

“Because You’re Exploring this Huge Abstract Jungle...”: One Student’s Evolving Conceptions of Axiomatic Structure in Mathematics

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ABSTRACT

For several decades, literature on the history and pedagogy of mathematics has described how history of mathematics is beneficial for the teaching and learning of mathematics. We investigated the influence of a history and philosophy of mathematics (HPhM) course on students’ progress through the lens of various competencies in mathematics (e.g., mathematical thinking and communicating) as a result of studying mathematical ideas from the perspective of their historical and philosophical development. We present outcomes for one student, whom we call Michael, resulting from his learning experiences in an HPhM course at university. We use the framework from the *Competencies and Mathematical Learning* project (the Danish KOM project) to analyze the evolution of Michael’s competencies related to axiomatic structure in mathematics. We outline three aspects of axiomatic structure to situate our analysis: Truth, Logic, and Structure. Although our analysis revealed that Michael’s views and knowledge of axiomatic structure demonstrate need for his further development, we assert what he experienced during the HPhM course regarding his mathematical thinking and communication about axiomatic structure is promising support for his future mathematical studies. Finally, we argue that a HPhM course has potential to support students’ progress in advanced mathematics at university.

Keywords: history of mathematics, undergraduate mathematics education, axiomatic structure, mathematical thinking, mathematical communication

INTRODUCTION

For several decades, literature focused on the history and pedagogy of mathematics has described the ways in which the history of mathematics is considered beneficial for the teaching and learning of mathematics. In an early contribution to the re-popularization of using history in teaching and learning mathematics,¹ Fauvel (1991) outlined 15 reasons for using history in mathematics education, including that it: helps to increase motivation for learning, gives mathematics a human face, changes pupils’ perceptions of mathematics, provides opportunities for investigations, and provides opportunity for cross-curricular work with other teachers or subjects (p. 4). From a critical perspective, it can be argued that much of the literature on the benefits of using history for the teaching and learning of mathematics focuses heavily on affective-motivational contributions and often lacks empirical support. For instance, although we strongly agree that learning of mathematics as a human endeavor is a significant contribution to students’ mathematical studies, we argue

¹ In the United States, one early effort to highlight the importance of using history in teaching mathematics was led by Phillip S. Jones and given in the 31st Yearbook of the National Council of Teachers of Mathematics (1969; revised in 1989).

Article History: Received 5 June 2020 ♦ Revised 17 July 2020 ♦ Accepted 23 July 2020

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that the ways that it contributes is a topic that requires empirical investigation. To exemplify efforts in this direction, we highlight the work of Barnett, Lodder, and Pengelley (2014) in the United States, in which they created materials based on primary historical sources for use in undergraduate mathematics classrooms by identifying a set of design goals that go beyond the “motivate, see, witness” (p. 10) orientation. To this end, Barnett et al. outlined reasons for using primary sources in undergraduate mathematics teaching that draw explicit attention to student learning, including:

- Provide practice moving from verbal descriptions of problems to precise mathematical formulations;
- Promote understanding of the present-day paradigm of the subject through the reading of an historical source which requires no knowledge of that paradigm;
- Promote reflection on present-day standards and paradigm of subject;
- Draw attention to subtleties, which modern texts may take for granted, through the reading of an historical source;
- Engender cognitive dissonance (*dépaysement*) when comparing a historical source with a modern textbook approach, which to resolve requires an understanding of both the underlying concepts and use of present-day notation (p. 10).

In this regard, we note that recent scholarship has also begun to focus on empirical research motivated by the need to provide the field with examples of what history of mathematics contributes to learning mathematics. For example, several scholars have discussed results that support the use of primary historical sources to achieve the student learning goals that were identified by Barnett and her colleagues (see, for example, Bernardes and Roque (2018), Clark (2012), Kjeldsen and Blomhøj (2012), Kjeldsen and Petersen (2014)).

In an effort to situate the history of mathematics more firmly in the work of mathematics education, Jankvist examined and organized the numerous reasons for and ways in which history of mathematics is used in mathematics education. Jankvist (2009, 2010, 2011) described the ideas of *history as a goal* and *history as a tool*, and as a way of discussing the differences between these two orientations, he introduced the notions of *in-issues* and *meta-issues* of mathematics. The term *in-issues* refers to the inner issues of mathematics, e.g., concepts, theories, methods, algorithms, etc. On the other hand, *meta-issues* refer to the various meta-perspective issues surrounding mathematics as a scientific discipline, including those related to its history, its sociology, its philosophy, and its epistemology. Thus, where history as a tool concerns the teaching and learning of in-issues of mathematics, history as a goal concerns the teaching of certain meta-issues of mathematics.

The two orientations to the use of history in mathematics teaching were highlighted by Jankvist and Kjeldsen (2011). Their theoretical and empirical analyses provided evidence and promise for two avenues that merit a closer look in the field of mathematics education:

- (1) Students’ development of mathematical competencies when using history as a tool in a setting where history of mathematics is not an integral part of the mathematics curriculum, and
- (2) Ways to anchor students’ meta-issue discussions and reflections in the related mathematical in-issues when using history as a goal in settings where historical insights are considered an integral part of the mathematics program. (p. 858)

Most recently, Furinghetti (2020) discussed several examples for which history of mathematics has been introduced at different school levels. In one collection of examples (focused primarily on topics found in upper secondary school or university), she emphasized “one important function of history in mathematics education is to foster understanding through an approach to the roots around which concepts developed” (p. 980). Although such scholarship on the use of history in teaching mathematics is useful in situating history of mathematics in mathematics education research, we claim that what is missing from the research literature regarding the alleged influence of the history of mathematics is inquiry on the actual impact on learning. In this regard, we investigated students’ progress within various competencies in mathematics (e.g., mathematical thinking and communicating) as a result of studying mathematical ideas from the perspective of their historical and philosophical development. Accordingly, in this article we focus on the outcomes resulting from one student’s learning experiences in a history and philosophy of mathematics course, a course which many mathematics majors around the globe take as either a required part of their curriculum or as an

elective. The research question that guided our inquiry is: *How do students' competencies² in mathematical thinking and communicating with regard to axiomatic structure change during a course on the history and philosophy of mathematics?*

In this article, we begin with a discussion on axiomatic structure in mathematics and its role in the learning of advanced mathematics at university (or, what is referred to as undergraduate mathematics in the United States). This discussion constitutes the theoretical background for our work. Here, we identify three aspects of axiomatic structure that guide our analysis of the case of Michael.³ Next, we describe the framework, *Competencies and Mathematical Learning* (also known as the Danish KOM Report), that we used to analyze the evolution of Michael's competency in mathematical thinking and communicating. We next provide the details of the research method, including descriptions of the context and participants. Following this, we present our analysis of the case of Michael regarding his progress in the two competencies of interest. Finally, we end with a discussion and concluding remarks on our claim for the role of a history and philosophy of mathematics course on students' experience with mathematics at university.

THEORETICAL PERSPECTIVES

Considerations of Axiomatic Method

A quick survey of the history of mathematics suffices to conclude that there have been different perspectives on mathematics. Although some consider mathematics as a universal language or argue that it is independent of time and context, mathematicians' approaches to similar problems in, for instance, different cultures throughout history, have been quite diverse. *What mathematics is* and *what a mathematician works on* are two questions that have led to differing philosophies of mathematics.

To exemplify the two extremes, the existence of mathematical objects is as much a fact of the existence of the objects with which a botanist works for a Platonist, but with the former residing outside space and time. What a mathematician does from this perspective is *discovery*. On the other hand, a Formalist will argue that there are *no* objects that a mathematician works on, only *axioms, definitions, and theorems*. Thus, a Formalist will create (mathematical) statements by working deductively from axioms (Hersh, 1998).

