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TEACHING RECURRENT SEQUENCES IN BRAZIL USING HISTORICAL FACTS AND GRAPHICAL ILLUSTRATIONS

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Abstract: The present work presents a proposal for study and investigation, in the context of the teaching of Mathematics, through the history of linear and recurrent 2nd order sequences, indicated by: Fibonacci, Lucas, Pell, Jacobsthal, Leonardo, Oresme, Mersenne, Padovan, Perrin and Narayana. Undoubtedly, starting from the Fibonacci sequence, representing the most popular in the referred scenario of numerical sequences, it was possible to notice the emergence of other sequences. In some cases, being derived from Fibonacci, in others, changing the order of the sequence, we thus have the historical study, with the use of mathematical rigor. Besides, their respective recurrence formulas and characteristic equations are checked, observing their roots and the relationship with the number of gold, silver, bronze and others. The results indicated represent an example of international research cooperation involving researchers from Brazil and Portugal.

Key words: Historical-mathematical investigation, Recurring Sequences, History of Mathematics.

1. Introduction

Linear and recurring sequences have many applications in several areas, such as Science, Arts, Mathematical Computing and others, thus showing an increasing interest of scholars in this subject. The Fibonacci sequence is the best known and most commented on by authors of books on the History of Mathematics, being, therefore, an object of study for many years, until today, safeguarding a relevant character (Rosa, 2012). Originated from the problem of immortal rabbits, this sequence is shown in many works succinctly and trivially, not providing the reader with an understanding of how we can obtain other sequences, from the evolution of Fibonacci sequence. (Alves, 2017).

Representing an inexhaustible source of many interesting algebraic identities, the Fibonacci numbers are presented in the form of a recurring sequence, as then being one of the most famous numerical sequences in mathematics and a popular topic for mathematical enrichment. Thus, this sequence appears in numerous mathematical problems and other ludic problems, highlighting a numerical text in which Fibonacci does an important job for the theory of numbers and solutions of algebraic equations (Posamentier & Lehmann, 2007).

Therefore, it is possible to notice the appearance of several recurring linear sequences, as a result of the mobilization of mathematical scholars from all over the world, with the aim of providing new ways of generalizing it. In view of this, this work has the interest of discussing the historical and epistemological process of evolution of linear and recurrent sequences, through a historical and mathematical bias (Plofker, 2009).

The deep interest of this research developed from a scientific cooperation involving Brazil and Portugal, was seen in view of works referring to these sequences, in which they treat their respective mathematical, historical and evolutionary processes individually and quickly, with a greater emphasis in the area of Pure Mathematics. On the other hand, it is very important for the students to understand the diversity of applications and examples in the environment, in which we can identify certain notions intrinsically related to these abstract entities. With this, a scientific dissemination of the recurring sequences known to date has been structured. however, disregarded by the authors of History of

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Mathematics books used in Brazil, with attention intermediating their historical and epistemological process, in addition to comparing their respective recurrence formula with Fibonacci numbers. Starting immediately, we have a brief description of what is a linear and recurring sequence, for a posteriori to develop a historical and mathematical study of these numbers.

2. Recurring linear sequence

A numerical sequence is understood as an infinite, ordered list of real numbers, where a term depends on its predecessor. A linear recursive sequence, on the other hand, is one in which it has an infinite number of terms, which are generated by a linear recurrence, called the recurrence formula, allowing the calculation of its predecessor terms (Zieler, 1959). It is emphasized that in any linear sequence, it is necessary to know its initial terms and that they can be arbitrarily chosen and determined.

Lima et. al (2006) present the orders of a recursive linear sequence, in which, from a mathematical point of view, for a first order, there is a recurrence expressed by x_{n+1} in function of x_n. It is said to be linear if, and only if, this polynomial function of the first degree. Homogeneous second-order sequences, with constant coefficients, have recurrences of the form $x_{n+2} + px_{n+1} + qx_n = 0$, in which $p, q \in \mathbb{R}$, with $q > 0$, as the case may be $q = 0$, there is a first-order recurrence. For each second order linear recurrence it will be associated with the second degree equation, called characteristic equation. Similarly, homogeneous third-order sequences, with constant coefficients, have recurrences of the form $x^3 + rx^2 + px + q = 0$:, where, with or e is associated with a third degree algebraic equation, indicated by, called the characteristic equation determined by this sequence.

