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A Synergy between History of Mathematics and Mathematics Education: A Possible Path from Geometry to Symbolic Algebra

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Abstract: This paper proposes an experimental path aimed at guiding upper secondary school students to overcome that discontinuity, often perceived by them, between learning geometry and learning algebra. This path contributes to making students aware of how the algebraic language, formalized in the most powerful form by Descartes, grafts itself onto the geometric language. This is realized by introducing a problem included in a text written by Abū Kāmil before the year 870. This awareness acquired by the students, when accompanied by some semiotic considerations, allows the translation of the problem from “spoken” algebra to “symbolic” algebra, and it represents the background for a possible use of the same problem within the framework of analytic geometry. This proposition manifests a didactic and popular efficacy that supports and favors the recognition of the object it is talking about in different contexts, helping to create a unitary vision of mathematics.

Keywords: mathematics education; history of mathematics; regular pentagon; geometry; algebra; analytic geometry

1. Introduction

This work proposes a didactic path, designed for an upper secondary school class, aimed at showing a connection between geometric knowledge and algebraic knowledge starting from the use of the first problem of the chapter on the regular pentagon and decagon of the *Kitāb fī al-jabr wa al-muqābala* (*Book on algebra and science of reduction and cancellation*) of Abū Kāmil [1].

The choice of a proposition taken from the history of mathematics is encouraged by the fact that knowledge of the latter and its use in teaching constitute a significant part of the cultural background of future mathematics teachers [2]. The knowledge of how a mathematical concept was born and evolved can contribute to a better understanding of that concept itself [3], and the study of historical sources contains a potential suitable to increase awareness of possible misconceptions, obstacles, and impediments related to various mathematical concepts and ideas [4].

Useful feedback on the training of teachers are found in Reference [5] with the practice-based theory of mathematical knowledge for teaching, in Reference [6–8] where the authors specify the positive elements of the teacher education through the history of mathematics, in Reference [9] where the importance of the historical and cultural dimensions in mathematics education is underlined, in Reference [10], concerning the support of history in changing individual’s epistemic beliefs about the nature of mathematical knowledge, and in Reference [11] for the implementation of teachers’ skills in the cultural analysis of content.

The same choice of Abū Kāmil’s proposition is also motivated by the fact that the teaching and learning of mathematics are favorably affected by a programming that takes into account the social context in which they develop [12,13]. A further motivation is given from the cultural elements [14–16] that can suggest suitable communicative means of the mathematical concepts and can address

towards adequate choices of specific contents [17–19]. Other authors [20–22] evidenced that history of mathematics certainly takes on a relevant role among the possible cultural elements to consider. Moreover, important considerations on the fact that the history of mathematics provides significant material for teaching in the classroom are included in the wide review [23].

Why does the problem of Abū Kāmil contains aspects of novelty for educational use among the available historical material?

With regard to the learning of mathematical language, it constitutes a particularly simple environment to effectively explore the potential of some forms and registers with which mathematics expresses itself, to ascertain how the changes between these registers can occur in a reliable and effective way and to evaluate the types of responses that these changes can produce in a problem solving.

The Abū Kāmil's problem constitutes an experiential environment of rare transparency to start students of secondary school to develop the ability to compose the mathematical structure that allows them to solve and then to generalize the problem, thus making a significant part of the path towards modeling in mathematics, that has for years been the subject of a constructively critical and argumentative discussion among researchers in mathematics education [24,25].

In the initial part of this path, starting from the problem posed by Abū Kāmil, students are invited to become aware that in the geometric field the figure performs the task of visualizing the object which the problem is centered on and which you can reflect on to understand and demonstrate their specific properties.

Subsequently, still following the path of Abū Kāmil, they can enter into the heart of the question constituted by the search for the measure of an element of the pentagon, its side in the specific case, as a function of the radius of the circumscribed circle. The question is, in fact, to establish numerical relationships that require a linguistic and narrative change with respect to the Euclidean one and contain the need for a new model that, as history shows, if realized, will have unpredictable consequences.

More specifically, the chosen problem is particularly suitable for showing the passages in which some properties, enumerated in Euclid's *Elements* in geometric language, are translated into the rhetorical language of Arabic algebra, in order to write the resolutive equation of the problem; subsequently those passages that realize a translation of the same properties from the language of spoken algebra in the language of symbolic algebra and, finally, those that lead to the generalization of the problem through the use of a parameter. These passages are sketched in the flowchart shown in Figure 1.

This problem is grafted on the geometric construction of regular polygons inscribed or circumscribed in a circle, the importance of which is evident in learning mathematics [26]. By reflecting on it, one can help to clarify the formative role of geometry and algebra, before going on to the real contents of analysis, as well as to the understanding of the significance of "demonstration/solution" within the study of algebra, considered much more than a mere tool for calculation.

Under a more general point of view, this kind of process of translating by the spoken algebra to symbolic algebra helps the students to understand better the "natural" historical evolution of mathematics through the progressive elimination of verbal components in the expression of algebraic procedures by realizing a more practical, fluid and effective language. Such an advantageous language, that represents a general tool for studying mathematical problems given in the past, and still remains, a strong impulse to science allowing the possibility to generalize both the problems and their solution through parameterization. All this is of utmost importance in the algorithmic expression of the mathematical problems and, therefore, in their numerical solution, as well as in the possibility to use computer science tools for a didactic aim.

