Preservice Teachers' Involvement in the Dynamic, Messy and Nonlinear Problem-Solving Process

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Abstract

This paper details an exploratory study of ten elementary preservice teachers (PSTs) involvement in the problem-solving process in a mathematics methods course. The dynamic, messy and nonlinear nature of this process was demonstrated by discussing PSTs' solution process of an open box construction problem. Effective approaches to solve this problem were discussed. Polya's (1957) four-step framework was presented to illustrate the problem-solving process as a more realistic and authentic endeavor, including the importance of using this framework with discretion. The eight Mathematical Practices (Common Core State Standard for Mathematics, CCSSM, 2010) were used to explore the PSTs' problem solution process.

Keywords: area models, elementary education, geometry, mathematical knowledge, mathematics curriculum, mathematics education, pedagogical content knowledge, preservice teacher education, problem-based mathematics learning, problem solving, real-world problems, student engagement, surface area, volume

Introduction

Problem solving provides a working framework to apply mathematics, and well chosen mathematics problems provide all students with the opportunity to solidify and extend what they know and stimulate their mathematics learning (NCTM, 2000). This framework affords students at any level the opportunity to augment their depth of knowledge of concepts and skills that are rich with meanings and connections. For preservice teachers (PSTs), it prepares them to help their future students. As a teacher or learner, it is important to use this framework with discretion and understand it as a dynamic, messy and nonlinear process. "Finding great problem-solving situations is a challenge, but it is crucial if we want to be effective" (Ortiz, 2016c, p. 10). An open box construction problem was used to analyze ten PSTs use of the problem-solving process based on the Common Core State Standards for Mathematics (CCSSM, 2010) eight Mathematical Practices. The PSTs were part of a bachelor's degree program in elementary education at a metropolitan area university in Florida, and enrolled in a mathematics methods course.

Four-step Problem Solving Framework

It is interesting to hear some people say how much they "hate" mathematics, but when asked about their jobs, they talk about how much they "love" their jobs, which in many cases turns out to be connected to mathematics. I think that this disconnection is symptomatic of how mathematics is in some cases presented in mathematics classrooms. In this paper, the following Polya's (1957) four-step problem-solving framework was used as a model for presenting mathematics as a more realistic, connected, learnable and authentic endeavor:

1. Understanding the problem [or situation]: The students should want to engage with the task, want to solve it and persevere in solving the problem.
2. Devising a plan: The student works on finding connections between the data and the unknown, consider auxiliary problems if an immediate connection cannot be found, and should eventually obtain a plan to solve the problem.
3. Carrying out the plan: The student implements the plan, checks each step of the solution plan, and persists in completing the plan.
4. Looking back: The student examines the solution obtained. It should be more than just checking the answer for correctness, and possible computational errors. This is a very important step, which is sometimes neglected, which is not a good idea.

Figures 1 (English version) and 2 (Spanish version) present an Infographic illustrating how the four problem-solving steps are interconnected in a nonlinear process (Ortiz, 2016a; Ortiz, 2016b). Polya's (1957) steps were discussed with the PSTs. Using a copy of the diagram, I ask them to think about the following questions: What do you think and wonder? What concepts, or ideas can you identify? What other strategies would you add? For example, we discussed the meaning of the sentence at the bottom of the diagram, "Go back to any of the steps as necessary" or "Ve a cualquiera de los pasos según sea necesario", as a reminder of the cyclical, dynamic and interconnected nature of this process.

Adaptations of the Four-Step Problem Solving Process

Polya's (1957) problem-solving steps have been adapted and renamed. For example, McGraw Hill Education (2020) offers a variation: 1. find out (what do you need to find out to solve the problem), 2. choose strategies (you can use logical reasoning and act out or use objects), 3. solve it (put cubes on the ducks to show the colors), and 4. read the problem again and check your work. The 3-Act Math format (Meyer, 2011; When math happens: 3-Act Math, n.d.) represents another adaptation involving images and videos: *Act One* **(**introduce the central conflict of your story/task clearly, visually, viscerally, using as few words as possible), *Act Two* **(**the protagonist/student overcomes obstacles, looks for resources, and develops new tools), and *Act Three* **(**resolve the conflict and set up a sequel/extension) (Meyer, 2011). The 3-Act Math Tasks foster students' curiosity, and advocate the use of real-world mathematics problems to make mathematics contextual, visual and concrete (Tap Into Teen Minds, 2018; Fletcher, n.d.). From business, the American Society for Quality (2020) indicates the following steps: 1. define the problem, 2. generate alternative solutions, 3. evaluate and select an alternative, and 4. implement and follow up on the solution. You might be able to add other adaptations.