In the subsequent section, we will see some aspects of a recurring homogeneous 2nd order sequence, which is usually the most discussed by the authors of History of Mathematics books.

2. 1. The Fibonacci sequence

The Fibonacci sequence was created by the mathematician Leonardo Pisano (1180-1250) known as Fibonacci or "son of Bonaccio". Born in Pisa, Italy (see Figure 1), Leonardo acquired a mathematical knowledge of the Arab world and in the areas of Algebra and Arithmetic, being remembered for the problem of the reproduction of immortal rabbits, thus generating the Fibonacci sequence (Santos, 2017) which is usually the most commented by authors of History of Mathematics. (Alves, 2017).

Figure 1. ***Leonardo Pisano (Fibonacci). Source: Silva (2017, p. 3).***

This sequence was studied, primarily, at the time of the Middle Ages in Europe, highlighting the movement of the crusades, with the bias of planning attacks against Muslims in Egypt (Posamentier & Lehmann, 2007). Fibonacci wrote 5 works, among which one of the proposed problems stands out, in which it marks the genesis of the Fibonacci method: "How many pairs of rabbits will be produced in a year, starting with a single pair, if each month generates one new pair that becomes productive from the second month, counting that no rabbit dies during that period? "(Boyer, 2006). When answering this question you will find a sequence of integers, a sequence called the Fibonacci sequence.

Definition 1. The Fibonacci sequence (F_n), has the fundamental recurrence, for every interger $n \in \mathbb{N}$ and for the initial values $F_0 = 0, F_1 = 1$, indicated by:

$$F_n = F_{n-1} + F_{n-2}, n \geq 2.$$

Generalizing the formula indicated in definition 1, we can also write it as follows:: $F_n = xF_{n-1} + yF_{n-2}, n \geq 2$, for every interger $n \in \mathbb{N}$. Based on this argument, we will explore other numerical sequences, where in some cases we can see the similarity with this mathematical formula.

Performing elementary algebraic operations, it is possible to obtain its characteristic equation given by $x^2 - x - 1 = 0$, in which its positive solution is given by the gold number (value of approximately 1.61) and the other solution is given by a negative number. There is also a way to represent these numbers geometrically. Thus, two squares are inserted with sides initially equal to 1, and then a new square is inserted according to the largest side of the new polygon formed with the junction of the two initial squares, and so on. This representation is called the Fibonacci spiral or the golden Fibonacci spiral, reconstructed in Figure 2 below, using the Geogebra software.

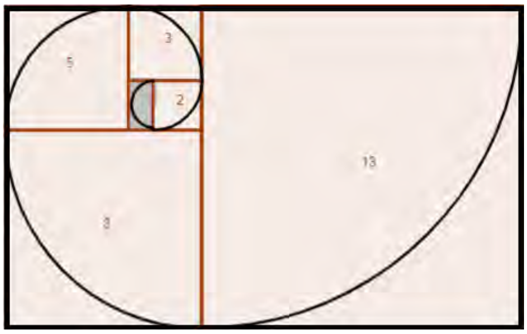

Figure 2. ***Spiral of Fibonacci. Source: Prepared by the authors.***

There are countless problems and applications of this sequence in the field of Sciences, for example, besides the problem of immortal rabbits, there is also the problem of drones. According to Waldschmidt (2018), male bees (drones) are born from unfertilized eggs. Female bees are born from fertilized eggs. Therefore, males have only one mother, but females have both mother and father. When a woman is born, she already carries a male egg. With that, it is possible to construct the representative scheme of this reproduction (see Figure 3).

