990), students also experience an inability to coordinate and use all the information simultaneously in constructing mathematical proofs. Güler (2014) addressed the difficulties faced by pre-service teachers in the proof process within five categories as follows: determining how to start the proof, using the mathematical language and notation, using definitions, forming the setup of the proof (fully determining the steps to be followed), and selecting the elements from the set.

Polat and Akgün (2016), who observed the proof process among pre-service teachers, stated that pre-service teachers could not decide on which definition to use when constructing proofs, do not make any plans before starting the proof process, and cannot decide on how to begin the proof. They examined the reasons for the difficulties experienced by pre-service teachers in the proof process under two headings, those originating from the individual and those originating from the mathematical subject, and they charted them as follows:

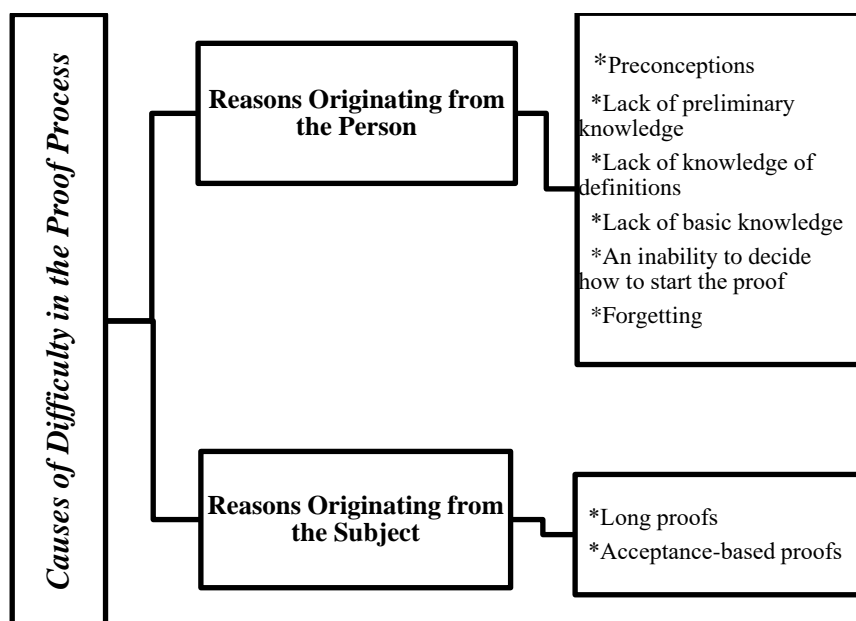


Figure 1. Reasons for experiencing difficulties in the proof process by pre-service teachers (Polat and Akgün, 2016).

According to Stewart and Thomas (2019), the reason why students face so many difficulties during the proof process is that each component in a proof is packed with different conceptual ideas. Therefore, it may be unrealistic to expect students to logically piece together the many definitions, other theorems, and various results that would help in completing a proof.

As can be seen from this literature review, the proof process is a complex process involving many difficulties that must be overcome step by step. In this study, each of these steps that make up a proof is called a basic component, and it is aimed to structure the mathematical proof process in abstract algebra into phases with the help of these basic components. This would make it possible to refine the proof process from relative complexity and address it within the framework of a certain order.

3 Method

A qualitative research method was used in this study, which aimed to phase the mathematical proof process in abstract algebra with the help of basic components by giving explanations about the process in accordance with expert opinions. Qualitative research methods are preferred when there is a theory gap in a subject, or when the existing theory is incapable of explaining the phenomenon. Moreover, the primary tool in data collection and analysis in qualitative research is the researcher himself or

herself (Merriam, 2013). In this study, in which the proof process is partitioned into phases, the researchers themselves were the main elements in the collection and analysis of data.

Among qualitative research methods, a case study design was utilized in this study. A case study is a qualitative research method that offers the opportunity to gain in-depth knowledge of the meaning of situations and events (Merriam, 1998). In this study, basic components were created in order to understand the proof process in depth, and as these basic components were identified, relevant documents and literature were used together with expert opinions.

3.1 Participants

In determining the basic components that make up the mathematical proof process in abstract algebra, three professors who conduct academic studies in the field of algebra were asked to make an examination and provide their opinions. These experts had at least 10 years of experience in teaching abstract algebra, number theory, and linear algebra.

3.2 Data Collection

In this study, document analysis was performed, and expert opinions were obtained in order to find the answer to the following questions: “What are the difficulties students face during the mathematical proof process?” and “What are the stages of the mathematical proof process in abstract algebra?”. Documents are important sources of information that must be used effectively in qualitative studies. A document analysis involves the examination of written materials about the cases or situations that are being investigated (Yıldırım & Şimşek, 2008).

Mathematical proofs are complex processes with many challenges that consist of several steps. Therefore, the proofs determined by the researchers were divided into steps so that the proof completion process could be addressed step by step with the complexity removed. In addition to literature review, the textbooks (Çallıalp, 2009; Karakaş, 2001; Taşçı, 2010) and course materials for abstract algebra, the notes of the lecturers teaching abstract algebra, the course notes of students were also examined and evaluated to identify the difficulties in the mathematical proof process. Accordingly, a draft basic component form involving those difficulties was constructed. The literature on the difficulties experienced in the mathematical proof process was briefly explained to three experts in algebra together with basic

information about the study, and they were asked to share their views on the draft basic component form and evaluate it. And then this draft form was reshaped in the light of their opinions. The final form of the 'basic component form', which was shaped by the document analysis and expert opinions, is presented in detail below in the "Findings" section.