The organization of the paper is the following. In next section, it is explained as the problem proposed by Abū Kāmil allows the students to "see" and "tell" a mathematical object with different languages and signs. Moreover, essential information about Abū Kāmil and his main work are given.

In the third section, the passage from the formulation of the proposition given by Abū Kāmil to a more general one, is described. Finally, a brief discussion and some conclusions are drawn.

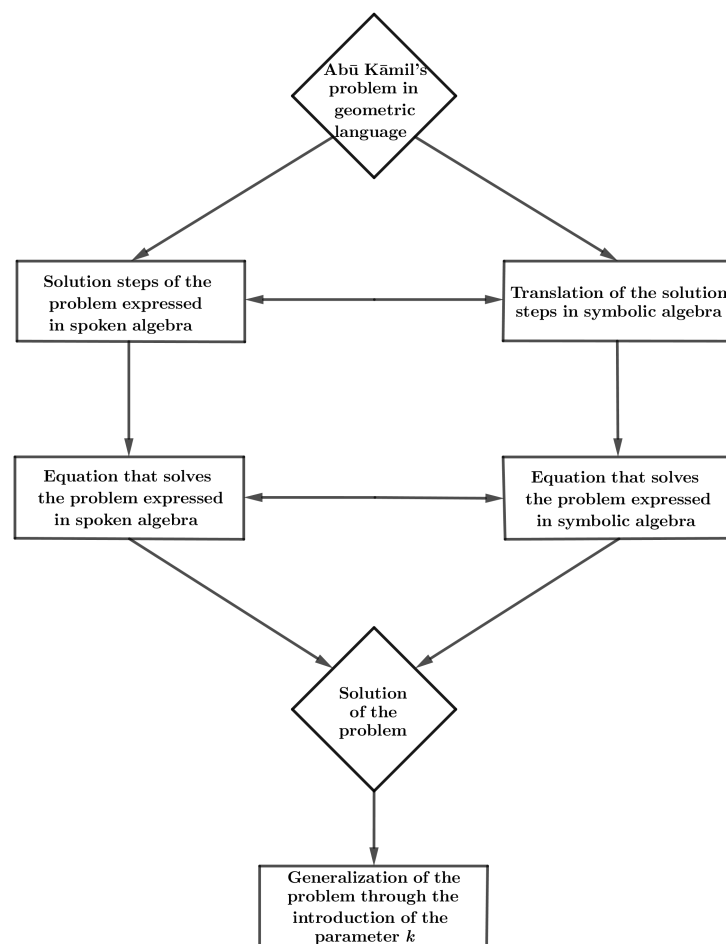


Figure 1. Flowchart showing the logical steps in the proposed path from geometry to algebra.

2. Materials and Methods

2.1. Theoretical Framework

The path proposed in this paper aspires to take into account the significant relationships between the construction of mathematical knowledge by students and the historical construction of mathematical knowledge highlighted in literature [27,28]. Moreover, the historical material presented in this article can take on a particular value of support for the teaching of mathematics within specific semiotic and epistemological considerations [16,29–33].

Abū Kāmil's chapter appears indeed as one of the first "written recordings" of a crucial moment of the semiotic evolution of the language of mathematics and of its expressive potential. It shows a didactic and informative efficacy that justifies its particular diffusion in a separate form from the rest of the work. The text by Abū Kāmil presents and solves some "simple" problems using the "signs of spoken algebra", not yet "signs of literal algebra", which for this reason naturally begin to ferry from the universe of the consolidated Euclidean narrative to that of the future Cartesian narrative [34].

As an element to graft the semiotic "metamorphosis", the geometric constructions that, in the context of Abū Kāmil's writing and, even now, in class work with the students, manifest their potential as a suitable element to support and encourage the recognition of the object one is talking about, take on a key role. This recognition [35] is fundamental for the realization of learning through a diversification of the way in which this object is "seen" and then "told" through a new sign [36,37].

In the text of Abū Kāmil, resonances and differences can be glimpsed between the “knowledge” underlying the use of the Euclidean language and that underlying the algebraic language, considering the knowledge as potential that emerges from human activity in a process of becoming to “materialize” into knowing (see Reference [16], p. 100). The first language (Euclidean) is strongly rooted in its spatial development, as the reading of any Euclid’s passage reveals, the second (algebraic) is projected to develop in successive states linked to temporally distinct “moments” [36], a prelude to the powerful mathematical tools indispensable to the expression of Newtonian physics and mathematics of the 18th century.

In this perspective, the proposed path develops. The change of register, introduced to the students “at the suggestion” of Abū Kāmil in order to speak of the pentagon with “new” signs in respect to those of Euclid, contains some prodromes to the modeling process which are crucial for the development of the skills of problem solving [38–41].

2.2. Historical Notes on the Writing of Abū Kāmil

The name Abū Kāmil Shuja ibn Aslam ibn Muhammad ibn Shuja (850–930), also known as a-Hasib al-Misri, or as the Egyptian “calculator”, frequently occurs in the first Western algebraic tradition, when reference is made to the transmission of algebra. The name slowly disappears in the later Latin algebraic tradition, until finally lost without trace. Abū Kāmil is especially remembered by later generations as the first commentator on *Kitāb al-jabr wa al-muqābala* (*Book of algebra and science of reduction and cancellation*) by Muhammad ibn Mūsā al-Khwārizmī, written between 813 and 830 [42]. Commentary by Abū Kāmil is the *Kitāb fī al-jabr wa al-muqābala* (*Book on algebra and science of reduction and cancellation*), written before 870 (see Reference [43] for a historical collocation of Abū Kāmil’s work on algebra among Arab mathematicians). In this treatise, the author devoted a chapter to problems on the regular pentagon and decagon inscribed in or circumscribing a circle in which the Latin translation by Gerard (from the 12th century) can be found at the Bibliothèque Nationale de Paris (Ms. lat. 7377A).