Figure 1

Problem-solving process diagram (Ortiz, 2016a)

Figure 2

Diagrama del proceso de solución de problemas (Ortiz, 2016b).

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Real-world Situations Involving the Problem-Solving Process

Furthermore, I presented the problem-solving process using real-world jobs. One example is about a lawyer who is defending her client. The lawyer does not know for sure if the client is innocent or guilty. She thinks about the situation and starts visualizing the possible defense. She intuitively beliefs her client is innocent and recognizes a possible strategy to defend her client, but needs more to demonstrate her clients' innocence to the jury. She might star with a hutch, but she also needs to immerse herself in the problem-solving process and "proof" her client's innocence. She needs to understand the case, devise a defense plan, carry out the defense plan, and look back to assess the effectiveness of the defense and in some cases going back to any of the other steps by re-understanding the situation or devising/implementing a new defense plan: Did she miss any evidence? Does she need to understand the case from a different point of view? How effectively she uses the problem-solving process could determine her success. Hunches, guesses, intuition and creativity are great, but she also needs convincing solutions.

After the initial discussion, PSTs provided and discussed their own real-world jobs examples. Such as a medical doctor who is trying to find what is a patient's illness: What are the symptoms? What diagnostic tests are needed? What are the possible treatment strategies? After a while, the prognosis might need to be changed and the treatment too, which implies going back and forth as the steps are implemented. Similarly, they talked about car mechanics, scientists, carpenters, engineers, architects, inventor, artists and teachers.

Mathematical Practices

This paper discusses PSTs' responses as they solved an open box construction problem using Polya's (1957) problem-solving steps, and connections to the eight *Mathematical Practices* (*MPs*) (CSSM, 2010): MP1. Make sense of problems and persevere in solving them; MP2. Reason abstractly and quantitatively; MP3. Construct viable arguments and critique the reasoning of others; MP4. Model with mathematics; MP5. Use appropriate tools strategically; MP6. Attend to precision; MP7. Look for and make use of structure; and MP8. Look for and express regularity in repeated reasoning.

BoxIn Company Problem

An open-box construction problem was presented to the participants organized in three groups of three or four. The following problem provided them opportunity to experience a dynamic problem-solving environment, which allowed them to present and argue about different points of view:

> The *BoxIn Company* is a shipping supply specialist. It has a section of a warehouse full of cardboards that are used to make boxes. This company has hired you, and one of your tasks is to explore the possible design of an open (no top) box, which needs to have one pair of opposite box sides each having the same dimensions as the base of the box. What would be the approximate dimensions and volume of this open box?

It provided a rich problem-solving opportunity related to two grade 6 domains (CCSSM, 2010):

- Geometry: Solve real-world and mathematical problems involving area, surface area, and volume, and
- Expressions and Equations: Apply and extend previous understandings of arithmetic to algebraic expressions.

Implementation of the Problem-Solving Process Example

The following is a discussion of each of the four problem-solving steps (Polya, 1957) based on PSTs' work related to the *BoxIn Company Problem* and *MPs* (CCSSM, 2010): *Understand the problem/situation.* At the beginning, the problem was kept very open ended. No cardboard dimensions or volumes were provided on purpose to challenge the participants' imagination and creativity. They talked about the different ideas involved in the problem, and asked about how it could be constructed. I explain that the open box has no top and that it could be made by cutting same size squares from each corners of the cardboard (see Figure 3). They understood that you might not know all the details of a problem, need more information along the way, and some information might be unnecessary. Also, they asked about what content or things will be placed in the box. I told them that we do not need to know this information now, and the company will decide which open box works best. Some of the initial strategies used by them were to read and reread for understanding, identify required information, visualize by using drawings, identify goal and unknown, and ask clarifying questions.

Figure 3
Paper with squares removed from corners.

The following is a discussion of the first seven *MPs* (CCSSM, 2010). *MP1* was especially prevalent in all four problem-solving-steps. PSTs were very involved and engaged during this activity. The level of engagement was a good indication of their sense of perseverance. Otherwise, we need to assess their level of interest and familiarity with the problem's.