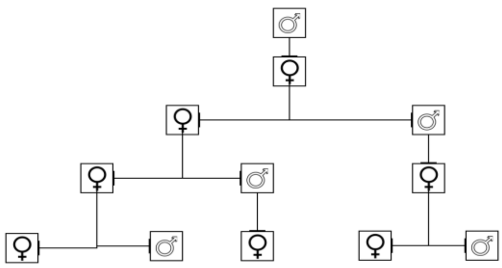

Figure 3. ***Tree reproduction of bumblebees and bees. Source: Prepared by the authors.***

It can be seen that, in the first generation, we have only 1 drone, in the second generation, we have a female and a drone egg. In the third generation, we have a female and a drone, plus an egg. In the fourth generation, we have two females and a drone, plus two eggs. With that, we can identify that for drones, we have the presence or the numerical correspondence of 1,1,2,3,5. For female bees, we have: 0,1,1,2,3. Thus representing the Fibonacci numbers.

These numbers are still present in several places, such as in the leaves of the trees, in the petals of the roses, in the fruits, in the human body, in the spiral shells of snails or even in galaxies. We can see the presence of the Fibonacci spiral on Monalisa's face (see Figure 4). The perfect rectangle is formed by creating rectangles in the corresponding dimensions of approximately 1.61, from each descending Fibonacci number (8, 5, 3, 2, 1, etc.), this spiral has been touching each side in the perfect rectangle . First the painter Leonardo da Vinci framed the woman in the painting, then inserts the spiral starting at the left wrist and then travels to the bottom of the image, which contrasts with the beauty of the face. Then it slides over the forehead and continues to rotate to the chin. He goes up, going through the slight dimple. Finally, it completes a rotation that ends at the tip of the nose. Let's see the image below in figure 4.

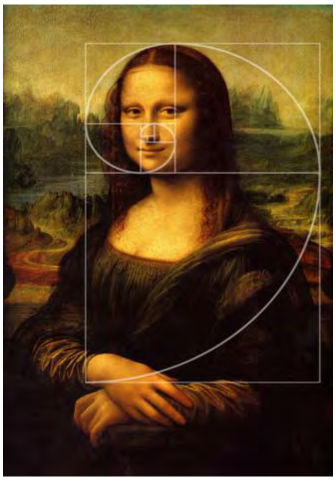

Figure 4. ***Monalisa's face and the Fibonacci spiral. Source: https://thefibonaccisequence.weebly.com/mona-lisa.html***

2. 2. The Lucas sequence

Lucas's sequence was created by French mathematician Édouard Anatole Lucas (1842-1891) (see Figure 5), who made some mathematical contributions such as the well-known Hanoi tower, providing methods for the process of teaching exponential function among other mathematical subjects . E. A. Lucas was a math teacher at some schools and served in the artillery army after France was defeated during the Franco-Prussian War (Silva, 2017).

Figure 5. ***Édouard Anatole Lucas. Source: Silva (2017, p. 21)***

Édouard Anatole Lucas proved a reciprocal form of Fermat's theorem and several tests for prime numbers based on linear and recurring sequences, thus being able to establish a relationship of the 12th prime number of Mersenne, consisting of a number of 39 digits that remained as being the largest prime number for many years. This number was still considered the largest prime number found without the aid of computational and technological resources (Eves, 1969).

Motivated by issues of divisibility and factoring, he studied the Fibonacci sequence and in one of his generalizations, created the Lucas sequence, where he changed only the two initial values to 2 and 1, remaining with the same recurrence seen in Definition 1 below, forming a second order sequence, linear and recurring.

Definition 2. The Lucas sequence (L_n), has the fundamental recurrrence, for every interger $n \in \mathbb{N}$ and the initial values $L_0 = 2, L_1 = 1$, indicated by:

$$L_n = L_{n-1} + L_{n-2}, n \geq 2.$$

Its characteristic equation is identical to that of Fibonacci, presenting the same relationship with the gold number. A. Lucas also formulated a problem of combinatorial analysis, given by the following question: how many ways can couples sit in different chairs around a circle, so that people of the same sex do not feel together and that no man stands by the side your wife? This problem was formulated primarily by Peter Guthrie Tait (1831-1901) and a few years later by Lucas, but in 1943, Irving Kaplansky (1917-2006) found a solution to the problem and published it in his work (Kaplansky, 1943).