Rather than positioning ourselves on any one side of the various perspectives on mathematics, we instead take the position of appreciating the importance and value of, for instance, inductive or empirical approaches to mathematics, as well as deductive-formal approaches. In this regard we highlight, as Putnam (1998) stated, the role of quasi-empirical methods and mathematical confirmation in establishing mathematical truth. Nevertheless, today, formalism, deductive approaches, and axiomatic method occupy a significant place in the study of mathematics, especially in advanced mathematics. However, despite the importance of axiomatic structure for advanced study of mathematics at university, it has not been an explicit part of the undergraduate mathematics curriculum.

Given the formal-abstract nature of undergraduate mathematics, insufficient training that addresses axiomatic structure can problematize students' engagement with advanced mathematics, especially when considered along with their prior mathematical experience at the K–12 level. As Witzke, Clark, Struve, and Stoffels (2018) put it, “change from an empirical-object oriented belief system to a formal-belief system constitutes a crucial obstacle in the transition from school to university” (p. 73).

At this point, it is important to note that beginning university students are not completely unfamiliar with some sort of axiomatic structure as a result of their earlier mathematical studies. However, as we will highlight later, those earlier experiences with axiomatic method are far from providing the mathematical background for the advanced mathematics at university. First of all, as Hintikka (2011) observed, axiomatization has been influential in mathematics and its pedagogy as modern mathematics has had structuralist orientations. Second, students gain familiarity with axiomatic structure through geometry courses taken in middle and high school.⁴ This argument is based on the fact that the teaching of geometry relies heavily on Euclidean geometry, and the elementary axioms (i.e., postulates) from Euclid are generally presented at the beginning of such a course.

² Here we use the structure of mathematical competencies as described by Niss and Højgaard (2011). See the “Framework for Analysis” section of this article for details.

³ We use pseudonyms for the names of the participants and the participating institution.

⁴ Since the study described in this article took place in the United States, geometry in school is typically taken in grade 8 (for advanced students) or grade 10.

However, it is difficult to argue that such exposure to Euclidean geometry and the corresponding postulates can provide the foundation for the formal study mathematics at university since, as Mueller (1969) argued, mathematical structure as conceptualized in modern mathematics was not present in ancient Greece. He further claimed that it is misleading to interpret Euclid's use of the verb 'postulate' to mean 'assume as true.' In Mueller's words,

...grammatically, at least, [three of the five Euclidean postulates] are not existence assertions like their modern counterparts.⁵ Nor are they descriptions of possibilities which might in fact be realizable, thereby rendering the descriptions false. They are what might be called licenses to perform certain geometric operations. (p. 290)

Thus, it is almost impossible to call the argumentation in Euclid's *Elements* formal. Therefore, some exposure to Euclid's axiomatization will not guarantee a conception of axiomatic structure that is supportive to learning university mathematics.

And what of Hilbert's conception of the axiomatic method for using it as a foundation for students' study of formal-abstract mathematics at university? Brown (1999) noted that the primary goal in developing theories based on Hilbert's formalism is not to describe reality or physical space, but, from a natural science perspective, to create useful theories to predict (mathematical) observations. Thus, while Euclid's mathematical theory began with existing mathematical objects, the formal axiomatic method of Hilbert is "flexible and leaves one the freedom not only for building various axiomatic theories (including incompatible ones) but also for making a choice of the background logic" (Rodin, 2014, p. 93). However, Corry (2007) observed that Hilbert, in his approach to geometry, should be identified as an empiricist rather than a formalist. Corry further argued that "there is no evidence that Hilbert ever saw axiomatics as a possible starting point to be used for didactical purposes" (p. 3).

The preceding discussion of the axiomatic methods of Euclid and Hilbert reveals important differences between the two. We interpret the discrepancies between these two conceptualizations of axiomatization, along with others, as representations of different philosophies of mathematics attempting to outline a suitable foundation of mathematics. This is at the core of why we are interested in how Michael "considers" axiomatic structure in mathematics, and not primarily in his "learning" or "understanding"⁶ of it. In the remainder of this section, we describe the perspective on axiomatic method that we believe is closest to that used in advanced mathematics at university, together with the aspects of axiomatic structure that we will highlight in the analysis of our data.

Axiomatic method

Davis and Hersh (1981) stated that an *axiom* is "a statement which is accepted as a basis for further logical argument. Historically, an axiom was thought to embody a 'self-evident' truth or principle" (p. 412). Regarding the "truth" of a theorem, a mathematician's task is then to ensure that a theorem is a result of "valid logical deductions" (p. 340) from the axioms. From the formalist standpoint, there is no search for meaning since the axioms do not need to be about anything, because mathematics "from arithmetic on up, is just a game of logical deduction" (p. 339). That said, in the very initial stage of developing the structure, an empirical study may be used as a foundation to decide the axioms.

In other words, the axiomatic method that we identify as relevant for modern mathematics is primarily about structure, but not meaning. As Hintikka (2011) observed, "the structuralist orientation of modern mathematics naturally leads to the use of axiomatization" (p. 70). He further argued that the view of axiomatization as "formulating a system of basic truths from which all the other truths of some body of knowledge can be derived purely logically" (p. 72) does not capture the whole idea of axiomatic method. According to Hintikka, an axiomatic system is also part of a metatheoretical study, so that deducing results from axioms is not the only goal of an axiomatic system. He observed, for instance, that a study of group theory includes "a metatheoretical study dealing with such questions as the taxonomy of the different kinds of groups, representation theorems, etc." (p. 72). Thus, one needs to have an idea of all structures satisfying the particular axiomatic system in her (mathematical) practice.

⁵ Mueller (1969) noted: "For any two points there exists exactly one straight line on which they both lie" as the modern counterpart of Euclid's postulate stating that: 'Let it be postulated to draw a straight line from any point to any point'" (p. 290).

⁶ Here we emphasize with quotes to indicate that the analysis of Michael's experience that follows is not focused on what we believe to be immeasurable in the context of the study presented here.

Table 1. Eight Mathematical Competencies, by Group (Niss & Højgaard, 2011)

To ask and answer in, with, and about mathematics	To deal with mathematical language and tools
Mathematical Thinking	Representing
Problem Tackling	Symbol and Formalism
Modelling	Communicating
Reasoning	Aids and Tools

Based on the preceding discussion, we outline three aspects of axiomatic structure to frame our analysis of Michael's views revealed in pre- and post-interviews: Truth, Logic, and Structure.⁷ This framing directs our attention to key notions of axiomatic structure that constitute the core of mathematical ideas in this regard.

Truth

The epistemological stance of a person in a specific situation dictates the meaning of truth regarding a certain phenomenon. Different disciplines argue for different methods—which are at times contradictory—of acquiring true, reliable information. In the same discipline or worldview, there may be multiple ways of establishing truth. Specifically, when considering the formal nature of mathematics, Truth refers to mathematical statements (i.e., theorems) which are proven deductively from the related axioms based on a valid logic. These theorems may or may not apply to real-life contexts (Davis & Hersh, 1981). That said, our interpretation of Truth is more of consistency rather than empirical verification.

Logic

Simply stated, Logic refers to the deductive logic used to derive theorems from a formal-abstract standpoint. In this regard, a mathematician is not primarily interested in whether the mathematical statements she derived from the axioms “make sense” or are “logical” in the traditional sense. A formal-abstract study of mathematics may rely on intuition (Mueller, 1969), or if concerned with the real world, should validate axioms empirically (Corry, 2007). However, the essential role of Logic is to ensure that the theorems of an axiomatic structure are validly deduced from axioms, which themselves need not even be “evident-truths” (Bourbaki, 1950).

Structure

For the purposes of this article, we do not deal with the philosophical questions of whether mathematics is the study of structures, or whether structure is inherent in mathematics itself. Instead, we refer to Structure as a set of axioms and Logic that accompanies these axioms. When considered along with Truth and Logic as we described above, a structure constitutes the objects that are worked on in order to create mathematical statements. For instance, a set and one binary operation satisfying the requirements (or axioms) of the definition of a group constitute an axiomatic structure. In this regard, Structure is the foundation of mathematical study in formal mathematics.