From the analysis of the historical literature available, it is possible to hypothesize an autonomous diffusion of this chapter with respect to the rest of the treatise, if we assume it was directed at readers already familiar with algebra and equations. It is possible to hypothesize that this could have happened during the period most favorable to the spread of algebra. Bearing in mind the considerable interest in the discipline following publication of the work of al-Khwārizmī and Abū Kāmil, the relevant period could lie between the years 950 and 1200 in the case of the Arab world, as seems most likely, or alternatively the years after 1100 in Europe, with the establishment of a Latin tradition in algebra and the birth of cultural centres on this subject in Europe, both among Jewish and Latin scholars. As to why this should have happened, it is possible to put forward the hypothesis that the contents of the chapter are easily placed in relation to certain propositions in books IV and XIII of the *Elements* by Euclid, that deal with the construction of regular polygons with a ruler and compass, and to the tradition built around this work.

Since the issue of the construction of polygons inscribed within or circumscribed by a circle is a fundamental part of basic mathematical training [44,45], as attested by numerous Arab mathematicians, one can believe that Abū Kāmil chose the construction of the pentagon and decagon as examples of the beauty and value of algebra in finding solutions to problems. Indeed, in the introduction to the second part of the treatise, after having displayed the indispensable elements for an understanding of algebra, Abū Kāmil turns his attention to that which “important” and “skilful” geometers have read in Euclid’s book and other works, explaining and commenting on it in his writing, starting from consideration of the measurement of the side of a pentagon and a decagon, inscribed within or circumscribed by a circle, and the measurement of the diameter of the circle in the two cases [1,46,47].

Al-Qūhī (second half of the X century) was inspired by this short chapter of Abū Kāmil when he wrote *On the construction of an equilateral pentagon in a given square*. This highlights the fact that Arab mathematicians recognized the importance of the chapter vis-à-vis the treatise on algebra.

The same chapter was a certain point of reference for Leonardo Pisano when writing his book *Practica Geometriae* [48].

3. Results

From the Editing of Abū Kāmil to the General Formulation of A Proposition

The propositions of the chapter of Abū Kāmil's treatise contain problems regarding regular pentagons and decagons inscribed in or circumscribing a circle (1–11), the triangles and squares in which figures are inscribed (12–15) and regular pentagons and decagons and how to determine the length of their sides when the area is known (16–20).

As mentioned above, to show students an example of connection between geometric knowledge and algebraic knowledge the first proposition is presented below in a didactic fashion. Figure 2 represents the original picture realized by Abū Kāmil (see Reference [1], pp. 524–525) to introduce the problem of finding the length of the side of the regular pentagon inscribed into a circle of diameter equal to 10.

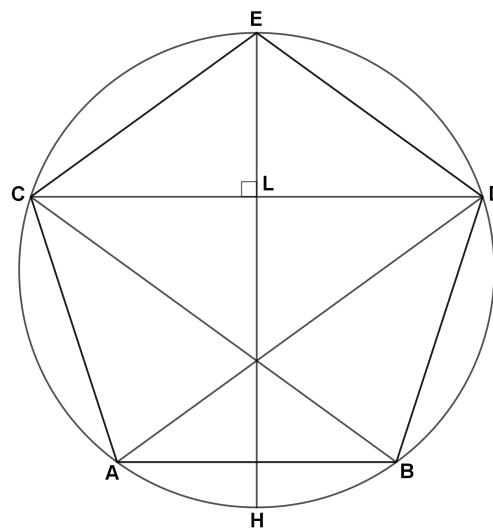


Figure 2. Abū Kāmil's construction to obtain the length of the side of the regular pentagon (see Reference [1], pp. 524–525).

Here, below, the English translation of the first of the twenty geometric problems that Abū Kāmil solves in the chapter concerning the regular pentagon and decagon of *Kitāb fī al-jabr wa al-muqābala*, is reported (see Reference [1], pp. 522–527, for a French translation of the original Arabic text):

Assume the known circle $ABDEC$ in which the diameter is ten in number, it is the straight line EH , and in which a regular pentagon is inscribed; it is the pentagon $ABDCE$. If we want to know what is the size of each of the sides of this pentagon, we draw the straight line CLD that is the chord of two-fifths of the circle, and we assume the straight line ED a thing. We know that the straight line EL is a tenth of a *māl* (Here, the authors used the transliteration of the original Arabic word. Literally, this word means a quantity of money (among other things). In modern mathematical translations, it is usually translated as “square”, as in Reference [1]. I preferred here to maintain the original word used by Abū Kāmil. I have referred to translations into a number of languages [1,46,47,49,50]), since the product of ED by itself is equal to the product of HE by EL ; the straight line DL is the root of *māl* minus one-tenth of one-tenth of *māl-māl* and straight line CL is equal to the straight line LD ; the straight line CD is therefore the root of four *māl* minus two-fifths of one-tenth of *māl-māl*. But we know that the sum of the product of AB and CD plus AB by itself is equal to the