MP2 was also involved as participants started to make sense of the problem (CCSSM, 2010). In this sense, the learning cognitive levels *concrete*, *representation* (*pictorial*), and *abstract* (*CRA* or *CPA*) were taken into account. For the *concrete* level, cubes were provided as a pedagogical material to model volume. For the *representational* level, rectangular paper, images

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and diagrams were used to represent the problem or record understanding. For the *abstract* level, they talked and wrote about their learning (verbal), and used symbols to record understanding (symbolic) (Ortiz, 2005). An additional cognitive level consideration was the *virtual level* involving apps and applets (Ortiz, 2017; Ortiz, Eisenreich, & Tapp, 2019), but this level was not included in this study.

MP3 encompasses participants arguing and critiquing their interpretation of the problem as they look for understanding (CCSSM, 2010). It was important to create an environment of respect and trust. This process involved PSTs' analysis of each other answers, misconceptions and errors (Pace & Ortiz, 2016). *MP4* was involved when they needed to use their mathematics knowledge to understand the problem (CCSSM, 2010). It was important to provide differentiation by using multiple entry points so that all of them were able to gain access to the problem (Van de Walle, 2004). *MP5* involved the use of appropriate tools strategically (CCSSM, 2010). They selected as a group the tools to use during the activity. If asked, I was open to provide other tools.

MP6 required attention to precision from the start of the problem-solving process (CCSSM, 2010). Proper use of definitions in discussions and reasoning were imperative. PSTs had the opportunity to revise the precision of the definitions as they moved through the process. For example, one group was incorrectly using the definition of perimeter for surface area. They decided to go back to the initial problem and reassess their definition of surface area. They also were careful with the definitions of surface area and volume.

When it comes to *MP7*, PSTs and students in general need to be able to look closely to the problem and discern if there was a pattern or specific structure and if this pattern or structure was important in the understanding of the problem (CCSSM, 2010). The *BoxIn Company Problem* required proper understanding of how an open box was constructed and how changes made in how the corners were cut affected the mathematical structure. In general, all the groups started with random corner square sizes, but eventually were able to revise their understanding, recognize the significance of the structure and shift perspective.

Devise a plan. One group suggested the use one cardboard size, and experiment with it. I asked them to only use whole numbers for measures. For consistency and initial explorations, they agreed to try the *8-inch by 11-inch paper*, scissors, rulers, calculators and graphing paper. The main challenge was to make sure they were able to delimitate the situation to set up the problem, and were engaged. *MP3* continued to be very relevant because they needed to construct viable arguments, and critique their reasoning. As it relates to *MP8* and devising a plan, some of them noticed that some calculation processes were repeated as the corners were cut and talked about possible short cuts to simplify the process. They paid close attention to their calculations and made an effort to find regularities. They decided to make box models and check for possible alternatives. At this time, other solution plans were expected; for example, the use of Excel worksheet to analyze and make graphs using the information from the different possible boxes, which was not considered.

Carry out the plan. By now PSTs understood that by cutting congruent squares from the corners, folding up the sides, and taping them could form a box without a top, and that the open box they made provided a specific capacity (see Figures 3 and 4). If necessary, they had the option to go back to any of the problem-solving steps.

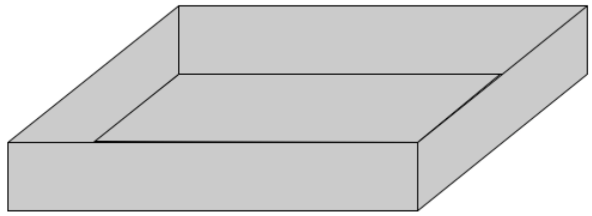

Figure 4

Paper folded (after cutting the corners) to form open top box.

In small groups, participants used the paper provided to make at least one box to see how it could work. For example, see Figures 5 the net involving a *4 by 7 by 2-inch open box using an 8-inch by 11-inch piece of paper*, and Figure 6 to see the folded box using these dimensions. They noticed that this open box did not work because it did not satisfy the require condition of "*one pair of opposite box sides each having the same dimensions as the base of the box*". For this one, we had that the two pairs of opposite sides of the open box did not have the same dimensions as the base: one pair resulted in *2 by 7-inch rectangles*, the other pair in *2 by 4-inch rectangles* and the base a *4 by 7-inch rectangle*. The participants used other corner sizes using the *8 by 11-inch piece of paper* to construct other open boxes. One group cut *1-inch corner squares*, which did not work either: one pair resulted in *1 by 9-inch rectangles*, the other pair in *1 by 6-inch rectangles* and the base a *6 by 9-inch rectangle*. Another group cut *3-inch corner squares*, which did not work either: one pair resulted in *3 by 5-inch rectangles*, the other pair in *3 by 2-inch rectangles* and the base a *2 by 5-inch rectangle*.