The resolution of this problem is given by counting the number of ways that exist when sitting a certain number of couples around a round table, with men and women alternating so that no couple stands beside their partner. Based on Kaplansky's mottos, Ferraz (2017) conducted a study around this problem, exploring the possible existing resolutions, based on formal theorems and mathematical properties.

A geometric representation of these numbers is given by the Lucas spiral, as shown in Figure 6, reconstructed using the Geogebra software, similarly to the Fibonacci spiral, but with the initial squares on the sides, respectively 2 and 1 (Michiel, 2001). We can see some of these geometric and arithmetic properties in figure 6. In the subsequent section, we will develop a study of properties on the Pell sequence.

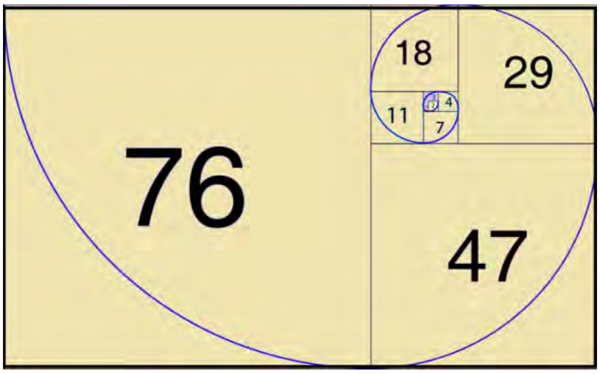

Figure 6. ***Sspiral of Lucas. Source: Prepared by the authors.***

2. 3. The Pell sequence

The name or term Pell attributed to the Pell sequence, originates from the important contribution of the English mathematician John Pell (1611 - 1685) (see Figure 5), known for being one of the most enigmatic mathematicians of the 17th century (Malcolm, 2000, p. 275). Pell acquired credit for the development of the study of Pell's equations, described by $x^2 - Ay^2 = 1$, with whole numbers x, y and A, not whole perfect square. Malcolm (2000) reports that John Pell lived in difficult conditions, constantly without money and resources, that would be a possible explanation for so few publications in his name. However, Waker (2011) states that Pell preferred to remain anonymous, so his publications were not as evident as the works of his contemporaries.

Por fim, Gullberg (1997, p. 29) esclarece que "embora a sequência de Pell foi nomeada após sua morte, nós não encontramos nenhuma outra boa publicação que enfatize sua real e extensiva contribuição." Diante disso, percebemos que quase são inexistentes as publicações e relatos sobre sua competência, por isso torna-se difícil determinar especificamente quais as suas contribuições. In figure 7 we present an image of the mathematician John Pell (1611 - 1685).

Figure 7. ***John Pell. Source: Walker (2011, p. 150).***

In Pell's best-known work, called An introduction to Algebra, he explains and discriminates rules for the handling and simplification of certain equations (Alves, 2016).

Definition 3. The Pell sequence (P_n), has the fundamental recurrence, for every integer $n \in \mathbb{N}$ and the initial values indicated by $P_0 = 0, P_1 = 1$, we have:

$$P_n = 2P_{n-1} + P_{n-2}, n \geq 2.$$

When performing some algebraic operations in the face of this recurrence, we can obtain its characteristic equation represented by $x^2 - 2x - 1 = 0$. Thus, we have that this second order sequence, has two roots, one known as a silver number (value of approximately 2.41) and the other root as being a negative number. Observing the generalized Fibonacci recurrence formula ($F_n = xF_{n-1} + yF_{n-2}, n \geq 2$), we will have that the first coefficient has an accurate value of 2 ($x = 2$), thus remaining with the same initial numerical values.

From the Pell numbers, we can make a geometric representation through the silver spiral, as shown in Figure 8, where the sides of the squares used to build the spiral are the numbers in this sequence. Its construction is based on the golden silver rectangle, tracing the Silver spiral (Teixeira, 2018). Thus, by constructing an ABCD square with side a, we fixpoint D of the previous square. Following the process, the CDEF square is constructed from side a, just adjacent to the first and, we fix the point F. Then, we build another square from side a, adjacent to the first, and we continue the procedure of fixing the point, for this case, point E. However, the process is repeated successively, until a sufficient number of squares are obtained for the construction of the silver spiral. Finally, the Geogebra software arc tool is activated, so that it is possible to build and visualize the spiral shown below.