FRAMEWORK FOR ANALYSIS: COMPETENCIES, CHARACTERISTICS, AND DIMENSIONS OF MASTERY

The report on *Competencies and Mathematical Learning* (also known as the KOM Report; Niss & Højgaard, 2011) provided the foundation for the framework we used to determine progress with regard to two different competencies related to axiomatic structure in our case analysis. In their report of the Danish KOM project, Niss and Højgaard described a mathematical competency as “a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge” (p. 49). In their description, Niss and Højgaard were careful to note that competencies cannot be “so sharply defined that there is no overlap” (p. 49), nor can any single competency “be acquired or mastered in isolation from the other competencies” (p. 50).

Niss and Højgaard (2011) identified eight mathematical competencies and divided these into two groups: one referring to *the ability to ask and answer questions in and with mathematics* and the second to *the ability to deal with mathematical language and tools* (see **Table 1**).

For this article, we focused on the mathematical thinking and communicating competencies. We made this purposeful selection for two reasons. First, we wanted to highlight a competency from each of the two groups

⁷ In this article, we capitalize the terms Truth, Logic, and Structure when we refer to them in the context of axiomatic structure in mathematics.

Table 2. Characteristics of Mathematical Thinking and Communicating Competencies^a

Competency	Characteristics
Mathematical Thinking	<ul style="list-style-type: none"> • Awareness of the types of questions characterizing mathematics • Ability to pose such questions • An insight into the types of answers • Recognize, understand, and deal with the scope of given mathematical concepts • Extend scope of a concept through abstracting properties • Understand implications of generalizing results
Communicating	<ul style="list-style-type: none"> • Study and interpret others' written, oral, or visual mathematical expressions • Express oneself in different ways and with different levels of theoretical or technical precision about mathematical matters, to different types of audiences

^aAs noted in Niss and Højgaard (2011, pp. 52-53, 67-68)

Table 3. Dimensions of Competency Mastery (Niss & Højgaard, 2011, pp. 72-73)

	Degree of coverage: Aspects of a competency	Radius of action: Contexts and situations	Technical level: Substance in the situation
Key descriptors of dimension:	How many aspects of competency are activated The extent to which independent activation takes place	Spectrum of contexts and situations in which the competency can be activated Of primary interest is the relation to mathematical contexts and situations; contexts and situations determined by problem formulations and challenges also of interest	How conceptually and technically advanced the entities and tools are that can be activated in a given competency

identified by Niss and Højgaard, since, as they articulated, “the ability to cope with and in mathematics can be said to consist of exactly these two capacities or ‘super competencies’” (p. 51). Second, the focus on mathematical thinking and communicating best fit the nature of the interview questions designed to capture students’ conceptions about axiomatic structure within mathematics.

In our analysis, we used the characteristics identified by Niss and Højgaard (2011) for each of the two competencies of interest (Table 2).⁸

Niss and Højgaard (2011) also identified three dimensions along which to measure mastery of a competency: degree of coverage, radius of action, and technical level (p. 72; summarized in Table 3). Within each dimension, Niss and Højgaard used terms such as modest, high, and higher, to characterize levels of mastery, and we adopted the same characterization terms for our analysis.

In the case analysis that follows, we identify progress or change in the two competencies highlighted here—mathematical thinking and communicating—based on the characteristics of these two competencies and the three dimensions of competency mastery presented in the KOM Report.

METHOD

In this section we describe the research context, including the research site and participants, as well as the data sources and rationale for the construction of our case of interest.

Description of Research Context and Participants

The focus of this article is the case of Michael, who was selected from four students who consented to participate in the research. There were 19 students enrolled in a history and philosophy (HPHM) course at Private Christian University (PCU) during the Autumn 2012 semester. Many undergraduate institutions in the United States require mathematics education majors to complete a course in the history of mathematics as part of their preparation program or offer such a course as an elective. Therefore, the relevant student population for a course in the history of mathematics can vary. For example, the course may carry particular prerequisites (e.g., successful completion of first-semester calculus) or may be intended for a more general audience, where the emphasis is more on the history and less on the mathematics.

The particular context for this study was selected for two reasons. First, the second author was acquainted with faculty members (one in mathematics and one in mathematics education) at PCU, and these connections facilitated the second author’s multi-day visits to get to know the students in the course, collect data, and

⁸ In the text (e.g., in analysis), the characteristic descriptors will appear in italics.

Table 4. Student Participants (PCU, Autumn 2012)

Participant	Year in school	Major	Highest level mathematics courses taken in high school (if known); college mathematics courses (taken at PCU)
Jenny	Senior	Elementary Education	Precalculus (high school); Intro to Mathematical Thinking (PCU)
Tabitha	Senior	History (Secondary History Education)	Intro to Mathematical Thinking (PCU)
Darren	Sophomore	Music (was considering Mathematics at time of study)	Advanced Placement Calculus BC (high school); was currently enrolled in two math courses (PCU)
Michael	Junior	Mathematics (recently changed major to Mathematics Education)	Mathematical Analysis II (high school); Intro to Mathematical Thinking (PCU)

conduct interviews. Additionally, students from a variety of majors took the HPhM course at PCU. Thus, the diverse student population (e.g., first-year through fourth-year undergraduate students, mathematics and non-mathematics majors) in this course allowed for the selection of information-rich cases (Patton, 1990) for our investigation.

The HPhM course at PCU was described as “a course surveying the history and philosophy of mathematics” (W. Brandon, personal communication) and was created to enable students to be able to:

- present a personal philosophy of mathematics;
- articulate the historical developments of mathematics and evaluate the reports of such developments;
- relate the various branches of mathematics to their historical development; and
- articulate major philosophies of mathematics. (W. Brandon, personal communication)

Furthermore, Dr. Brandon (course instructor) stated that he designed the course to explore the development of and relationships among significant mathematical ideas and concepts, as well as the cultural and philosophical influences on them (a schedule of topics addressed in the HPhM course is provided in **Appendix 1**). Additionally, Dr. Brandon sought to implement the course from the perspective of three themes: the complex number system, the concept of infinity, and axiomatic structure.

The HPhM course at PCU met three times (50-minute class sessions) per week during a 15-week semester. Although the course was intended for mathematics and non-mathematics students alike, the course was not devoid of significant mathematical content. A typical class session comprised the following:

- Brief discussion on a comment or question related to assigned readings, discussed in either small table groups or as a whole class (approximately 10 minutes). Example from 8 October 2012 session: “At your tables, dig in deep and see what you can come up with on Zeno’s paradoxes.”
- Interactive lecture on a relevant historical problem or examination of a historical method or technique (approximately 20 to 30 minutes). Example from 8 October 2012 session: “Topic: What everyone should know about calculus. Three branches of calculus include: adding up an infinite number of things (series), the derivative, and integral calculus.”
- Concluding discussion (again, either in small table groups or whole class) to process content from interactive lecture (approximately 10 to 20 minutes). Example from 8 October 2012: “In your table groups, put the genius of the integral-derivative connection into your own words.”

Students from the HPhM course were recruited to participate in the research at the beginning of the Autumn 2012 semester. The course instructor introduced the opportunity to participate in the research study during the first meeting of the course. Then, at the end of the second class meeting, the second author introduced herself to the students and explained that the primary goal of the research was to investigate changes in students’ mathematical thinking and ability to communicate about mathematical concepts (e.g., the concept of infinity, the complex number system, and the axiomatic structure of mathematics) that occur as a result of studying the history and philosophy of these concepts. Finally, the nature of the interviews was explained and students’ questions about the research, interview process, and their potential participation were answered. A brief description of the four students who volunteered to participate in the pre- and post-instruction interviews is given in **Table 4**.

Research Methodology and Data Collection

For the purpose of this article, we selected the case of Michael, who was a mathematics major and had just changed his major to mathematics education at the time the study was conducted. We were particularly interested in Michael because he was just beginning to take mathematics courses for the major and was not taking another mathematics course during the Autumn 2012 semester (unlike Darren, who was enrolled in one other mathematics course). In selecting Michael, we believed that shifts in his mathematical thinking and communicating about mathematics would be more easily attributed to the HPhM course content and less likely to be commingled with possible overlapping content from other mathematics courses. We also believed that in Michael's case—given the attributes described above—we would be able to detect more nuanced shifts, particularly with respect to the other dimensions in Niss and Højgaard (2011) framework, including the various competency characteristics and the three dimensions of mastery.