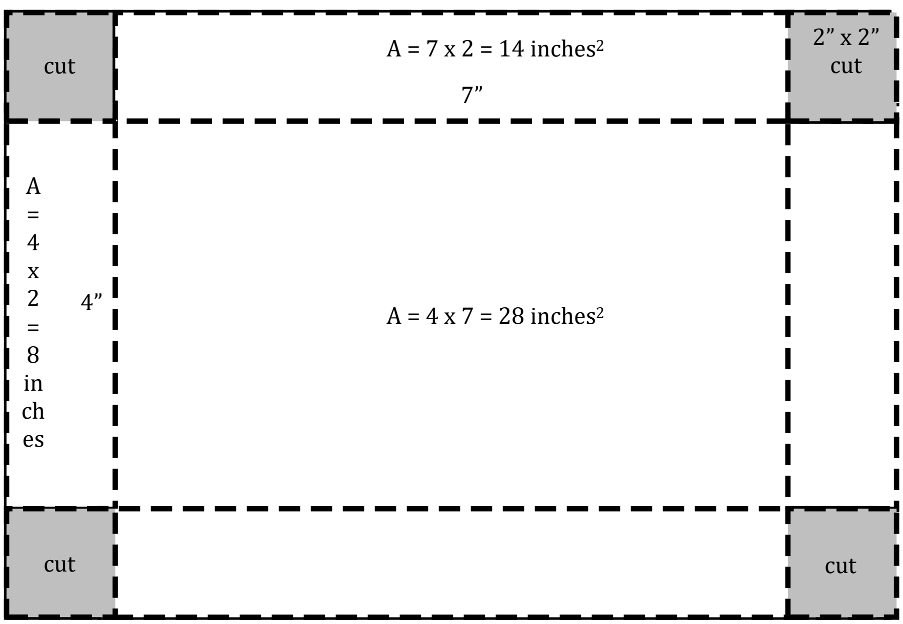

Figure 5

Group work for a 4 by 7 by 2-inch open box using an 8-inch by 11-inch piece of paper.

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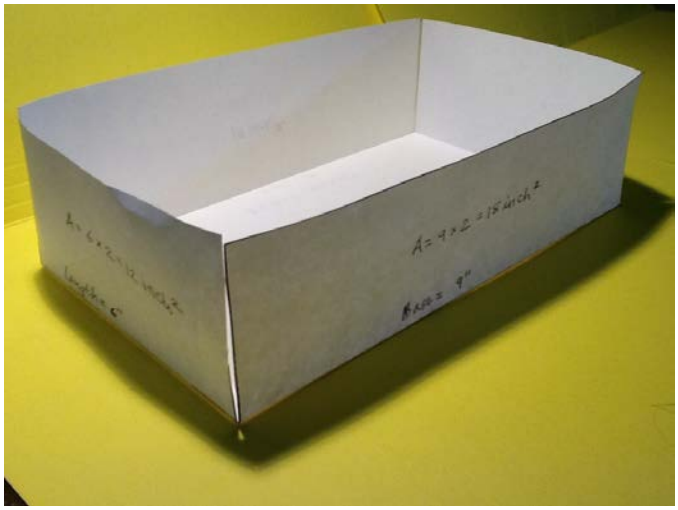

Figure 6

Model for a 4 by 7 by 2-inch open box using an 8-inch by 11-inch piece of paper.

After discussing the finding in small and whole groups, the class decided to check their understanding of the problem, and their plan. By now, all of them had a good grasp of the problem. They realized that the size of the paper needed to be adjusted to make it work. Some of the techniques used were making a model, drawing diagrams or nets, asking questions, allowing for mistakes, and sharing and discussing findings.

The groups planned to be more flexible with the size of the piece of paper, and cut the piece of paper to fit what they wanted. One group tried a *6 by 11-inch piece of paper* and cut *2-inch corner squares*: one pair of opposite sides resulted in *3 by 3-inch rectangles*, the other pair in *2 by 3-inch rectangles* and the base a *2 by 3-inch rectangle*. They constructed a *2 by 2 by 7-inch open box* that met the required condition with a volume of *28 cubic inches* (Figure 7). This answer provided evidence of an open box with two opposite sides with the same dimensions.