3.3 Credibility and Transferability

The credibility and transferability issues for the findings were pursued in the following way: After 'the draft basic component form' was prepared, two mathematicians checked 'the draft basic component form' and after 'the basic component form for proofs' was prepared, five mathematicians checked 'the basic component form for proofs'.

One of the methods used to increase the credibility of a study is to obtain expert opinions. While determining the basic components, the opinions of three experts were obtained, and the process was directed in this direction. In this study, textbooks and course materials for abstract algebra, the notes of the lecturers, the course notes of students, notes taken during the study, and the relevant literature were examined and evaluated in the light of expert opinions. In this way, the credibility of the study was increased through a diversity of methods.

3.4 Analysis of the Data

In case studies, the researcher must rely on his or her own instincts and skills (Merriam, 2013). The data for the determination of the basic components that make up the mathematical proof process were analyzed by descriptive analysis in line with the researchers' instincts and skills and are presented with a holistic approach. Purpose in descriptive analysis to present the obtained findings in an organized and interpreted manner (Yıldırım & Şimşek, 2008). In the descriptive analysis of the data, the expert opinions on the draft of the basic component form, which was created as a result of document analysis, were taken as a basis. The opinions of the three expert academicians on the draft of the basic component form are quoted in detail.

The literature addressing the difficulties experienced in the mathematical proof process is explored step by step and analyzed as follows:

Table 1. Difficulties experienced in the mathematical proof process according to literature review

The Challenges Encountered in the Mathematical Proof Process	The Role of the Challenges Encountered in the Mathematical Proof Process in the Draft of the Basic Component Form
Understanding the logic of the proof	Determine the hypothesis and determine the judgement
Knowing how and where to start the proof	Use of hypothesis
<ul style="list-style-type: none"> • Being able to decide what kind of method and conceptual information to use • Knowing how to use preliminary knowledge, definitions, properties and concepts to create the whole structure of the proof • Understanding and using the mathematical language and notation • Forming the setup of the proof (fully determining the steps to be followed) 	Process steps
Being able to conclude the proof	Reaching proof

Figure 2, created based on Table 1, was submitted for expert examination. Figure 3 was created in accordance with the expert opinions.

3.5 Ethical Issues

In this study, the purpose and method used are presented to the reader with a detailed explanation. The experts whose opinions were used were informed about the research process and their personal information was kept confidential, but some academic information about the experts was included in order to ensure the credibility and transferability of the study. Furthermore, the researchers in this study did not act biasedly in the process of data analysis and have reported the findings that they obtained without making any changes to them.

4 Findings

In this section, it is aimed to determine the basic components of the mathematical proof process. To this end, the proof process was schematized as a draft basic component form after examining textbooks about theories and proofs in abstract algebra, the course notes of the lecturers and the course notes of students together with the literature revealing the challenges encountered in the mathematical proof process.

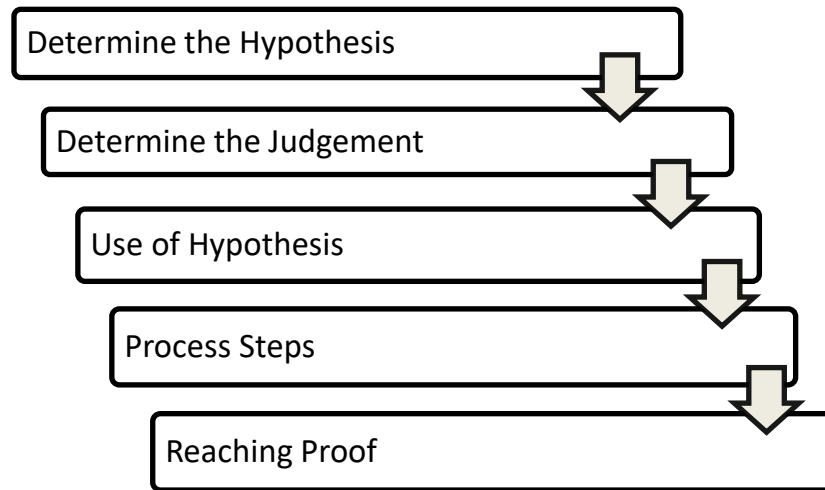


Figure 2. Draft basic component form.

The draft of the basic component form was submitted for the review of the three expert academicians. The literature on the difficulties experienced by students in the mathematical proof process was also provided to the experts, and their opinions on the draft form were obtained. One of the experts (P1) stated that the division of the process steps here should be expanded by being detailed in a non-hierarchical order, such as the use of definitions, use of properties, use of knowledge and theorems, and the performing operations. The second expert (P2) expressed similar opinions and stated that the step of using the hypothesis should be included in this non-hierarchical order. The third expert (P3) stated that the step of determining the method should be added after the step of determining the judgement.