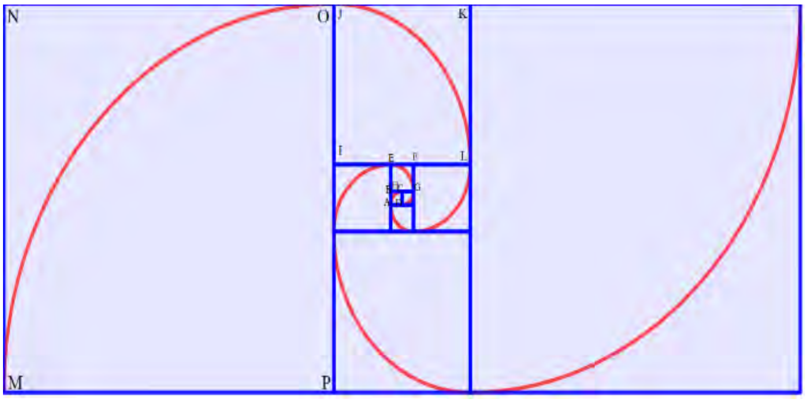

Figure 8. ***Espiral de prata. Source: Teixeira (2018, p. 47)***

2. 4. The Jacobsthal sequence

Receiving this name as a reference to the German mathematician, remembered in Figure 9, with the name of Ernest Erich Jacobsthal (1882-1965), the Jacobsthal sequence, is a second order sequence, with recurrence very similar to the Fibonacci sequence. Jacobsthal was a specialist in Number Theory and a former student of Ferdinand G. Frobenius, being one of the first to study the Fibonacci polynomials (Siegmund-Schultze, 2009).

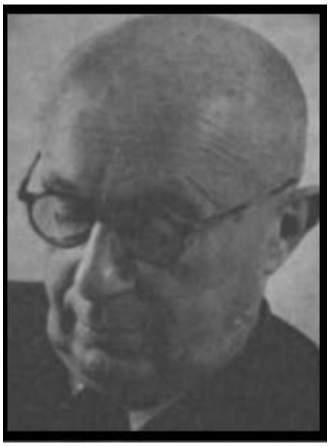

Figure 9. ***Ernes Erich Jacobsthal (1882 – 1965). Source: Siegmund-Schultze (2009).***

This recursive linear sequence is considered to be a particular feature of the Lucas sequence, being widely used to solve mathematical problems regarding the content of combinatorial analysis. Thus, we have a mathematical problem proposed and solved by Craveiro (2004) in relation to tiling, in which Jacobsthal's numbers are used: the amount of tiles q_n that can be calculated for a rectangle ($3xn$) using two types of tiles, one of which is dimensions ($1xt$), in white and the other ($2x2$) in the red, obeys Jacobsthal numbers.

Defining $q_0 = 0$, for a rectangle ($3x1$), we have, for a rectangle ($3x2$), $q_2 = 3$, that is, three types of possible tiles, for a rectangle ($3x3$), $q_3 = 5$ following the Jacobsthal sequence.

Based on the recurrence of the generalized Fibonacci sequence ($F_n = xF_{n-1} + yF_{n-2}, n \geq 2$), we can then change or modify the second coefficient of the formula to 2 ($y = 2$), and remain with the same initial values as the Fiboancci numbers. Alves (2017), points out that this study was previously carried out by the mathematician, highlighting some properties of numerical sequences, based on Fibonacci, giving rise to Jacobsthal numbers.

Definition 4. The Jacobsthal sequence (J_n), has the fundamenal recurrence, for every integer $n \in \mathbb{N}$ and the initial values $J_0 = 0, J_1 = 1$, is given by:

$$J_n = J_{n-1} + 2J_{n-2}, n \geq 2.$$

There are many mathematical properties that are being developed on this sequence, we highlight, in turn, its characteristic equation, obtained from Definition 4, resulting in: $x^2 - x - 2 = 0$ presenting two real roots, which are indicated by the values 2 and -1. The value of the positive root, 2, has a relationship with the copper (metallic) number (approximate value of 2), a member of the family of metallic averages (Spinadel, 1999). This sequence also presents several applications, of which we can exemplify the use of these numbers in the computing area, changing the directives for the execution flow of a program.