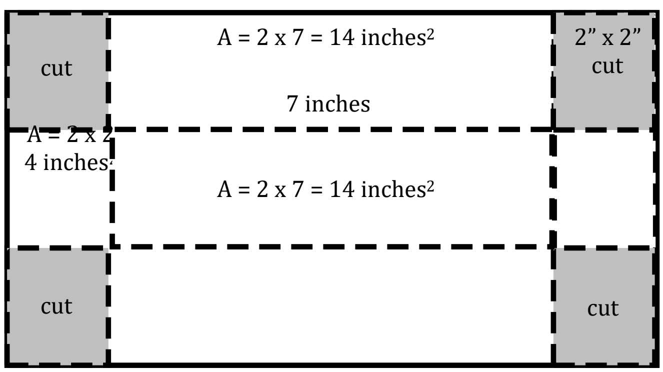

Figure 7

Group work for a 7 by 2 by 2 inch box using a 6-inch by 11-inch paper.

Similarly, another group used graphing paper to illustrate their answer (see Figure 8) using scale drawing (*1 graphic paper square = 1 square inch*), and *3 by 3 by 2-inch open box* and *volume of 18 cubic inches*. It had one pair of opposite sides with *3 by 3-inch rectangles*, another pair with *2 by 3-inch rectangles* and the base with a *2 by 3-inch rectangle*. The group noticed that two opposite sides had to be squares and with the same dimensions as the square corners.

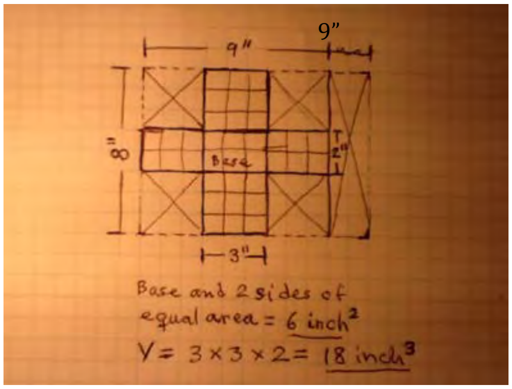

Figure 8
Group work for 3 by 3 by 2-inch open box using graphing paper in 8 by 9-inch rectangle drawn on the graphing paper.

Look back. As a whole group, we discussed the findings of the small groups, and revisited the problem-solving steps (Polya, 1957): What new understanding of the problem do we have? What possible answers do we have so far? Are the answers we have found so far correct and do they make sense? How could we used what we learned to solve similar problems? In general, we decided to keep looking for a generalization. We knew that the dimensions of the corner square had to be the same dimensions as a pair of the opposite sides of the open box. If the size of the corner square was x inches long, then the one of the sides of the open box must be $x + x + x = 3x$ *inches long* or a multiple of 3: 3, 6, 9, 12, … They used 8 inches for the other side, which was familiar to them. Figure 9 provided a visualization of this generalization, where a pair of the opposite sides of the cardboard were *8 inches long*, and the other two sides were 3*x inches long*: one pair resulted in *x by x-inch rectangles*, the other pair in *x by (8 – 2x)-inch rectangles* and the base a *x by (8 – 2x)-inch rectangle*. This open box meet the criteria of having two opposite side with the same dimensions (*x by (8 – 2x)-inches*) as the base of the open box (*x by (8 – 2x)-inches*).

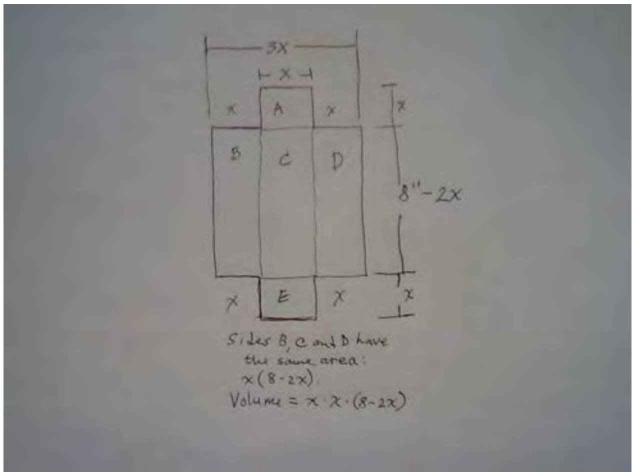

Figure 9
Group work for 3x by (8 – 2x) by x-inch open box using a 3x by 8-inch paper.