product of CD by itself, because the product of AB and CD plus AC by BD is equal to the product of AD by BC ; or AC by BD is equal to AB by itself, AD by BC is equal to CD by itself, and CD by itself is four $māl$ minus two-fifths of one-tenth of $māl-māl$. Eliminate from this the product of AB by itself, that is a $māl$; thus remain three $māl$ minus two-fifths of one-tenth of $māl-māl$, equal to the product of AB by CD . We then divide three $māl$ minus two-fifths of one-tenth of $māl-māl$ by the straight line AB , which is a thing; we get the straight line CD , three things minus two-fifths of one-tenth of a cube. But we have shown that the straight line CD is the root of four $māl$ minus two-fifths of one-tenth of $māl-māl$. We multiply three things minus two-fifths of one-tenth of a cube by themselves; we get nine $māl$ and one part of six hundred-twenty-five parts of cube-cube minus six parts of twenty-five parts of $māl-māl$ equal to four $māl$ minus two-fifths of one-tenth of $māl-māl$. Reducing from this, we have a fifth of $māl-māl$ equal to five $māl$ and a part of six hundred twenty-five parts of a cube-cube. Dividing all that you have by a $māl$, you get five “dirhams” and one part of six hundred twenty-five parts of $māl-māl$ equal to one-fifth of $māl$. Make whole $māl-māl$ for which you have a $māl-māl$, and so multiply it by six hundred twenty-five, so all that you have by six hundred and twenty-five; we have a $māl-māl$ plus three thousand one hundred twenty-five “dirhams” equal to one hundred twenty-five $māl$. Divide the $māl$ into two halves, get sixty-two and a half; multiply them by themselves, you get three thousand nine hundred six and a quarter; subtract three thousand one hundred twenty-five, it remains seven hundred eighty-one and a quarter. We remove the root of this one from sixty-two and a half, we take the root of what remains, we have the straight line ED which is one of the sides of the pentagon.

All this can be equally and more synthetically expressed in the modern algebraic language as (see Reference [1], pp. 115–117):

Let $ABDEC$ be the known circle of diameter $EH = 10$ and let $ABDEC$ be the regular pentagon inscribed within it. Draw the chord DC that subtends $\frac{2}{5}$ of the circumference (because each side of the pentagon subtends $\frac{1}{5}$ of the circle). Assume $ED = x$ (that is the side of the pentagon we want to find). We know that $EL = \frac{1}{10}x^2$ since $ED^2 = EH \cdot EL$ (for Euclid’s first theorem applied to the right triangle EDH , in which ED is a cathetus, and EL is the projection of ED on the hypotenuse EH).

For Pythagoras’ theorem applied to the right triangle EDL , we have: $DL = \sqrt{ED^2 - EL^2} = \sqrt{x^2 - \frac{1}{10} \frac{1}{10} x^4}$; moreover $CL = DL$; therefore, $CD = \sqrt{4x^2 - \frac{2}{5} \frac{1}{10} x^4}$.

We know that $AB \cdot CD + AB^2 = CD^2$ because, for Ptolemy’s theorem applied to the quadrilateral $ABDC$, $AB \cdot CD + AC \cdot BD = AD \cdot BC$, or $AC \cdot BD = AB^2$, $AD \cdot BC = CD^2$, and $CD^2 = 4x^2 - \frac{2}{5} \frac{1}{10} x^4$. Let us subtract from this $AB^2 = x^2$; thus, we remain with $3x^2 - \frac{2}{5} \frac{1}{10} x^4 = AB \cdot CD$. Let us divide the last relation by $AB = x$, and we obtain $CD = 3x - \frac{2}{5} \frac{1}{10} x^3$.

Let us multiply $3x - \frac{2}{5} \frac{1}{10} x^3$ by itself, and we obtain:

$$\left(3x - \frac{2}{5} \frac{1}{10} x^3\right)^2 = 9x^2 + \frac{1}{625} x^6 - \frac{6}{25} x^4.$$

Since we have shown that $CD^2 = 4x^2 - \frac{2}{5} \frac{1}{10} x^4$, by equating the two expressions for CD^2 we have just found, we get $\frac{1}{5} x^4 = 5x^2 + \frac{1}{625} x^6$. If we divide this relation by x^2 , we obtain: $5 + \frac{1}{625} x^4 = \frac{1}{5} x^2$.

By multiplying the last relation by 625, we get: $x^4 + 3125 = 125x^2$. This is a biquadratic equation, in which the acceptable solution (for our problem) is:

$$ED = x = 5\sqrt{\frac{5 - \sqrt{5}}{2}} \simeq 5.88.$$

Table 1 contains the steps of the solution of the problem under consideration, given through a one-by-one correspondence between Abū Kāmil's solution expressed in "spoken" algebra and a "translation" in symbolic algebra. The left part of the table shows how, drawing on the knowledge contained in Euclid's *Elements* on the pentagon and expressed in the Euclidean geometric register, the problem proposed by Abū Kāmil takes expression and form in the register of rhetorical algebra and how the solution is found through its rules. In other words, a first expression of modeling of the problem and its solution is realized. The right part of the table contains a further expressive passage, the "translation" of the solution in the register in which today's mathematical culture requires students to learn to express themselves, the one to which Descartes invited us in his *Géométrie*, that of formal algebra.