The literature that reveals the difficulties experienced in the mathematical proof process and the expert opinions about the basic component form were analyzed as follows:

Table 2. Developmental process of the basic component form based on literature and expert opinions

The Challenges Encountered in the Mathematical Proof Process	Expert Opinions	The Role of the Challenges Encountered in the Mathematical Proof Process in the Draft of the Basic Component Form
Understanding the logic of proofs	No opinion	Determine the hypothesis and the judgement
Being able to decide what kind of method and conceptual information to use	<i>The step of determining the method should be added after the step of determining the judgement (P3),</i>	Determine the proof method
Knowing how and where to start the proof	<i>The step of using the hypothesis should be included in a non-hierarchical order (P2)</i>	Use of hypothesis
Knowing how to use definitions to create the whole structure of the proof	<i>The step of using definitions should be included in a non-hierarchical order (P1, P2)</i>	Use of definition
Knowing how to use properties to create the whole structure of the proof	<i>The step of using properties should be included in a non-hierarchical order (P1, P2)</i>	Use of property
Being able to decide what kind of conceptual information to use to create the whole structure of the proof	No opinion	Use of conceptual knowledge
Knowing how to use preliminary knowledge and basic information to create the whole structure of the proof	<i>The step of using knowledge and theorems should be included in a non-hierarchical order (P1, P2)</i>	Use of knowledge
Forming the setup of the proof (fully determining the steps to be followed)	<i>The step of performing operations should be included in a non-hierarchical order (P1, P2)</i>	Perform Operations
Being able to conclude the proof in accordance with mathematical language and notation	No opinion	Complete the proof

Based on the expert opinions and the challenges encountered in the mathematical proof process, the basic component form was reshaped as follows.

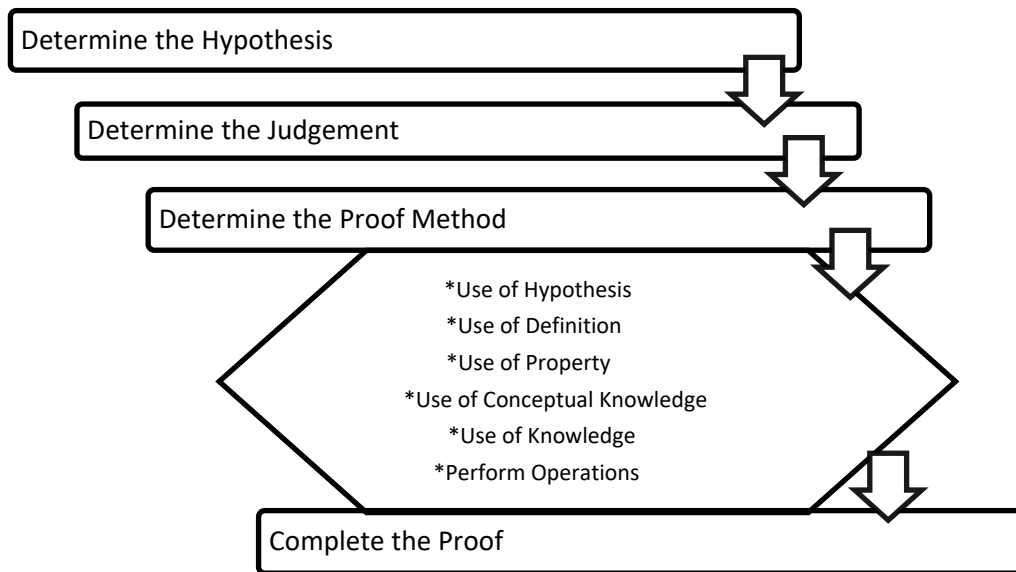


Figure 3. The basic component form for proofs.

These steps are non-hierarchical, and there is no obligation to apply every step in every proof. In other words, it is not necessary to determine the hypothesis in the proof of a theorem that only determines the judgement, such as “Prime numbers are infinite.” Similarly, while no definition is used in certain proofs, there is no need to use any conceptual knowledge or properties in some others.

It is expected that an assumption that will provide the basis for a proof, or in other words that will lay the foundation of the proof, will be established in the stage of determining the hypothesis. For example, for the proposition $\Rightarrow q$, the expression p is the hypothesis and it is important to be able to determine hypothesis p and write it in mathematical language and notation in order to start proving this proposition.

At the stage of determining the judgement, it is expected that the judgement to be achieved based on the hypothesis will be determined, creating the basis of the proof in this way. For example, for the proposition $p \Rightarrow q$, the expression q is the judgement, and it is important to write judgement q in mathematical language and notation in order to be able to shape the proof of this proposition.

In the basic component form, comprising the process steps of the draft basic component form as finalized by expert opinions, the components are the use of hypothesis, use of definitions, use of properties, use of conceptual knowledge, use of knowledge, and perform operations. These are used according to the content of the

proof and in a non-hierarchical order. Therefore, in some proofs, all these basic components are used, while only one or some of them are used in others.

The writing of expressions to help construct a proof based on the hypothesis is expected in the step of using the hypothesis, and it is expected that an auxiliary theorem will be used, which should be known at the time or else should be information gained during the flow of the proof in the step of using knowledge. The use of a property that helps to build the proof is essential in the step of using properties, while the use of an expression that is present in the hypothesis/ judgement or that is mentioned anywhere in the proof and expected to help in the construction of the proof is essential in the step of using definitions. It is expected that the appropriate concepts and the information related to these concepts will be selected and used correctly in the step of using conceptual knowledge. For example, using the definition of subgroups or prime numbers was addressed as the use of definitions, selecting and using the concepts of unit elements and inverse elements in accordance with their properties was addressed as the use of conceptual knowledge, and using group properties when performing an operation was addressed as the use of properties. The component of performing operations aims to reveal the state of taking the proof to a certain level with various algebraic operations. In order to perform an algebraic operation, it may sometimes be necessary to use a definition, sometimes a property, and sometimes conceptual knowledge. At this point, deficiency in one of these areas will make it impossible to successfully complete the proof process.