In turn, Barry (2003) made a representation of Jacobsthal's numbers, through the construction of Pascal's triangle, based on the recurrence derived from Definition 4, thus obtaining the following relationship $2^n = J_n + J_{n+1}$. Thus, we have in Figure 10, in which the sum of the underlined numbers in each line, represent the terms of the Jacobsthal sequence, highlighting that in the first line, since there is no underlined number, we have the initial value equal to 0.

1

$\underline{1}\ 1$

$1\ 2\ \underline{1}$

$1\ 3\ \underline{3}\ 1$

$\underline{1}\ 4\ 6\ \underline{4}\ 1$

$1\ 5\ \underline{10}\ 10\ 5\ \underline{1}$

$1\ \underline{6}\ 15\ 20\ \underline{15}\ 6\ 1$

$\underline{1}\ 7\ 21\ \underline{35}\ 35\ 21\ 7\ 1$

$1\ 8\ \underline{28}\ 56\ 70\ \underline{56}\ 28\ 8\ \underline{1}$

Figure 10. ***Jacobsthal's Pascal Triangle. Source: Barry (2003, p. 56).***

In the next section, we will study some properties of a sequence recently introduced in the scientific literature. We highlight the contribution of works by Portuguese researchers on the subject (Catarino & Borges, 2020).

2. 5. The Leonardo sequence

Few are the works related to the Leonardo sequence, thus highlighting the pioneering works of Catarino and Borges (2020), Shannon (2019), Alves and Vieira (2019) and Vieira, Alves, and Catarino (2019), in which they define these numbers as second order. In these studies carried out in Portugal and Brazil, the absence in relation to the historical process of these numbers is notable, reporting only their mathematical evolution. In fact, no research was found confirming the creator of this sequence, but it is believed that, because it has the name of Leonardo's sequence and great mathematical similarities, it was created by the mathematician Leonardo Pisano (1180-1250), the same creator of the Fibonacci sequence.

Definition 5. The Leonardo sequence (Le_n), has the fundamental relation, for every interger $n \in \mathbb{N}$ and the initial values $Le_0 = Le_1 = 1$, defined by:

$$Le_n = Le_{n-1} + Le_{n-2} + 1, n \geq 2.$$

Comparing it to the Fibonacci sequence, we can see that the value 1 was added at the end of the recurrence, in addition to its initial values being equal to 1, not more than 0 and 1. The numerical properties of these numbers are analogous to the Fibonacci properties. Thus, Catarino and Borges (2020) established another relationship of recurrence, from Definition 5, as being $Le_n = 2Le_{n-1} - Le_{n-3}, n \geq 3$. The characteristic equation of this sequence can be written based on this recurrence, transforming it into a third-order sequence. Thus, we have $x^3 - 2x^2 + 1 = 0$ and the three real roots, one equal to 1 and the other two equal to the roots of the characteristic Fibonacci equation, highlighting the presence of the gold number (value of approximately 1.61) as a result of one of the positive roots.

The term denoted by (Le_n) represents the number of nodes in the order Fibonacci sequence, which is a complete binary tree with two branches. The branch on the right is the Fibonacci tree of order, and the branch on the left is the Fibonacci tree of order $n-1$ (Knuth, 1998).

Shannon (2019) and Vieira, Alves and Catarino (2019), follow the mathematical evolution of these numbers, introducing imaginary units. In the work of Alves and Vieira (2019), a representation is made by means of Newton's fractal of this sequence, around its respective characteristic equation. First of all, the convergence relationship between the neighboring terms of the sequence and its relationship with the gold number was then discussed mathematically. After that, the Google Colab tool was used to search for the values of the roots of its characteristic polynomial, thus generating the fractal. In Figure 11, we have the 2D fractal on the left, and its 3D representation on the right. It is noteworthy that the 2D image is similar to a bird. The 3D image allows you to easily find the values of the roots of the equation, highlighting the gold number.