Table 1. The first of the twenty geometric problems that Abū Kāmil solves in the pentagon and decagon chapter. The left part contains the problem expressed in rhetorical algebra. The right part contains a translation in the register of symbolic algebra.

| Step | Statement and Solution Steps of the Problem Expressed in Spoken Algebra | Translation of the Statement and of the Solution Steps in Symbolic Algebra |
|------|--|--|
| 1 | Determine the length of a chord of a fifth of a known circle starting from its diameter. | Find the side of the regular pentagon inscribed in a given circle. |
| 2 | Assume the known circle $ABDEC$ in which the diameter is ten in number, it is the straight line EH , and in which a regular pentagon is inscribed, it is the pentagon $ABDEC$. | Let $ABDEC$ be the known circle of diameter $EH = 10$ and let $ABDEC$ be the regular pentagon inscribed within it. |
| 3 | Draw the straight line CLD that is the chord of two-fifths of the circle. | Draw the chord DC that subtends $\frac{2}{5}$ of the circumference. |
| 4 | Assume the straight line ED is a thing. | Assume $ED = x$. |
| 5 | We know that the straight line EL is a tenth of a $māl$, since the product of ED by itself is equal to the product of EH by EL ; | We know that $EL = \frac{1}{10}x^2$ since $ED^2 = EH \cdot EL$; |
| 6 | the straight line DL is the root of $māl$ minus one-tenth of one-tenth of $māl-māl$ and straight line CL is equal to the straight line DL ; | $DL = \sqrt{x^2 - \frac{1}{10} \frac{1}{10} x^4}$ and $CL = DL$; |
| 7 | the straight line CD is therefore the root of four $māl$ minus two-fifths of one-tenth of $māl-māl$. | therefore, $CD = \sqrt{4x^2 - \frac{2}{5} \frac{1}{10} x^4}$. |
| 8 | But we know that the sum of the product of AB and CD plus AB by itself is equal to the product of CD by itself, because the product of AB and CD plus AC by BD is equal to the product of AD by BC ; | We know that $AB \cdot CD + AB^2 = CD^2$ because $AB \cdot CD + AC \cdot BD = AD \cdot BC$; |
| 9 | or AC by BD is equal to AB by itself, AD by BC is equal to CD by itself, and CD by itself is four $māl$ minus two-fifths of one-tenth of $māl-māl$. | or $AC \cdot BD = AB^2$, $AD \cdot BC = CD^2$, and $CD^2 = 4x^2 - \frac{2}{5} \frac{1}{10} x^4$. |
| 10 | Eliminate from this the product of AB by itself, that is a $māl$, thus remaining three $māl$ minus two-fifths of one-tenth of $māl-māl$, equal to the product of AB by CD . | Subtract from this $AB^2 = x^2$, thus remaining with $3x^2 - \frac{2}{5} \frac{1}{10} x^4 = AB \cdot CD$. |
| 11 | We then divide three $māl$ minus two-fifths of one-tenth of $māl-māl$ by the straight line AB , which is a thing; we get the straight line CD , three things minus two-fifths of one-tenth of a cube. | Divide by $AB = x$, and we obtain $CD = 3x - \frac{2}{5} \frac{1}{10} x^3$. |

Table 1. Cont.

| Step | Statement and Solution Steps of the Problem Expressed in Spoken Algebra | Translation of the Statement and of the Solution Steps in Symbolic Algebra |
|------|--|--|
| 12 | But we have shown that the straight line CD is the root of four $māl$ minus two-fifths of one-tenth of $māl-māl$. | We have shown that $CD = \sqrt{4x^2 - \frac{2}{5}\frac{1}{10}x^4}$. |
| 13 | Multiply three things minus two-fifths of one-tenth of a cube by themselves; we get nine $māl$ and one part of six hundred-twenty-five parts of cube-cube minus six parts of twenty-five parts of $māl-māl$ equal to four $māl$ minus two-fifths of one-tenth of $māl-māl$. | Multiply $3x - \frac{2}{5}\frac{1}{10}x^3$ by itself, and we obtain: $\left(3x - \frac{2}{5}\frac{1}{10}x^3\right)^2 = 9x^2 + \frac{1}{625}x^6 - \frac{6}{25}x^4.$ |
| 14 | Reducing from this, we have a fifth of $māl-māl$ equal to five $māl$ and a part of six hundred twenty-five parts of a cube-cube. | Reduce this and we get $\frac{1}{5}x^4 = 5x^2 + \frac{1}{625}x^6$. |
| 15 | Dividing all that you have by a $māl$, you get five “dirhams” and one part of six hundred twenty-five parts of $māl-māl$ equal to one-fifth of $māl$. | Divide all by x^2 , and we obtain $5 + \frac{1}{625}x^4 = \frac{1}{5}x^2.$ |
| 16 | Make whole $māl-māl$ for which you have a $māl-māl$, and so multiply it by six hundred twenty-five, so all that you have by six hundred and twenty-five; we have a $māl-māl$ plus three thousand one hundred twenty-five “dirhams” equal to one hundred twenty-five $māl$. | Multiply all by 625, and we have $x^4 + 3125 = 125x^2.$ |
| 17 | Divide the $māl$ into two halves, get sixty-two and a half; | Divide $125x^2$ by half, and we have $\left(62 + \frac{1}{2}\right)x^2$; |
| 18 | multiply them by themselves, you get three thousand nine hundred six and a quarter; | multiply $\left(62 + \frac{1}{2}\right)$ by itself, and we obtain $3906 + \frac{1}{4}$; |
| 19 | subtract three thousand one hundred twenty-five, it remains seven hundred eighty-one and a quarter. | subtract 3125 and we are left with $781 + \frac{1}{4}$. |
| 20 | We remove the root of this one from sixty-two and a half, we take the root of what remains, we have the straight line ED which is one of the sides of the pentagon. | Take away the root of $781 + \frac{1}{4}$ from $62 + \frac{1}{2}$ and we get the root of the remainder; thus, we have the line ED that is one of the sides of the pentagon $ED = \sqrt{62 + \frac{1}{2}} - \sqrt{781 + \frac{1}{4}} \simeq 5.88.$ |