At the stage of completing the proof, students are expected to complete the proof according to mathematical language and notation by using all the data or information obtained.

It is believed that the basic proof components determined in this form will help students successfully undertake the proof completion process by following a specific order.

4.1 Examples of Proofs Divided into the Basic Components

Examples of proofs that are divided into the basic components of the mathematical proof process in abstract algebra are presented below. The proofs below have been taken from the textbooks (Çallıalp, 2009; Karakaş, 2001; Taşçı, 2010) and course materials for abstract algebra, the notes of the lecturers, the course notes of students. These proofs are divided into the basic components by researchers and confirmed by five mathematicians that checked 'the basic component form for proofs'.

An example of a proof in which the components of “use of property”, “use of definition”, “perform operations”, “use of conceptual knowledge”, and “complete the proof” are used is given in the following theorem.

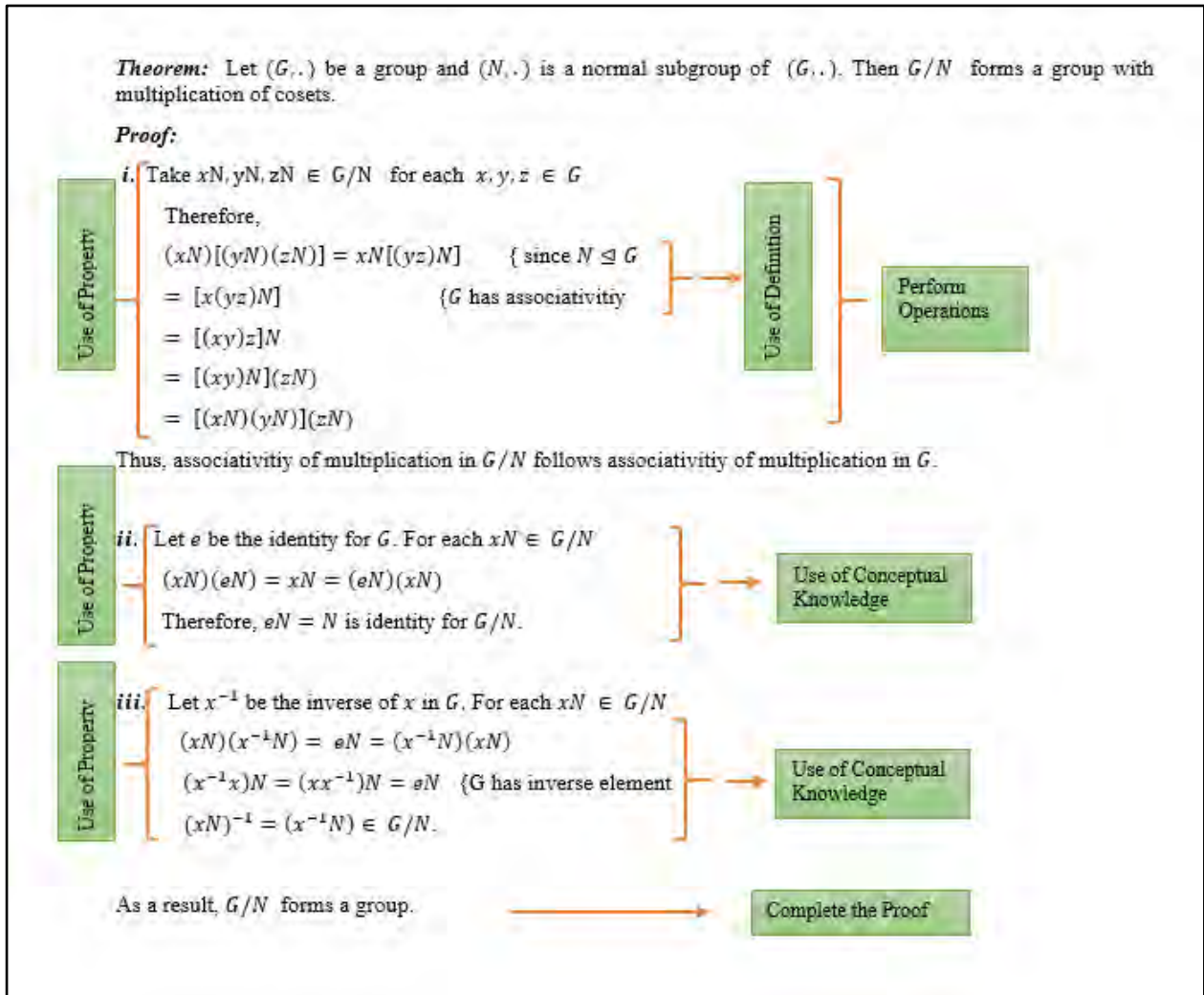


Figure 4. First example for a proof divided into basic components.

In the proof of the above theorem, in cases i, ii, and iii, the subgroup properties are called “use of property”. In step i, the process of performing operations is discussed using the definition of the normal subgroup. In step ii, knowledge about the concept of the unit element is used. In step iii, knowledge about the concept of the inverse element is used. Based on all these steps, the completion of the proof is written as the final sentence.

An example of a proof in which the components of “determine the proof method”, “determine the hypothesis”, “use of hypothesis”, “use of definition”, and “complete the proof” are used is given in the following theorem.