We employed qualitative case study for several reasons. First, case study enables researchers to study particular phenomena and to gain a better understanding of it (Stake, 2000). It is important to note that a single case is “not undertaken primarily because the case represents the other cases or because it illustrates a particular trait or problem, but because, in all its particularity *and* ordinariness, this case itself is of interest” (p. 437). Second, given the scarcity of empirical research on our phenomena of interest, we use case study to explore in depth the role of a history and philosophy of mathematics course on students' experience with university-level mathematics. In this regard, qualitative case study does not seek generalizability; instead, the power of one case is that it can “shed light on, [and] offer insights about, similar cases” (Rossman & Rallis, 2003, p. 105).

The three-part interview instrument for this study comprised 18 items (**Appendix 2**). The instrument was constructed with the assistance of a research mathematician, and, based upon the intended foci for the PCU HPhM course, included items about the concept of infinity (five items), the complex number system (six items), and axiomatic structure in mathematics (seven items). Before using the interview protocol in this research study, we pilot-tested the instrument with two upper division mathematics majors (who would not be participating in the research) and two professors at PCU (one mathematics education professor and one mathematics professor) for content, readability, and pacing. As previously mentioned, for the purpose of this article, we focus our analysis and discussion on Michael's pre- and post-course interview responses to the seven items about axiomatic structure in mathematics.

Interviews⁹ were conducted at the beginning (during the first week) and at the end (during the final week) of the Autumn 2012 semester. Livescribe Smartpen technology was used during each interview. The Smartpen enabled the interviewer (the second author) to capture audio as well as written documentation of the students' responses to each of the tasks. Interview transcripts, once transcribed, were analyzed using the competencies of interest from the KOM Report, as well as reference to content details of the HPhM course. Instructor notes on course content (e.g., lecture notes, assigned reading, homework assignments, class session activities), and supplementary artifacts, including observation field notes from three site visits during the semester and student work, informed the analysis.

ANALYSIS OF THE CASE OF MICHAEL

We present the analysis for Michael's case in three parts, which are structured according to three different components of the seven interview items focused on axiomatic structure (Part 3 in **Appendix 2**): Truth, Logic, and Structure. In the analysis, we describe the progression (i.e., evolution) in Michael's mathematical thinking and communicating by using his actual words from the task interviews. In this way, we believe we were able to capture the progress Michael achieved, while still able to detect and connect this progress to the characteristics and dimensions of mastery as provided by Niss and Højgaard (2011).

In our analysis we used particular items from the interview protocol to identify and characterize changes in Michael's views regarding each aspect of axiomatic structure. We developed analytical questions to guide our inquiry. Furthermore, we focused on selected interview items for each aspect of axiomatic structure in the competencies of interest for the analysis of mastery dimensions. In doing so, we aimed to illustrate how we observed progress along the lines of the three dimensions of mastery.

⁹ The interviews conducted were mathematical task interviews, which call for participants to articulate their methods, interpretations, and thought processes while working on mathematical tasks, similar to “talk-aloud” interviews.

Analysis of Axiomatic Structure Items: Truth

For Truth, we share specifics of Michael's progress only with respect to mathematical thinking, as his progress was most discernible for this competency.¹⁰ We asked the following analytical question to describe shifts in his mathematical thinking:

In what ways does Michael's mathematical thinking about Truth progress within the content frame of Axiomatic Structure?

We found evidence of progress in Michael's mathematical thinking about Truth relative to the two characteristics, *deal with the scope of given mathematical concepts* and *insight into the types of answers*, of the related competency in KOM Report. Beginning with *insight into the types of answers*, for example, Michael repeated his view on the role of others in establishing Truth in both items (1 and 2 of Part 3 of the interview protocol) of the two interviews.

In the pre-interview, Michael noted that the role of "experienced mathematicians" would be used to check whether a mathematician had "some error or problem." Furthermore, if "what he [the mathematician] said is actually describing like a true equation, then it will be accepted." It is evident that Michael's view on Truth is closer to, for instance, mathematical confirmation or verification in establishing the validity of a theorem or mathematical statement in general, as Putnam (1998) described. Although Michael mentioned that a mathematician "put[s] forward a proof" when "somebody demonstrates an idea" in his qualification of the process of establishing Truth, there is no evidence of a structural approach in how he expressed this.

In the post-interview, Michael associated Truth with "absoluteness" while explicitly referring to mathematics as "this deductive thing." Michael noted the relation between deduction and absoluteness in science and history when he was asked to compare the way truth is established in science, mathematics, and history. We note that this is consistent with how Hintikka (2011) discussed axiomatic method, in that it can be found in other disciplines as a method. When asked about how mathematics was different than science and history, Michael noted that mathematics was more absolute than the others. His qualification of absoluteness in mathematics, compared to history, is explicitly related to the existence of axiomatic structure in mathematics, in addition to a deductive way of proving. That is, in history, deductions are made based on analogies where the information is based on historical documents, which may or may not be accurate. However, in mathematics, a mathematician begins with axioms, which "everyone knows is just, like, obvious." Michael continued his explanation in the post-interview:

See, with history, establishing the truth of a statement...you kind of start with the analogy instead of axioms; you start with like primary documents or something similar...like a book that somebody wrote their observations and then from there you kind of make inferences and use Logic to say, well if this has happened then this must have caused this. That is how this happened. It's letting you work up to like, different historical statements, which might not be...I don't want to say directly true, but directly provable, but through the process of provable Logic.

It is notable that Michael attributed more absoluteness to mathematics, but not to history. As we interpret this, he thought that there would be a slight doubt about the final results in history, as the information from which we deduce a conclusion may be subjective. However, in mathematics, he asserted that it is "through...provable Logic" based on statements that are obvious to everyone, that we deduce our results.

With regard to *the scope of given mathematical concepts*, we note that Michael demonstrated progress in his consideration of context and use of examples. In this regard, we highlight his explicit reference to the "whole thing with Cardano and Tartaglia,"¹¹ to discuss whether mathematical results were created or discovered when he attempted to compare mathematics with science and history. Given that the part of the course related to Cardano and Tartaglia took place 14 class sessions (or almost five weeks) before the class session that was directed by the question, "What is Truth? How does it relate to math?" (as written by the

¹⁰ This choice is consistent with how the KOM Report distinguished between the competencies of mathematical thinking and communicating. However, we recognize that others may identify progress in the communicating competency based on a conceptualization of the relationship between mathematical thinking and communication that differs from the KOM Report framework.

¹¹ The priority dispute involving Cardano (1501–1576) and Tartaglia (1499–1557) on the question of the solution of cubic equations has appeared in many sources. See Katz (2009), for an example.

instructor in his notes), we identified a strong contribution of the HPhM course on Michael's mathematical thinking related to axiomatic structure, in that it provided him with a context in which to discuss his views on this issue.

Next, we focus on Michael's responses to interview item 2 (of Part 3 of the interview protocol) to illustrate the progress in his mathematical thinking in terms of the three dimensions of competence mastery from the KOM Report.

Analysis of mastery dimensions: Degree of coverage

Some of what we have already shared regarding Michael's progress in his mathematical thinking illustrates progress in the mastery dimension of degree of coverage (Niss & Højgaard, 2011). For instance, it was Michael's *independent activation* of the mathematical thinking competency to introduce the relevant aspects of the Cardano and Tartaglia dispute in his response, as this story was not related in any explicit sense to Truth in the course. Additionally, we highlight other aspects of the mathematical thinking competency activated in this second interview item as evidence for Michael's higher degree of coverage in the post-interview compared his degree of coverage in the pre-interview.