In this way, students can be stimulated to observe how Arabic algebra, as a numerical problem solving technique, uses the properties of the figure considered, expressed by Euclid in the geometric language of the *Elements*, in order to write the equation that solves the proposed problem and then to express themselves with the signs of our current mathematical culture.

In order to facilitate understanding of the learners, in Table 2 a translation of the symbols from spoken algebra to symbolic algebra used in Table 1, is shown.

Students of secondary school can be pointed out that Abū Kāmil starts from Euclid’s result (*Elements* IV, 11), which allows him to think of the given figure as “built”. This gives meaning to the search for the length of its side. This measure constitutes new knowing, produced by diversification through semiotic change. The “dual” and crucial role of Euclidean constructions becomes evident.

Euclid teaches to build and attributes the dignity of existence to objects that become “realizable” and therefore “measurable” with the new creative role in the signs of algebra [34].

Table 2. Table useful to convert from rhetorical algebra into symbolic algebra for the fundamental operations.

| # | Expressions in Spoken Algebra | Expressions in Symbolic Algebra |
|----|--|---|
| 1 | a fifth (or one-fifth) | $\frac{1}{5}$ |
| 2 | two-fifths | $\frac{2}{5}$ |
| 3 | a tenth (or one-tenth) | $\frac{1}{10}$ |
| 4 | one-tenth of one-tenth | $\frac{1}{10} \cdot \frac{1}{10} (= \frac{1}{100})$ |
| 5 | two-fifths of one-tenth | $\frac{2}{5} \cdot \frac{1}{10} (= \frac{1}{25})$ |
| 6 | one-part of six hundred-twenty-five parts | $\frac{1}{625}$ |
| 7 | six parts of twenty-five parts | $\frac{6}{25}$ |
| 8 | sixty-two and a half | $62 + \frac{1}{2}$ |
| 9 | a thing (or one-thing) | x |
| 10 | a <i>māl</i> (or one- <i>māl</i>) | x^2 |
| 11 | a cube (or one-cube) | x^3 |
| 12 | <i>māl-māl</i> | $x^2 x^2 (= x^4)$ |
| 13 | cube-cube | $x^3 x^3 (= x^6)$ |
| 14 | five dirhams | 5 |
| 15 | three thousand one hundred twenty five dirhams | 3125 |
| 16 | three thousand nine hundred and a quarter | $3906 + \frac{1}{4}$ |
| 17 | seven hundred eighty one and a quarter | $781 + \frac{1}{4}$ |

Both the drawn figure and the beginning of Abū Kāmil’s speech are in the Euclidean register for the terms introduced, for the use of capital letters to indicate points and segments, and for the proposal to “build” the straight line *CLD*. When Abū Kāmil assumes *CD* equal to “a thing”, the register changes: in fact, he passes from the geometric register to the algebraic register.

At this stage, students can observe and learn, on an elementary example, how the change of register instantly generates the beginning of the process that will then be named mathematical *modeling* by the teacher.

While Abū Kāmil indicates with the term “draw” the movement made in Euclid’s “space”, in which the construction takes place, with the term “assume” he introduces the reader to the algebraic “narrative” and, therefore, to the possibility of taking a measurement. It is in this passage that one can see the beginning of those semiotic transformations described in Reference [29,36,37].

Abū Kāmil continues to transform the “objects” from the Euclidean to the algebraic register, justifying each passage with appropriate Euclidean propositions, up to obtaining for the square of *CD* two expressions in which the “thing” appears. By matching the two expressions, he obtains the equation that allows him to solve the problem. Abū Kāmil so completes the operation that, several centuries later, will be taken up by Descartes in his *Géométrie* (1637). That is, he writes the equation that solves the problem by equating the two different “narratives” that “tell” the same *CD* object multiplied by itself. At this point, he resolves the equation in the algebraic register with the passages he learned from the work of al-Khwārizmī.

Having acquired the foregoing, students can now move towards a general formulation of the statement and the solution of the same proposition, introducing the concept of parameter and reflecting

on the articulated relationship between parameter and unknown. Introducing the concept of parameter and reflecting on the relationship between parameter and variable leads one to imagine dynamic transformations of the figure. In the transition from the particular case to the generalized case, students can be guided to consciously complete the linguistic transformation, to comment on its effects and descriptive effectiveness, gradually acquiring familiarity with forms of increasing complexity and flexibility in interpreting, reflecting, and analyzing.

In the example, if one indicates the diameter of the circle with the parameter $2r$ ($r > 0$), one can rewrite the proposition in the following way:

Find the side of the regular pentagon inscribed within a circle of diameter $2r$.

With the aim to facilitate the understanding of the students, in Figure 3, a pictorial representation of this problem is given.