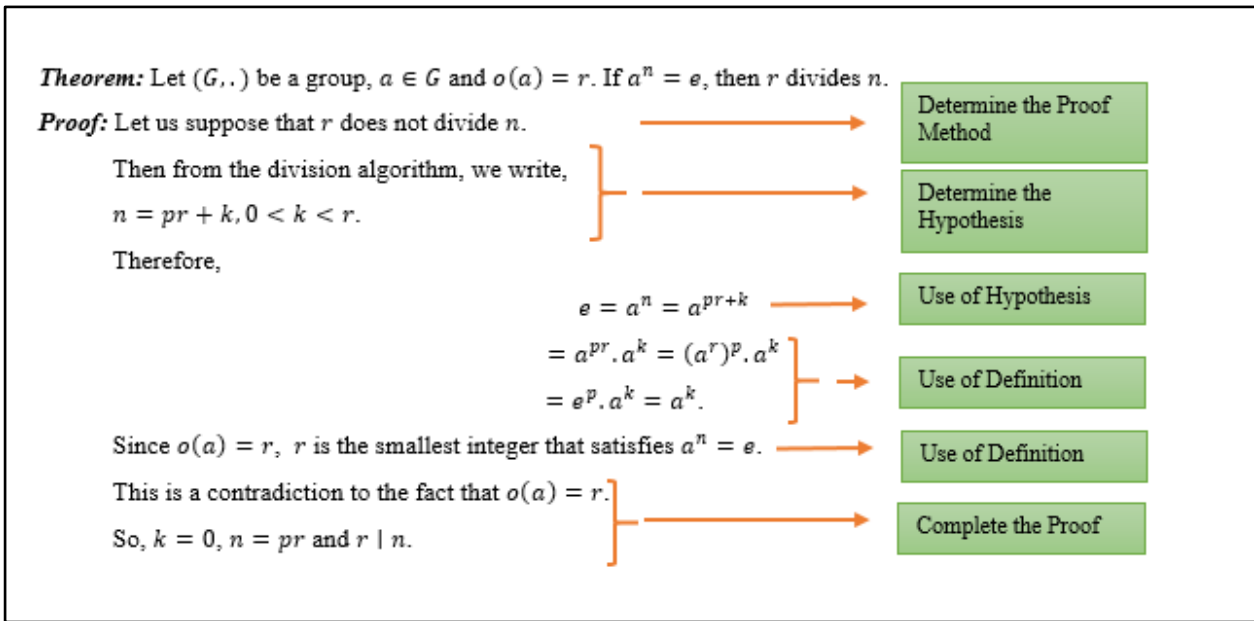


Figure 5. Second example for a proof divided into basic components.

The proof of the theorem presented above was started by determining the method, and then the hypothesis was determined, and an equation was established using this hypothesis. In the established equation, the proof was constructed using the definition of the unit element $o(a) = r \Rightarrow a^r = e$ and an inference was made by using the definition of order. All of this knowledge was interpreted together, and the proof was completed.

An example of a proof in which the components of “determine the hypothesis”, “determine the judgement”, “use of definition”, and “complete the proof” are used is given in the following theorem.

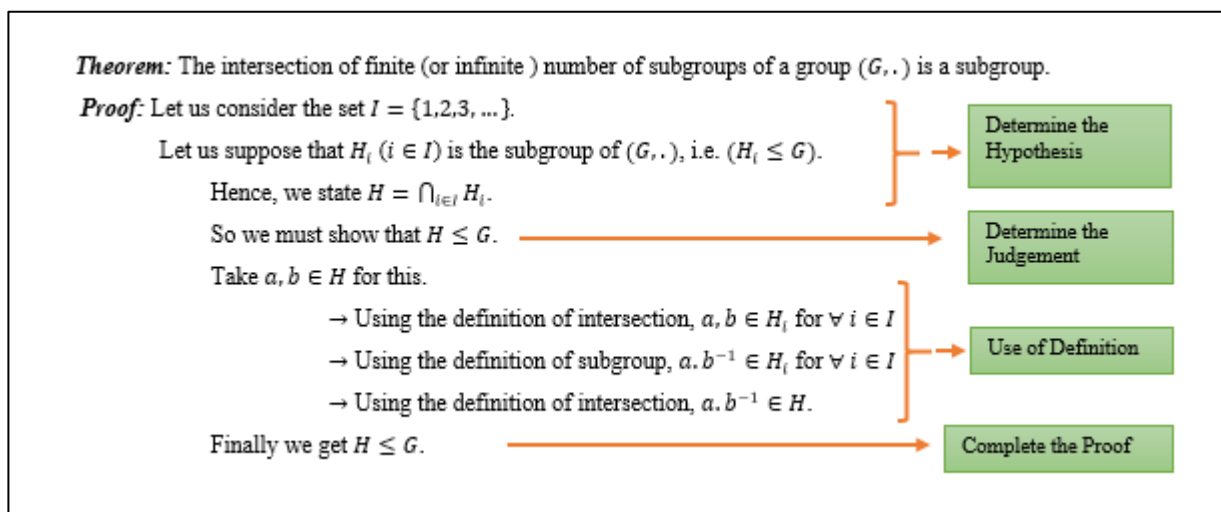


Figure 6. Third example for a proof divided into basic components.

In the proof of the theorem presented above, the hypothesis and judgement were determined based on the statement of the theorem and proof was started in that way. Necessary comments were made using the definitions of intersection and subgroup, and the proof was completed in light of these comments.