When prompted to talk about the similarities and differences among mathematics, science, and history, Michael activated another characteristic of the mathematical thinking competency, *an insight into the types of answers*, that one might offer regarding his conclusion on mathematics being more absolute than history. In the previous quote we shared, Michael stated that the reason we would doubt history was that the evidence for historical information (e.g., primary source documents) might be unreliable. In the next quote occurring several minutes later in the post-interview, he appeared to respond to a potential criticism of what he previously stated and refined his answer.

But if you look [at], ...talk about history, which is like the past, so then...*we can't change if you define history as past so then it is just as absolute*. It is just a matter of ...you being able to get back...will you be able to get to that point, like there's certain points in history, but it's too far back; we can't really deduce what happened...soundly. I think...if somebody, oh, like the whole thing with the Archimedes [in week 4 in the semester], if somebody—if...we would have never known what happened if somebody—if they hadn't discovered it—and been able to go through and do the whole infrared imaging we would have never known the things that Archimedes did.¹² But it seems like evidence can't really be destroyed because we can almost always work back up. Does that make sense? So, I guess it is more absolute. Math. (emphasis added)

We argue that this quote demonstrates Michael's progress in the mathematical thinking competency because he could *extend the scope* of the discussion on Truth: what he concluded about the absoluteness of history was about how we define it, which is illustrated by the italicized statement in the quote above.

Analysis of mastery dimensions: Radius of action

We also interpret Michael's discussion of Truth in science, mathematics, and history as evidence of an increase in the *spectrum of contexts and situations [in which] the competency can be activated*, corresponding to progress in the mathematical thinking competency. To support his argumentation on whether the practice of mathematicians is discovery or creation, Michael referred to the historical controversy between Cardano and Tartaglia, which was another context that Michael drew upon in his interview response and which was a dimension of mathematical thinking activated by the HPhM course.

Analysis of mastery dimensions: Technical level

The post-interview provided evidence that demonstrated the entities and tools Michael used to share his thinking were conceptually and technically more advanced when compared to his responses in the pre-interview. For instance, in the post-interview, when Michael concluded with identifying the similarity between science, history, and mathematics regarding Truth, he talked about the scientific method in each discipline that results in the need to be verified by repeated experiments, supporting primary documents, or logical derivation. Furthermore, his approach in the post-interview to each discipline is more refined and consistent

¹² Here Michael is referring to the discussion that took place in the HPhM course about the discovery and the imaging study of Archimedes' Palimpsest, which began in 1999.

with an axiomatic approach as evidenced by his need to provide his definition of history as a discipline to share the way he reasoned about “establishing the truth of a statement with history.”

Analysis of Axiomatic Structure Items: Logic

Our analytical question for Logic is as follows:

In what ways does Michael view Logic as a tool in making sense of Truth?

Simply put, Michael shifted his view of Logic as the foundation of the study of mathematics in the pre-interview to Logic as a tool to reason deductively in mathematics. In particular, in the post-interview, he no longer claimed that mathematical statements need to make sense or be logical in the traditional meaning of these terms. In this way, Michael’s later thinking (e.g., at the end of the course) about the role of Logic is consistent with the formalist perspective. When asked about the relation between Logic and mathematics, he explicitly stated the change in his views on the role of Logic. Michael summarized his views in the post-interview:

This one has changed a lot, I guess, from the beginning of the year because I see Logic...not as much as the foundation, but as a tool that you use in math; to get from your axioms to your final result like your theorem. It’s more of a ladder than it’s like the floor. Because like the previous question [task interview item 5 of Part 3 of the interview protocol], they can both be true...but if Logic is your floor, then you have a problem. Because you are contradictory. But if they are different buildings in the same town of mathematics...I think I just took that analogy too far [laughter]. Yeah, if they are different buildings, then you could take the ladder wherever you want and as long as the ladder brings you to where you need to be then it’s okay.

Michael’s initial view on Logic as the foundation led him to argue in the pre-interview that Euclidean and non-Euclidean geometries were inconsistent and they could not exist at the same time in mathematics. In other words, in the pre-interview, he had to reject, for instance, Euclidean geometry as part of mathematics if he was to (simultaneously) talk about non-Euclidean geometry. As he noted in the pre-interview, while referring to Aristotle’s three fundamental laws of Logic, “two opposing truths can’t exist at the same time, in the same way, in the same sense. And whenever people try to push relativism [...] they always ignore that last clause: ‘in the same sense.’”

In this regard, Michael’s views on Logic in the post-interview are consistent with the perspective of axiomatic structure that we outlined in this article: To deduce theorems from axioms without looking for meaningful connections between axioms and the final result in the traditional sense (Hintikka, 2011). Thus, we conclude that Michael was able to *recognize, understand, and deal with the scope of Logic* and was able to *express [himself] in different ways and with different levels of technical precision* about Logic within the content frame of axiomatic structure better than he was able to do in the pre-interview.

Next, we provide our analysis of Michael’s progress related to the three dimensions of mastery for each competency for interview item 6 of Part 3 of the interview protocol, as the representative item for Logic.

Analysis of mastery dimensions: Degree of coverage

In the post-interview, we found evidence to show that Michael activated several aspects of the mathematical thinking competency that were not activated in the pre-interview. We argue that this activation allowed him to *deal with [the] scope of mathematics* that contradicts Logic in the traditional sense. For instance, the following post-interview excerpt demonstrates how Michael could reason on the ‘non-intuitive’ aspects of mathematics:

There are a lot of things that are non-intuitive about math and there are a lot of things like we talked about with *i*. They’ve, just like—they’re paradoxical and almost contradictory. [...] If you have Logic as your foundation, ...then it’s just...you’re going to break it up because there’s so much that defies Logic in math. Even with Russell’s paradox, you can’t have a set, like, the two sets, the one that contains...How’s it go? The one that contains all the [set] within itself?

Ultimately, Michael ended up with providing a consistent or non-contradictory perspective to his views on Logic. This was possible by activating another feature of the mathematical thinking competency in the post-

interview, that of *extend the scope of a concept through abstracting some of its properties*. In this regard, we bring attention to Michael's success in using analogies and metaphors to tackle the non-intuitive aspects of mathematics that he did not achieve in the pre-interview. We note that the characteristics of the mathematical thinking competency that were activated in the post-interview provide evidence for a discernible progress in degree of coverage in the post-interview. We also observe that Michael provided historical examples and/or philosophical discussion that were rooted in the course without prompting in the interview, which indicates his *independent activation* of the mathematical thinking competency in this dimension.

Furthermore, we believe that the examples Michael used in the post-interview to share his view on Logic provide evidence for his progress in the degree of coverage dimension in mastery of the communicating competency. First, during the pre-interview, Michael discussed his views on the question, "What is mathematics?", rather than the question that was actually posed in the interview item. (He stated in the post-interview that he was thinking of Logic as the foundation before he took the HPhM course.) Second, we believe that some of the examples he shared in the post-interview belong to Michael himself, and not to the reading and course activities of the related week in semester or the classroom discussion. At the end of his response for interview item 6, Michael explicitly referred to the example on paintings from the book *The Mathematical Universe* (Dunham, 1994) that he was assigned to read as part of a book review assignment for the course. He also repeated what the instructor stated during the class session to illustrate his view on Russell's paradox. However, prior to either of these examples, Michael had already provided his own analogies and metaphors. In other words, there is evidence that *independent activation* took place regarding this competency.

Analysis of mastery dimensions: Radius of action

Compared to Michael's pre-interview responses, we identified an increase in the *spectrum of contexts and situations* within the mathematical thinking and communicating competencies in Michael's post-interview responses. In the post-interview, Michael reasoned about the existence of Euclidean and non-Euclidean geometries as branches of mathematics. We found improvement in how he shared his thinking: the analogies he used were both precise and refined and enabled him to articulate his thinking more accurately. Therefore, Michael's radius of action regarding these two competencies in the post-interview was greater than the radius of action in the pre-interview.

Analysis of mastery dimensions: Technical level

The shift in Michael's views on Logic from being the foundation of mathematics to a tool that is used in mathematics demonstrates, along with his use of examples and metaphors, a conceptual and technical advancement of the related competencies. Michael's views on Logic are consistent with the literature on axiomatic structure, and he successfully made use of tools to express his views in the post-interview. We conclude that Michael's mathematical thinking and communicating competencies were more technically advanced in the post-interview compared to the pre-interview.