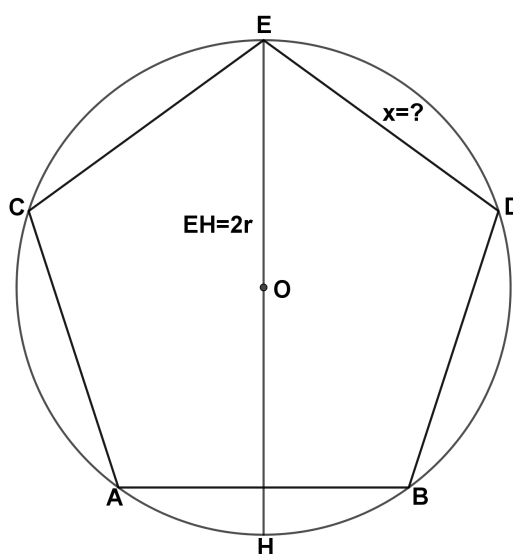


Figure 3. Pictorial representation of the problem proposed to students in its general form.

By going over the steps contained in the right part of Table 1, one can determine the acceptable solution for the problem: $x = r\sqrt{\frac{5-\sqrt{5}}{2}}$.

The new formulation allows us to observe that the relationship between the side x of a regular pentagon and the radius r of the circle circumscribing it is constant and independent of the pentagon considered: $\frac{x}{r} = \sqrt{\frac{5-\sqrt{5}}{2}}$.

In the given semiotic and epistemic reading, this generalization, with the introduction of parameters, has the characteristics of a new core of “knowledge” produced by the “determination” constituted by the moment of the particular solution produced in numerical terms. In this sense, it is configured as an example similar to those described in Reference [3,16].

When a proposition (statement and solution) is formulated in a more general language, a reflection naturally takes on what character the geometric construction offered in the *Elements* assumes with the parameterization.

Considering a “parameter”, that is a “variable” measure among the positive values that can be assumed, gives the construction a more universal reading, that is, a reading that includes all possible cases. Thus, that particular figure, which expresses a specific case, is not considered, if not exceptionally, in its specificity, but rather in the context of the general formulation of the problem: that figure contains in itself all possible cases. Consequently, to look at “that” figure implies a rational and intellectual vision, for which the physical “looking” is only the starting point of observation.

On the sidelines to the solution of Abū Kāmil's problem, it is also possible to reflect on a possible extension of its use within the teaching of plane analytic geometry. This immediately leads to the realization that "treating" the equations does not automatically imply being in the context of the "analytical method". The equation treated in this path has its own consistency, and it helps to understand what it means to operate in algebra. But, if the teacher wants to lead students to give a meaning to this specific equation in the context of analytic geometry, he should "re-read" inside its geometric texture.

In this regard, students can be noted that the ontological character of the solution $x = r\sqrt{\frac{5-\sqrt{5}}{2}}$ of the problem is a length (that of the side of the pentagon). This solution expresses the length of the side of the pentagon in units of the radius of the circumscribed circle; that is, the side of the pentagon, which has the dimensions of a length, is proportional, through the dimensionless constant $\sqrt{\frac{5-\sqrt{5}}{2}}$, to the radius of the circumference, which also has the dimensions of a length. There is therefore a linear relationship between the side of the pentagon and the radius of the circle. Being the latter positive, the equation that describes the dependence between the x side and the radius r will be represented, in the Cartesian plane xr , as a half-line coming out of the origin.

Then, wanting to interpret the problem proposed in the context of analytic geometry, it can be observed that the length of the side of the pentagon depends, "is a function" of the radius of the circumscribed circle. One can express this by writing $x = kr$ (k constant, $r > 0$ and $x > 0$), which represents the half-lines of the first quadrant, each point r, x of which is a solution of the problem. Thus, one is faced with infinite solutions (infinite are the points of a half-line), and the half-line is the "locus" of the solutions of the problem.

In the same chapter of the book *Kitāb fī al-jabr wa al-muqābala*, Abū Kāmil further calculated the length of the sides of the regular decagon (see Reference [1], pp. 526–529) and of the regular polygon with 15 sides (see Reference [1], pp. 546–551) both inscribed in a circle of radius 10. Therefore, the linear relation $x = kr$, already found for the regular pentagon inscribed in a circle, can be generalized in the form: $x_n = k_n r$ in the case of a polygon with n sides. Here, below, the values for the cases $n = 5$, $n = 10$ and $n = 15$ are given:

$$\begin{aligned} x_5 &= k_5 r; & k_5 &= \sqrt{\frac{5-\sqrt{5}}{2}} \simeq 1.18 \\ x_{10} &= k_{10} r; & k_{10} &= \frac{1}{2}(\sqrt{5}-1) \simeq 0.62 \\ x_{15} &= k_{15} r; & k_{15} &= \frac{1}{2} \left[\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{3}}{2}(\sqrt{5}-1) \right] \simeq 0.42. \end{aligned}$$

Such a situation is given a representation in terms of analytic geometry in Figure 4, in which some solutions of the generalized problem, $x_n = k_n r$, for the three values of k_n corresponding to $n = 5$, $n = 10$ and $n = 15$, are shown in the first quadrant of the Cartesian plane. The black half-line, in particular, represents the solution for the side of the regular pentagon inscribed in the circle of radius r and the point S is the solution found by Abū Kāmil, obtained in the case $r = 5$. Analogously, the blue and red half-lines represent the solutions for the regular decagon and polygon with 15 sides, respectively, and the points Q and P are the solutions found by Abū Kāmil.