The example of a proof in which the components of “determine the hypothesis”, “use of hypothesis”, “use of definition”, “use of conceptual knowledge”, “use of knowledge”, and “complete the proof” are used is given in the following theorem.

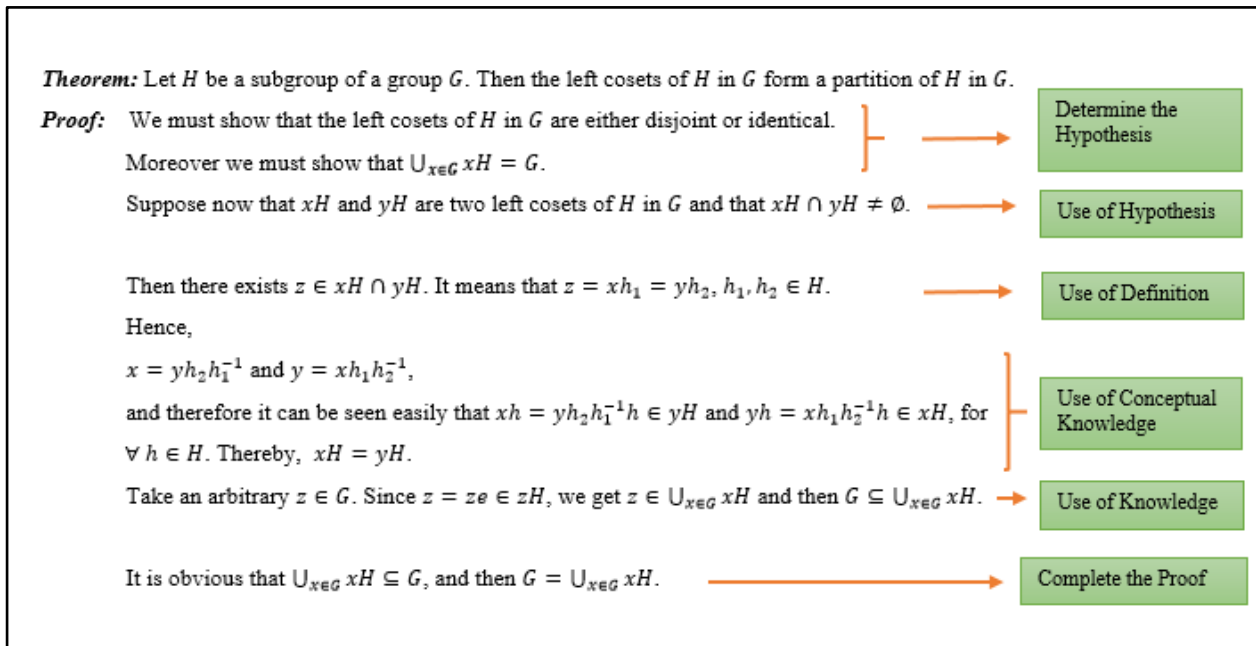


Figure 7. Forth example for a proof divided into basic components.

In the proof of the theorem presented above, the hypothesis was determined based on the statement of the theorem, and by using this hypothesis, left cosets xH and yH , the intersections of which are different from the empty set, were established. Then, using the definition of intersections, the proof was advanced by using conceptual knowledge about the concept of cosets. Finally, it was concluded that $G \subseteq \bigcup_{x \in G} xH$ by using the information achieved while determining the hypothesis and the flow of the proof, and the proof was completed using all this information.

5 Results and Discussion

The purpose of this study is to stage the mathematical proof process in abstract algebra according to its basic components. Therefore, a literature review was carried out regarding how the mathematical proof process was experienced by students and teacher candidates and what kinds of difficulties they faced in this process. The proof process was then staged as basic components based on the difficulties identified according to the literature and the opinions of experts.

Proofs were addressed step by step with a non-compulsory and non-hierarchical structure with the basic component form (Figure 3) that was created in line with the challenges that students encounter in the mathematical proof process. The final basic component form was established in accordance with the opinions of algebra specialists. It is not necessary for each of these components to be included in a proof. The basic components of a proof may vary according to the statement and the proof structure of the theorems. Similarly to this study, Boero (1999) and Leron (1983) dealt with proof generation with non-linear steps.

One of the difficulties encountered in the mathematical proof process is not knowing how and from where to begin the construction of the proof (Karaoğlu, 2010; Moore, 1990; Morah, Uğurel, Türnüklü, & Yeşildere, 2006; Polat & Akgün, 2016; Yeşilyurt Çetin & Dikici, 2020). In the basic component form presented in Figure 3, the process of starting the proof is addressed in two sub-steps: determining the hypothesis and determining the judgement. The step of determining the hypothesis includes the establishment of an assumption that constitutes the basis of proof, while the step of determining the judgement includes the determination of the judgement to be achieved based on the hypothesis. Similarly, according to Boero (1999), the first two stages of mathematical proof construction are generating an assumption and formulating the statement according to shared textual conventions.

Another difficulty encountered in the proof process is determining the proof method and strategy (Doruk & Kaplan, 2015; Güler, 2013; Karakuş & Dikici, 2017; Weber, 2001), and it is necessary to check the accuracy of the selected method while examining the proof process. In this study, the step of determining the proof method in the basic component form involves determining the appropriate proof method based on the use of the theorem.