Analysis of Axiomatic Structure Items: Structure

In our analysis for Structure, we track Michael's progress in the mathematical thinking and communicating competencies, and for this purpose we asked the following analytical question:

In what ways does Michael's reliance on or notion of Structure evolve?

Based on the conceptualization of Structure from the perspective of axiomatic method that we have articulated, we claim that there is a hierarchical order from axioms to theorems through deductive reasoning. In addition to investigating Michael's mathematical thinking and communicating competencies from these perspectives, we also prompted Michael to share his views on some of the elements of such a Structure and their relations regarding roles within it.

Michael stated in the pre-interview that he was familiar with terms such as theorem, postulate, and conjecture from his high school geometry course. When analyzed from the content frame of axiomatic structure, Michael demonstrated an empiricist view of mathematics. On the other hand, he indicated that a theorem is something that can be proved, but he did so without a reference to axioms. Even when we explicitly asked for his definitions for such notions as axiom or theorem, he stated (during the pre-interview) that a "theorem would be something with accepted propositions." Interestingly, Michael had just provided his definition for axiom and he did not link "accepted proposition" to axiom. Thus, we conclude that the particular elements of Structure were not hierarchically formed in Michael's thinking.

Based on Michael's pre-interview response regarding definition, axiom, conjecture, theorem, lemma, law, and property and their role in mathematics, we assert that they were not connected for him in a way that they constitute a Structure:

I think of it almost as a filing system. Because you're exploring this huge abstract jungle and you're trying to classify everything and figure everything out, but you want to do it in an organized way.

In our interpretation, this organization or "filing system" refers to the classification of existing mathematical objects, but not for the purpose of creating new mathematical statements. In this sense, we conclude that Michael was only at the characteristic level of *recognition* regarding his mathematical thinking competency and did not demonstrate an *understanding* (with regard to characteristic level, in this case) of Structure.

Finally, we discuss the three dimensions of competency mastery using interview item 7 (of Part 3 of the interview protocol) as the representative item for Structure.

Analysis of mastery dimensions: Degree of coverage

We argue that Michael demonstrated a higher degree of coverage for both competencies of interest in the post-interview than he did in the pre-interview. For mathematical thinking, he was able to *deal with the scope* of the terms related to axiomatic structure since he could successfully articulate descriptions for each and how they are related within a structure. For example, in the pre-interview Michael struggled to situate the different types of mathematical statements (e.g., axiom, property, conjecture, lemma) and their relationship to each other; whereas in the post-interview the scope of terms—and how they are related to each other—was much more extensive, and included axiom, law and property, definition, conjecture, theorem, and lemma. Additionally, we argue that Michael's attempts to provide examples from a geometry course (high school) and a discrete mathematics course (university), in the sense that they were not related to the HPhM course and the context of the interview item, provide evidence for the *independent activation* of the mathematical thinking competency.

One additional aspect of the communicating competency was activated in the post-interview but not in the pre-interview and was evidence of Michael's increased degree of coverage. Michael was able to interpret the model that illustrated axiomatic structure, which was provided by the instructor during the HPhM course. Additionally, Michael could recreate this model (**Figure 1**) during the post-interview. In other words, Michael could *interpret others' mathematical expressions*. To explain his view about the elements of the model and the relations among each of them, Michael referred to the metaphors and notions that were discussed. Thus, we can attribute Michael's progress in this aspect to the influence of the HPhM course.

Analysis of mastery dimensions: Radius of action

Regarding the mathematical thinking competency, we contend that Michael exhibited a broader radius of action in the post-interview compared to that in the pre-interview. We observed Michael trying to provide his judgement on how to identify the sum of interior angles of a triangle (which to us was incorrect) to demonstrate his thinking on the notions of definition and axiom. Although it may seem controversial to describe this as an example of progress for Michael, we believe that it does represent a shift for him, or an extension in his radius of action, in which he was able to delve into a philosophical discussion without explicit prompting to do so.

With respect to radius of action in the communicating competency, we could not identify clear instances of substantial progress. However, we did identify some indicators which demonstrate potential contributions of the HPhM course. For instance, Michael drew a diagram (**Figure 1**) to demonstrate the relationship among the concepts of interest (e.g., axiom, property, conjecture, theorem), which was introduced when studying Euclid during the course.

Analysis of mastery dimensions: Technical level

It is clear that the technical level of Michael's mathematical thinking competency is higher in the post-interview items regarding Structure. Unlike the pre-interview during which he described the various types of mathematical statements (e.g., axiom, conjecture, theorem) as a "filing system" through which you can "figure everything out but you want to do it in an organized way," in the post-interview Michael described the relationships among these mathematical statements as a constructive process where you begin with axioms and "your final result is the theorem."

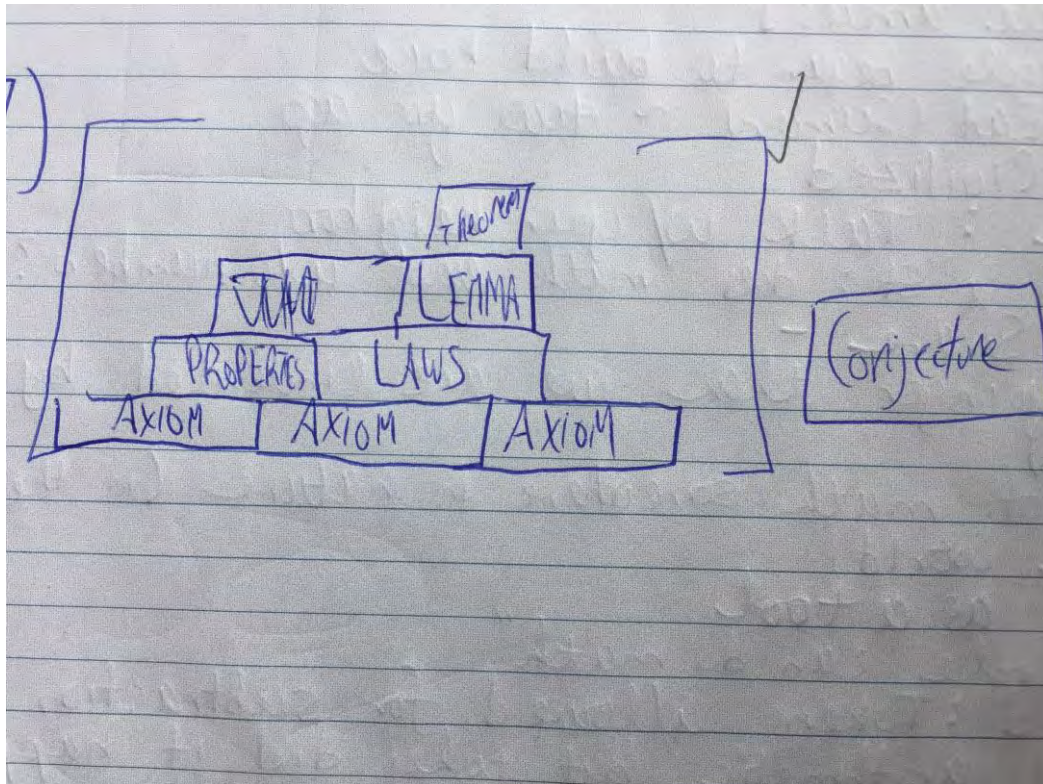


Figure 1. Michael’s Recreation of a Model of Axiomatic Structure

With respect to the communicating competency, we argue that Michael’s use of an organizing diagram successfully demonstrates a high technical level. Furthermore, we believe Michael’s sophisticated use of metaphors in the post-interview can be interpreted as a progress in the technical level dimension of the communicating competency. In this regard, we highlight Michael’s explicit reference in the post-interview to each of these elements being human constructions, in the lines of (Davis & Hersh, 1981). To us, with this approach, his conceptions of axiomatic structured evolved from what he referred with “filing system” or “huge abstract jungle.” On the other hand, we note that the diagram Michael used to demonstrate the role of and relationships among the various statements (**Figure 1**) is consistent with the views of Structure in the literature which we have shared. Thus, we conclude that Michael achieved a more advanced technical level of the communicating competency in the post-interview than he was able to in the pre-interview.