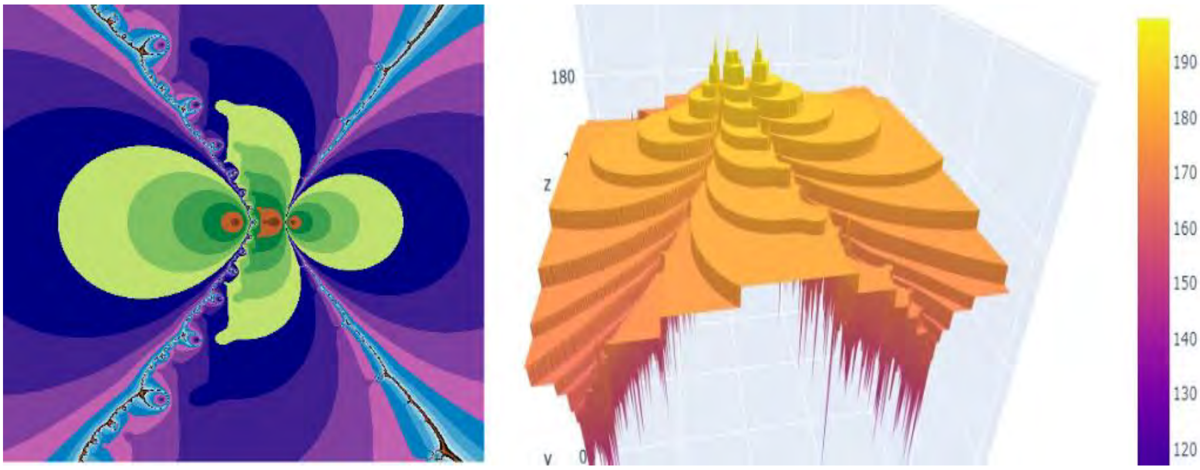

Figure 11. ***Fractal de Leonardo. Source: Alves e Vieira (2019, p. 6).***

2. 6. The Oresme sequence

The Oresme sequence was created by philosopher Nicole Oresme (1320 - 1382), born in Germany, whose image is remembered in Figure 12. Oresme, was one of the most eminent scholastic philosophers of the 14th century, obtaining the title of master and, still assuming the post of the bishop in the city of Lisieux. Among many functions, he was an economist, philosopher, physicist, psychologist, astronomer, astrologer and theologian (Clagett, 1964).

Figure 12. ***Nicole Oresme. Source: Mendonça e Neto (2016).***

According to Horadam (1974), mathematically, the Oresme sequence is important for at least three reasons. The first is the exhibition of a graphic representation of qualities and speeds, although there is no mention of the (functional) dependence on one quality on another, as found in Descartes. The second, for being the first person to conceive the notion of fractional powers and to suggest a notation. The third, for having found the sum of the following rational numbers $\frac{1}{2},\frac{2}{4},\frac{3}{8},\frac{4}{16},\frac{5}{32},\frac{6}{64},...,\frac{n}{2^n}$, composing an infinite series of rational numbers. Thus, it is believed that N. Oresme used primitive mathematical ideas, what we now know as the improper integral, to perform the sum of the infinite series, obtaining a value of 2. Furthermore, the aforementioned enigmatic sequence has a character, also, of interest in the biology, insofar as, from the first two terms, we can estimate the number of parents, grandparents and determine this proportion in any generation (Mendonça & Neto, 2016).

Let us look at its corresponding linear recurrence relationship.

Definition 6. The Oresme sequence (O_n), has the fundamental recurrence, for every interger $n \in$ and the initial values indicated by $O_0 = 0, O_1 = \frac{1}{2}$, we have the relation:

$$O_n = O_{n-1} - \frac{1}{4}O_{n-2}, n \geq 2.$$

From this formal definition, we can then obtain the characteristic equation of this second order sequence, as being $x^2 - x + \frac{1}{4} = 0$, presenting two real and identical roots of value $\frac{1}{2}$. Observing its recurrence and comparing it with the generalized Fibonacci formula, we realize that the coefficient y has a corresponding value of $-\frac{1}{4}$ and only an initial value is changed to $\frac{1}{2}$.