We went a step further by using *y inches long* for the other pair of opposite sides of the open box and the opposite sides of the carboard as $2x + y$ (see Figure 10): one pair of opposite sides resulted in *x by y-inch rectangles*, the other pair in *x by x-inch rectangles* and the base a *x by y-inch rectangle*. This met the criteria of two opposite sides with dimensions equal to the base of the open box. The volume of this open box is $x \bullet x \bullet y$ or x^2y.

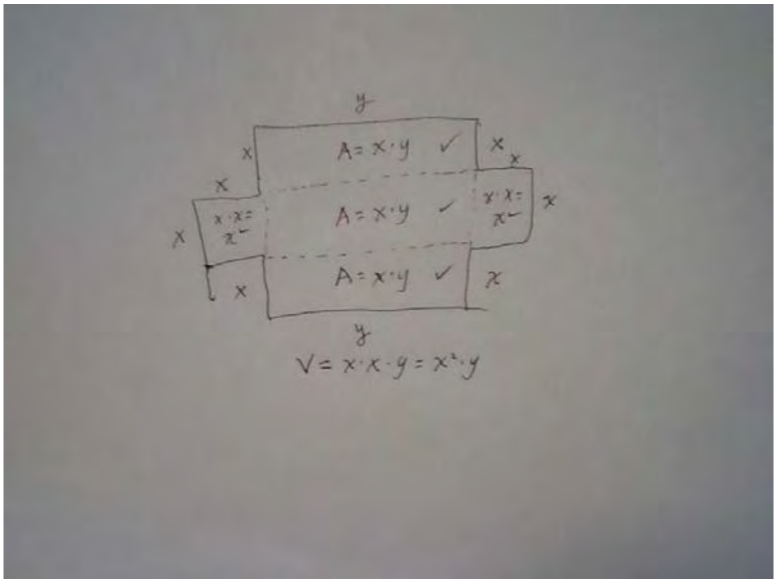

Figure 10

Diagram for x by x by y-inch open box from student B work using a 3x by (y + 2x)-inch paper.

This volume formula provided a way to play with the possible open boxes we can construct. One student noticed that if you select a specific volume and value for x, such as volume equal to *20 cubic inches* and x=2, then $V = x^2y$ or $20 = 2^2 \bullet y$; $20 = 4y$; $20/4=y$; $y = 5$. The dimensions of a pair of the opposite sides of the cardboard need to be $3x$ or 6 inches long, and the other pair needs to be $2x + y$ or $2(2) + 5 = 9$ inches long. Similarly, another group tried 30 cubic inches for the volume, then $2 \bullet 2 \bullet y=30$ or $4y=30$. We cannot find a whole number value for y that gives 30 as a product, but we can use a fraction (y=7½).

With these ideas, they realized that they could tailor make the open box if they had enough information for the *BoxIn Company*, or at least could check if the open box could be made given specific boundaries. As with *MP3* (CCSSM, 2010), it is very relevant to emphasize the need for constructing viable arguments and critiquing the reasoning of others.

Conclusion

PST and in-service teachers need to become adept at providing proper problem-solving experiences to their students. They need to start by experiencing this dynamic, messy and nonlinear process themselves. These experiences will help them with proving their students with appropriate problem solving experiences, and understanding of where to help or provide a hint. You might prematurely give away the solution to the problem. You do not want to be too helpful and diminish their "Eureka!" moments.

Most word problems presented to students are already neatly set up. Most real-life problems are very messy. In some cases, they might start to solve a problem without a complete understanding of the problem, but this challenge should not stop them from trying and wanting to find a solution. They might only have a hunch or intuitive sense of knowing the answer. They might tell you, "I know the answer, but I don't know how to show how I got it." In some cases, this is a legitimate point of view, and one that should not be taken slightly since it is an important part in of the messiness involved in a nonlinear process. If it is not used properly, the four-step approach could become too linear and might prevent students from being more creative, subjective and inventive. Using your imagination, being able to try your hunches, having an intuitive visualization of the solution must be allowed and carefully considered during this challenging process. As indicated by Schoenfeld (1985), it is possible to teach learners to use general strategies such as those suggested by Polya (1957), but that is insufficient. We need more than going through the motions of a four-step problem-solving approach.

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