The proof process is found to be influenced by the fact that students have adequate conceptual knowledge (Karaoğlu, 2010; Moore, 1990) and understand mathematical definitions and how to use them (Bayazit, 2009; Moore, 1990; Polat & Akgün, 2016;

Şahin, 2016). Students' lack of preliminary basic knowledge also makes this process more difficult (Polat & Akgün, 2016). Therefore, whether students' preliminary knowledge and conceptual knowledge are sufficient and whether they can use their knowledge and the definitions are also factors that shape the mathematical proof process. The component of "use of definition" in the basic component form presented in Figure 3 involves the use of definitions for the purpose of the proof. The component of "use of knowledge" involves the use of preliminary knowledge that must be possessed or the information achieved in the proof process, while the component of "use of conceptual knowledge" involves accurately selecting and using mathematical concepts and information related to these concepts. The component of "use of hypothesis" involves the use of this hypothesis in the flow of the proof. In addition, the component of "use of hypothesis" in this study is similar to the third phase of Boero's (1999) study, "exploration of the content of the conjecture".

It is thought that the components of "use of property" and "perform operations", which were added to the basic component form in accordance with the opinions of three academicians, experts in the field of algebra, shape the proof process. The component of "use of property" involves using a mathematical property that is expected to help construct the proof, while the component of "perform operations" involves carrying the proof to a certain level with various algebraic operations.

The component of "complete the proof" requires that all the information obtained in the proof process is addressed in a certain order, that the necessary inference is made, and that the proof is completed with expressions suitable for mathematical language and notation.

It is thought that structuring the proof of a theorem into phases with the help of basic components and addressing the proof process of students step by step with these components will provide convenience in both the teaching and the investigation of the proof process. It is hoped that the teaching of proofs within a specific non-hierarchical order using the basic component form presented in Figure 3 will facilitate understanding and allow teachers to identify which step in the proof process is difficult for a student. Therefore, it is suggested that including the basic component form and the proofs prepared for this form in textbooks would provide convenience for students in the proof process. In addition, similar studies can be conducted on whether the basic component form revealed in this study can be applied to other mathematics courses other than abstract algebra.

As Selden, Selden, & Benkhalti (2018) suggests, if certain stages of the proof are requested from the students, the success in proving can be increased. Proofs can be staged with the basic components set out in this study. In addition, students may be asked to complete some missing components instead of completing a proper proof. In this case, the teacher can decide about which component will be missing according to the use of this component in the proof. For example, in a proof about homomorphism, the 'use of property' in which the homomorphism property is used is left incomplete and the student can be expected to complete. Or, while teaching such kind of proof, it can be emphasized that the most important thing in making such a proof is the 'use of property'. Thus, students focus on a major part of the proof but not a whole proof, and they are exposed to staged proofs rather than long and intimidating proofs.

Acknowledgements

The authors would like to thank the experts who allocated their time to this study, shared their opinions and thoughts with the researchers, and contributed to the final shape of the basic component form.

The authors would like to express their appreciation to the anonymous reviewers and the editor Johannes Perna for making useful suggestions regarding the presentation of this paper.

References

- Agustyaningrum, N., Husna, A., Hanggara, Y., Abadi, A. M., & Mahmudi, A. (2020). Analysis of mathematical proof ability in abstract algebra course. *Universal Journal of Educational Research*, 8(3), 823–834. <https://doi.org/10.13189/ujer.2020.080313>
- Altıparmak, K., & Öziş, T. (2005). An investigation upon mathematical proof and development of mathematical reasoning, *Ege Journal of Education*, 6(1), 25–37. Retrieved from <https://dergipark.org.tr/tr/pub/egeefd/issue/4918/67296>
- Bayazit, N. (2009). *Prospective mathematics teachers' use of mathematical definitions in doing proof* (Doctoral dissertation). Florida State University, Florida.
- Boero, P. (1999). Argumentation and mathematical proof: a complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on The Teaching and Learning of Mathematical Proof*, 7,8. Retrieved from <http://www.lettredelapreuve.org/OldPreuve/Newsletter/990708Theme/990708ThemeUK.html>
- CadwalladerOlsker, T. (2011). What do we mean by mathematical proof?. *Journal of Humanistic Mathematics*, 1(1), 33–60. <https://doi.org/10.5642/jhummath.201101.04>
- Ceylan, T. (2012). *Investigating preservice elementary mathematics teachers' types of proofs in geogebra environment* (Master's thesis). Ankara University, Ankara.