One therefore arrive, naturally, at *modeling* the locus of solutions of the generalized problem.

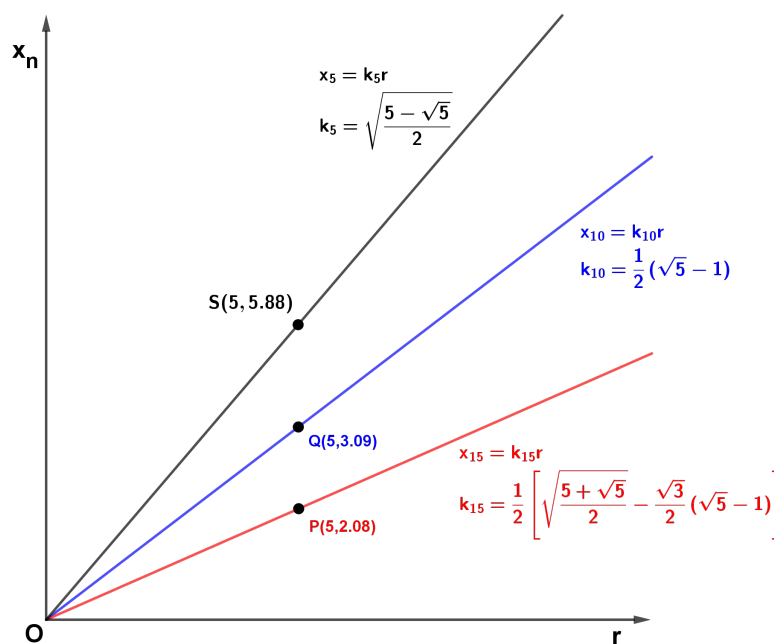


Figure 4. Diagram showing the solution of the generalized problem $x_n = k_n r$ in the first quadrant of the cartesian plane for three different values of n : $n = 5$, black half-line; $n = 10$, blue half-line; $n = 15$, red half-line. The points S , Q , and P lying on the three half-lines represent the solutions found by Abū Kāmil in the case $r = 5$, for the three cited values of n , respectively.

4. Discussion

The didactic proposal presented in this article allows to concretely follow: the realization of a metamorphosis of objects, described in the Euclidean register, into the same objects, told in the algebraic-rhetorical register; the translation of the same objects in terms of symbolic algebra; the transition from the particular formulation of a problem to its general formulation, hinting at its possible use within analytic geometry. All this helps to recognize, in different contexts, results obtained in a specific context, making possible an intellectual experience of a continuum of meaning instead of a discontinuity, which is often found in mathematics teaching, between learning geometry and learning algebra.

The reference to results present in the *Elements*, as well as to the theorem of Ptolemy and that of Pythagoras, within the proof of the proposition, contributes to creating a unitary vision of mathematics and an awareness of the strength inherent in a new semiotic transformation.

The use of history of mathematics to improve learning abilities of the students has been largely discussed in several works. The proposed path could even be used in the framework of the Mathematical Knowledge for Teaching. In particular, comparing the approach of the present work with Reference [2], some similarities with their case study n. 1 emerge, although they used a different (and more modern) geometrical example excerpted from Viète's theory of equations. For instance, the model they show in Figure 1, that provides an analysis of meta-discursive rules in the excerpt from Viète versus modern day meta-discursive rules, can be compared to Table 1 of the present work. This shows how examples of this kind could be of fundamental importance not only for the learning process of students but for the training of teachers, as well. This kind of approach could be the starting point for further developments that will be given in future works.

5. Conclusions

In this work, a possible didactic path has been presented, that could be followed in a secondary school class, in which it is shown a relationship between geometric knowledge and algebraic

knowledge. The starting point for this didactic path was the first proposition of a chapter of the book *Kitāb fī al-jabr wa al-muqābala* written by the Arab mathematician Abū Kāmil before the year 870. Starting from Abū Kāmil's proof, based on Euclidean geometric properties and expressed in the language of spoken algebra, and afterwards in the language of symbolic algebra, the learners can apprehend to "translate" from a mathematical language to another. This is a paradigmatic example to experiment with the changes in registers with which mathematics is expressed and to assess how such a procedure can be received by the students. This kind of approach can help the learners to be trained with the process of generalizing a problem and constitutes a starting point in mathematical modeling. This is the basis for modern science, as well as also a fundamental component for the application of computer science to the solution of scientific problems.

This didactic path has the further advantage of making the students aware of the huge power of symbolic algebra with respect to spoken algebra. Not only the language of symbolic algebra is more compact and versatile, when compared to spoken algebra, since it can treat similar situations with the same formalism, but also the use of symbolic algebra is a necessary prerequisite for mathematical modeling. Last but not least, the use of symbolic algebra makes the communication of scientific results almost independent on the language spoken by the persons, resulting in a more "universal" approach to scientific advancements.

Reflecting on the link between the formulation of a problem and its geometric construction [44,45], understanding how the "feasibility" of the geometric construction supports the possibility of "narrating" the objects of the image in algebraic language up to identify an equation that models the problem, understanding the character of the root that solves an equation [51], identifying one of the first historical sources of the long path that will lead to express similar problems in the context of the analytical method, etc., all opens culturally broader horizons in which successive mathematical theories will develop and opens the students' mind towards new situations that stimulate their creative potential.

6. Patents

There is no patent resulting from the work reported in this manuscript.

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