DISCUSSION

In this research we investigated the potential of a HPhM course to impact students’ progress in the competencies of mathematical thinking and communicating (Niss & Højgaard, 2011) related to the notion of axiomatic structure. We developed three primary analytical questions to aid our determination of Michael’s progress within the two competencies:

- In what ways does Michael’s mathematical thinking about Truth progress within the content frame of Axiomatic Structure?
- In what ways does Michael view Logic as a tool in making sense of Truth?
- In what ways does Michael’s reliance on or notion of Structure evolve?

These questions provided a lens through which to analyze Michael’s responses to 14 interview items (seven items about axiomatic structure for both a pre-course and post-course interview) in order to address the research question: *How do students’ competencies in mathematical thinking and communicating with regard to axiomatic structure change during a course on the history and philosophy of mathematics?*

The primary goal of the research was to provide an interpretation of the contribution of a HPhM course through the identification of one student’s progress in the competencies of mathematical thinking and

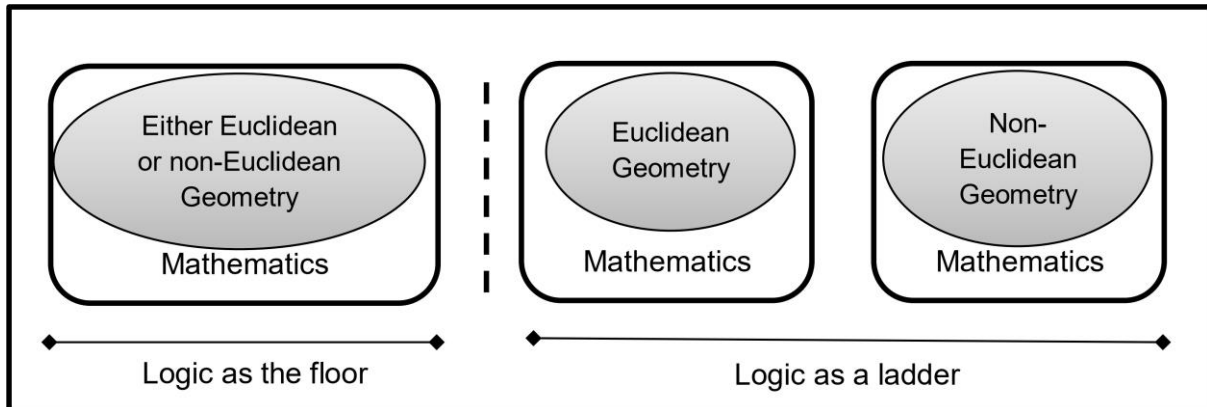


Figure 2. Michael's Changing Conception of Mathematics and the Role of Logic

communicating. To do so, we used the competency descriptions (including accompanying characteristics) and dimensions of competency mastery outlined by Niss and Højgaard (2011). In the discussion that follows we summarize the essence of the ways in which Michael thought about and communicated his ideas with regard to axiomatic structure in mathematics, and how his ways of doing so evolved.

To begin, we return to our conception of the essence of modern mathematics to situate our discussion of Michael's progress toward a formalist orientation of mathematics. As Hintikka (2011) observed, "the structuralist orientation of modern mathematics naturally leads to the use of axiomatization" (p. 70). When it comes to the ways that mathematicians establish *truth* or create *knowledge*, we acknowledge the role of quasi-experimental methods and mathematical experiments (Putnam, 1998). However, regarding axiomatization and from a formalist perspective on mathematics, what Michael thinks and how he talks about Structure is at the core of our investigation on his progress in this discussion, where his views on Truth and Logic also assist us in constructing a holistic representation of his views on Structure, and thus, axiomatic method.

We acknowledge that Michael did not develop a full-fledged modern conception of axiomatic method. For instance, his statement "axioms—everyone knows, [are] just, like, obvious" is contrary to the Bourbaki (1950) perspective that the traditional meaning of axiom as "evident truth" for everyone is no longer applicable in the modern conception of axiomatic structure. Despite this statement of Michael about axioms, which may be originally viewed as a flaw in his conceptualization of axiomatic structure, we argue that he demonstrated an appreciable (modern) conceptualization in the post-interview. We further contend that he embraced an axiomatic-structural way of thinking about mathematics. For example, Michael rejected the co-existence of Euclidean and non-Euclidean geometries as branches of mathematics in the pre-interview. However, in the post-interview he stated:

I think the problem would be how you define mathematics. [...] there's that difference between Euclidean geometry and non-Euclidean geometry. [...] They're not consistent with each other but on the realm they're internally consistent.

Thus, rather than looking for meaning in the traditional sense, Michael considered the field of mathematics as a structure whose meaning depends on how you define that structure. (See **Figure 2** for our representation of Michael's consideration of mathematics in two interviews.)

In addition to a structural orientation in thinking about mathematics, Michael was explicit about the nature of building axiomatic structure in his post-interview response (**Figure 1**): he placed axioms as the base and theorems represent the final result, leading to a characterization of deductive thinking that he related to absoluteness. **Figure 2** also depicts how Michael framed Logic in the pre-interview as the foundation of mathematics (as the floor; on the left in **Figure 2**) that one uses to derive or deduce theorems. In the post-interview, while describing Logic as a tool (as a ladder; on the right in **Figure 2**), Michael contrasted his previous view by stating that "he took that analogy too far." Given his structural approach to mathematics, framing Logic as a tool rather than relating it to meaning, and recognition of the role of deductive thinking in this endeavor, we argue that Michael demonstrated significant progress in his consideration of axiomatic structure. Michael explicitly identified the contribution of the course in several instances during the post-interview. The following quote is a primary example of such an acknowledgement regarding his views on the

relation between Logic and mathematics. When prompted about whether he thought that Logic was the foundation of mathematics before the HPhM course, he stated:

... I think the thing with this class...has changed me. [...] If you have Logic as your foundation...then it's just...you're going to break it up because there's so much that defies Logic in math. (emphasis added)

CONCLUDING REMARKS

Clearly, Michael's views and knowledge of axiomatic structure demonstrate the need for his further mathematical development. Whether Michael achieved sufficient mastery of the other mathematical competencies necessary for success in advanced undergraduate coursework is a question beyond the scope of this article. Furthermore, our study is an attempt to underscore how students' study of history and philosophy of mathematics can contribute to a critical dimension of advanced mathematical study like axiomatic structure which is not included in the curriculum, rather than drawing grand conclusions about the effectiveness of the HPhM course on Michael's mathematical development in general. We believe that the results of our empirical investigation and claims about Michael's progress have the potential to inform curriculum development and classroom instruction. The reader will be able to transfer the findings of this case study upon "[determining] the degree of similarity between the study site and the receiving context" (Mertens, 2005, p. 256).

In this respect, although Michael's conceptualization of axiomatic structure differs from that in the literature we discussed, we assert that the progress he experienced during the HPhM course regarding his mathematical thinking and communication about axiomatic structure is promising support for his future mathematical studies. We further contend that the shift towards a formalist tendency in the ways Michael argued about mathematics (see **Figure 2** for a demonstration of this shift) indicates a greater degree of preparedness for a study of advanced mathematics.¹³ Given the importance of such a perspective for the study of mathematics, we have highlighted the significance of a course focused on the study of history and philosophy of mathematics, for the learning of mathematics.

Axiomatic method in mathematics is not generally taught as a part of the undergraduate mathematics curriculum in an explicit way. Yet, we, as mathematics educators, do not know whether or how often instruction in undergraduate mathematics classrooms implicitly includes teaching of axiomatic structure since, for instance, research on how students conceptualize axiomatic structure is scarce, and research on instructors' teaching practices is only now gaining attention following the call by Speer, Smith, and Horvath (2010) on the issue. Therefore, we argue that a HPhM course, or similar efforts which include reading and studying history and philosophy of mathematics, each have potential to support students' "success" in advanced mathematics at university.