- Çallıalp, F. (2009). *Örneklerle Soyut Cebir [Abstract Algebra with Examples]*. İstanbul: Birsen Publisher.
- Demiray, E. (2013). *An investigation of pre-service middle school mathematics teachers' achievement levels in mathematical proof and the reasons of their wrong interpretations* (Master's thesis). Middle East Technical University, Ankara.
- Derek, M. (2011). *Teaching and learning of proof in the college curriculum* (Master's thesis). San Jose State University, Washington.
- Doruk, M., & Kaplan, A. (2015). Prospective mathematics teachers' difficulties in doing proofs and causes of their struggle with proofs. *Journal of Bayburt Education Faculty*, 10 (2), 315–328. Retrieved from <https://dergipark.org.tr/tr/pub/befdergi/issue/17275/180470>
- Faizah, S., Nusantara, T., Sudirman, S., & Rahardi, R. (2020). Exploring students' thinking process in mathematical proof of abstract algebra based on mason's framework. *Journal for the Education of Gifted Young Scientists*, 8(2), 871–884. <http://dx.doi.org/10.17478/jegys.689809>
- Goldberg, A. (2002). What are mathematical proofs and why are they important?. Retrieved from <http://www.math.uconn.edu/~hurley/math315/proofgoldberger.pdf>
- Güler, G. (2013). *Investigation of pre-service mathematics teachers' proof processes in the learning domain of algebra* (Doctoral dissertation). Ataturk University, Erzurum.
- Güler, G. (2014). Analysis of the proof processes of pre-service teachers regarding function concept. *International Journal of Education and Research*, 2(11), 161–176. Retrieved from <https://www.ijern.com/journal/2014/November-2014/14.pdf>
- Güler, G., & Dikici, R. (2014). Examining prospective mathematics teachers' proof processes for algebraic concepts, *International Journal of Mathematical Education in Science and Technology*, 45(4), 475–497. <https://doi.org/10.1080/0020739X.2013.837528>
- Güler, G., Özdemir, E., & Dikici, R. (2012). Pre-service teachers' proving skills using mathematical induction and their views on mathematical proving. *Kastamonu Education Journal*, 20(1), 219–236. Retrieved from <https://dergipark.org.tr/en/pub/kefdergi/issue/48696/619520>
- Hanna, G. (2000). Proof, explanation, and exploration: An overview. *Educational Studies in Mathematics*, 44, 5–23. <https://doi.org/10.1023/A:1012737223465>
- Jones, K. (2000). The student experience of mathematical proof at university level, *International Journal of Mathematical Education in Science and Technology*, 31(1), 53–60. <https://doi.org/10.1080/002073900287381>
- Karakaş, H. İ. (2001). *Matematiğin Temelleri [Fundamentals of Mathematics]*. Ankara: Metu Press.
- Karakuş, D., & Dikici, R. (2017). The opinions of the students of secondary education mathematics teaching on mathematical proof methods. *The Journal of International Education Science*, 13, 194–206.
- Karaoğlu, Ö. (2010). *The performance of pre-service mathematics teachers in proving theorems supported by key points and arguments* (Master's thesis). Gazi University, Ankara.
- Leron, U. (1983). Structuring mathematical proofs. *The American Mathematical Monthly*, 90(3), 174–185. <https://doi.org/10.1080/00029890.1983.11971184>
- Merriam, S. B. (1998). *Qualitative research and case study applications in education. revised and expanded from: Case study research in education*. San Francisco: Jossey-Bass Publishers.
- Merriam, S. B. (2013). *Qualitative Research: A guide to design and implementation* (Trans. Ed. S. Turan) Ankara: Nobel Academic Publishing.
- Moore, R. C. (1990). *College students' difficulties in learning to do mathematical proofs* (Doctoral dissertation). University of Georgia, Athens.
- Moore, R. C. (1994). Making the transition to formal proof, *Educational Studies in Mathematics*, 27, 249–266. <https://doi.org/10.1007/BF01273731>
- Moralı, S., Uğurel, I., Türnüklü, E., & Yeşildere S. (2006). The views of the mathematics teachers on proving. *Kastamonu Education Journal*, 14(1), 147–160. Retrieved from <https://dergipark.org.tr/en/pub/kefdergi/issue/49106/626665>

- Polat, K., & Akgün, L. (2016). Pre-service mathematics teachers' opinions about proof and difficulties with proving. *The Journal of Academic Social Science Studies*, 43, 423–438.
- Raman, M. J. (2002). *Proof and justification in collegiate calculus* (Doctoral dissertation). University of California, Berkeley.
- Sarı, M. (2011). *Undergraduate students' difficulties with mathematical proof and teaching of proof* (Doctoral dissertation). Hacettepe University, Ankara.
- Selden, A., & Selden, J. (2008). Overcoming students' difficulties in learning to understand and construct proofs. In M. Carlson & C. Rasmussen. (Eds.), *Making the connection: Research and teaching in undergraduate mathematics*, (pp. 95–110), Mathematical Association of America.
- Selden, A., Selden, J., & Benkhalti, A. (2018). Proof frameworks: A way to get started. *PRIMUS*, 28(1), 31–45. <https://doi.org/10.1080/10511970.2017.1355858>
- Stewart, S., & Thomas, M. O. (2019). Student perspectives on proof in linear algebra. *ZDM*, 51(7), 1069–1082. <https://doi.org/10.1007/s11858-019-01087-z>
- Şahin, B. (2016). Examination of process of proving on divisibility of mathematics teacher candidates. *Journal of Bayburt Education Faculty*, 11(2), 365–378. Retrieved from <https://dergipark.org.tr/en/pub/befdergi/issue/28762/307847>
- Taşçı, D. (2010). *Soyut Cebir [Abstract Algebra]*. Ankara: Öziş Typography.
- Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge, *Educational Studies in Mathematics*, 48, 101–119. <https://doi.org/10.1023/A:1015535614355>
- Yeşilyurt Çetin, A., & Dikici, R. (2020). Examination of pre-service mathematics teachers' ability to make algebraic proof. *Online Journal of Mathematics, Science and Technology Education (OJOMSTE)*, 1(1), 75–85.
- Yıldırım, A., & Şimşek, H. (2008). *Sosyal bilimlerde nitel araştırma yöntemleri [Qualitative research methods in social sciences]*. Ankara: Seçkin Publishing.