Disclosure statement

No potential conflict of interest was reported by the authors.

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¹³ Lack of such preparedness for a formal study of mathematics at the undergraduate level is one component often associated with the transition problem. Several mathematics educators (e.g., Witzke et al., 2018) have suggested that the development of mathematics from a historical perspective can assist students in the transition from an empirically-oriented study of mathematics in school to a formal study of mathematics at university.

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APPENDIX

1: HPHM Course Schedule of Topics (PCU, Autumn 2012)

Class Session	Content
1	Syllabus What is mathematics?
2-3	Argue for/against math as universal vs. arbitrary system Evidence for universal, arbitrary, or both
4	What is number? What are operations?
5	What is mathematics? <i>continued</i>
6	View on infinity (Aristotle, Zeno)
7	Incommensurability (Pythagoras, problems of antiquity, twin primes, Goldbach)
8	Examine Euclid and the concept of "axiom" Self-evident TRUTH without proof
9	Proof (Pythagoras' Theorem, $\sqrt{2}$)
10	Greek Mastery: What did Greeks contribute?
11	Archimedes
12	Math Moves East: Did the West do a disservice to the method? - Diophantus - Heron's Formula
13	Exam 1
14	Early Renaissance Mathematics (Fibonacci, Cardano) (Read Cardano article: Anglin "The Secret of the Cubic")
15	Math & God (Discuss Cardano)
16	Intro to Number Theory - Fermat (Fermat's Last Theorem; primes) - Bridge between Archimedes & Newton What is proof?
17	Pascal, Fermat, & Bernoulli Pascal's Triangle; finding patterns
18	Descartes – Cartesian Coordinate System Examine questions like: - What is a line? - Equation of a line? - Development from Greek view of math, to development in East (early Renaissance), to Descartes - Graph of inequality; conics, etc.

19	Exploring the Geometry of Calculus <ul style="list-style-type: none"> - Newton and Leibniz - Three “unrelated” topics (instantaneous rate of change; infinite series; integral)
20	Class Session 19, continued
21	Class Session 19, continued: <ul style="list-style-type: none"> - Infinite series - Laplace
22	Review reading Review Table of Contents of “Letters to a German Princess” (Euler)
23	A taste of Euler Zones of Proximal Development (ZPDs) in math history: pre-post Euler Math symbols you know Famous results of Euler
24	Gauss: General <ul style="list-style-type: none"> - Child prodigy in math - Sample of Gauss’s work: Number theory systematized; figurate numbers; normal curve; Fundamental Theorem of Arithmetic; Fundamental Theorem of Algebra; complex numbers, etc. Math ability: Nurture or Nature?
25	Gauss: Euclid’s 5 th Postulate (Gauss’s teacher: “Only a crazy person would doubt the 5 th .”) What is truth? What does math have to do with truth? Structure of Euclid
26	Non-Euclidean Geometry
27	A Look Back Mathematicians and Mathematics (1400–1800)
28	Exam 2
29	Truth? <ul style="list-style-type: none"> - What is Truth? How does it relate to math? - Is Euclidean Geometry / non-Euclidean Geometry truth? - What are they?
30	Math “Oddities” Is Geometry “always true”? What do you mean by that? Consider commutativity Other arithmetic?
31	Quest for Rigor <ul style="list-style-type: none"> - Greek - Axioms of arithmetic (Peano) - Rational numbers - Definition of limit
32	Set Theory – Dealing with Intuition <ul style="list-style-type: none"> - Defining infinite set - Following rules (see example in instructor notes), but does not set well - What are these mathematicians doing? - Discovery? Intuition?
33	What is Proof? What is Axiomatic Structure? <ul style="list-style-type: none"> - What should we postulate? - Hilbert and formalism (reject Platonism; formulated methods in 1900s) - Begin from tautology
34	Hilbert, continued: Hilbert’s Challenge <ul style="list-style-type: none"> - Four schools
35	Gödel <ul style="list-style-type: none"> - Timeline of mathematical development of truth - Discuss “truth” (1931): Absolute truth; axioms; logic; Four schools
36	Review of Key Ideas: Students pose questions and consider possible views <ul style="list-style-type: none"> - What is math? - Where is math? - What is number? - Does infinity exist?

	- How is truth determined in math?
	- What is proof?
	- What does a mathematician do?
	- What is important in math?
	- Can anyone do math? Who?
	- Is math a science? Art? Both? Neither?
37	Book reviews and newspaper assignment (in-class work)
38	Last day for newspaper assignment
39	“The Proof” (What does a mathematician do?)
40	Discussion of “The Proof” (the movie) Philosophy of Math paper
41	“Story of 1”
42	Present final exam / newspapers

2: Task Interview Protocol

Directions

Please solve the following problems (or respond to the questions) and while you do so, talk aloud and write your ideas using the Smartpen to share what you are thinking about as you work. Certain materials will be available for you during the interview: blank white paper, graph paper, pencils, pens, and graphing/scientific calculator. The Smartpen will be used to capture your work.]

Part 1: Infinity

- (1) Do you believe the equation $0.999\dots = 1$ is true? Justify your response.
- (2) Determine the limit of the quotient $\frac{f(x+h)-f(x)}{h}$ as h approaches zero (0) for $f(x) = x^2 + 3$.
- (3) What is the sum of the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots + \frac{1}{n^2}$ as n approaches infinity (∞)?
- (4) Suppose we are given an infinite set of numbered tennis balls and two bins of unlimited capacity. Now suppose we place balls 1 and 2 in the first bin and then immediately move ball 1 to the second bin. Next, we place balls 3 and 4 in the first bin and move ball 2 to the second bin. Then, we place balls 5 and 6 in the first bin and move ball 3 to the second bin. This process continues ad infinitum. How many tennis balls are found in the bins when the process is finished?
- (5) Accept or reject the following statements. Provide reasons for your position.
 - (a) I think of infinity as a number.
 - (b) I think of infinity as a process.
 - (c) I think of infinity as something else.
 - (d) “Infinity” and “the infinite” mean the same thing.
 - (e) $\frac{\infty}{\infty} = 1$

Part 2: Number

- (1) Draw a Venn diagram to show the relationships among the different types of numbers that comprise the number system.
- (2) Using a graphical system (of your own construction) to plot points that represent the locations of the following:

$$\frac{4}{3}; 0.875; -\sqrt{27}; -3 + 2i; 7\pi; \frac{6+i}{6-i}; \sqrt[3]{-27}; \frac{e}{4}$$

- (3) There are several contexts for which a value of -6 would make sense. Describe an example of such a context. How does this illustrate the value, -6 ?
- (4) Consider: $2; \sqrt{2}; 2i; 2e$. Explain whether each (number):
 - (a) describes something,
 - (b) exists (i.e., it *is* something), or
 - (c) both describes something and exists.
- (5) Explain the difference between a “numeral” and a “number”
- (6) How many numbers exist between 0 and 1? Explain.

Part 3: Axiomatic Structure

- (1) In mathematics, how does a statement become accepted as “true”?
- (2) How is mathematics the same as science and history in establishing the “truth” of a statement? How is mathematics different from science and history in establishing the “truth” of a statement?
- (3) The Pythagorean Theorem states, *In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.* Define “theorem” and explain why the statement given above is a theorem.
- (4) How do conjectures (like the Twin Prime Conjecture or Goldbach’s Conjecture) differ from theorems?
- (5) Is it possible in mathematics for the following two statements to both be true? If yes, how? If not, why not?
 - (a) The sum of the interior angles of a triangle is always 180 degrees.
 - (b) The sum of the interior angles of a triangle is always greater than 180 degrees.
- (6) What is the relationship between logic and mathematics?
- (7) Describe each of the following and the role each plays in mathematics:
definition, axiom, conjecture, theorem, lemma, law, property

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