2. 7. The Mersenne sequence

Composed or originally from the emblematic Mersenne numbers ($M_n = 2^n - 1$), the Mersenne sequence is named after the French mathematician Marin Mersenne (1588 - 1648), shown in Figure 13. From mathematical studies, especially in number theory, Mersenne became known for his contributions to the so-called Mersenne cousins. Rosa (2012) reports that Franciscan Mersenne offered his home for meetings with contemporary scientists such as Descartes, Galileo, Fermat, Pascal and Torricelli, with an interest in discussing and studying mathematics and scientific subjects.

Figure 13. ***Marin Mersenne. Source: http://mathshistory.st-andrews.ac.uk/Biographies/Mersenne.html.***

Despite the fact that the mathematician was not a composer, interpreter or artist, Marin Mersenne established a theory based on practice, defending an equal temperament during the construction of the instruments and, when explaining, in a rational way, Mersenne's tunings still reveal his concern and musical feeling with the temperament when dividing the octave into 12 equal semitones, thus obtaining the harmonic monochord, called the tempered scale (Rosa, 2012).

In the context of Number Theory, Mersenne numbers are described in the form $M_n = 2^n - 1$, with $n \in \mathbb{N}$. However, more recently, Portuguese researchers Catarino, Campos & Vasco (2016), defined a recurrence, based on the recurring formula of the Fibonacci sequence. Below we see a recurrence proposed by Portuguese researchers and recently introduced in the scientific literature.

Definition 7. The Mersenne (M_n), has the fundamental recurrence, for every integer $n \in \mathbb{N}$ and the initial values $M_0 = 0, M_1 = 1$, defined by (Catarino, Campos & Vasco, 2016):

$$M_n = 3M_{n-1} - 2M_{n-2}, n \geq 2.$$

Through the inherent algebraic manipulations of Definition 7, we have its characteristic equation for this second order sequence, given by $x^2 - 3x + 2 = 0$. The two solutions for this polynomial are the real roots 2 and 1, highlighting that the first value is related to the copper number (value of approximately 2). This relationship is given when defining a subsequence, given by dividing a term in the sequence by its predecessor. Given this, we have that this subsequence converges to the copper number. Making a comparison with the generalized Fibonacci formula ($F_n = xF_{n-1} + yF_{n-2}, n \geq 2$), we have that the coefficient value, for this case, has the value 3 and the coefficient has the value -2, remaining with the same initial values.

In the subsequent section, we will discuss a sequence that corresponds to and relates to the Fibonacci sequence, especially when we relate the Fibonacci sequence to objective reasons and measures in the context of Architecture and aesthetics. However, while the golden ratio corresponds to the study of certain 2D proportions, the Padovan sequence corresponds to the modern 3D architectural model.

2. 8. The Padovan sequence

With the end of the Second World War, in 1945, there were great losses, highlighting the destruction of the churches for this work. In view of this, the Benedictine monk and Dutch architect Hans Van Der Laan (1904-1991) and his brother, began the process of reconstructing these churches, eventually discovering a new standard of measurement, given by an irrational number (Voet & Schoonjans, 2012). This number was first studied by the French mathematician Gérard Cordonnier (1907 - 1977), calling it a radiant number. Therefore, this sequence is also known as Cordonnier sequence, but this study was interrupted due to his death. Thus, with a value of approximately 1.32, this number is known as a plastic number or radiant number, considered ideal for carrying out work on geometric scales and spatial objects. It is worth noting that the architect Richard Padovan, considers this sequence still to be a discovery by Hans Van Der Laan. We then have in Figure 14, one of the churches rebuilt based on the property of this number, thus showing an application of the plastic number, discussed in the work of Marohnie & Strmecki (2012).

The plastic number and the gold number are two the only solutions of the morphic numbers (Spinadel & Buitrago, 2009; Ferreira, 2015), where the first represents the convergence relationship between the neighboring terms of the Padovan sequence, and the second the same relation to the Fibonacci sequence.

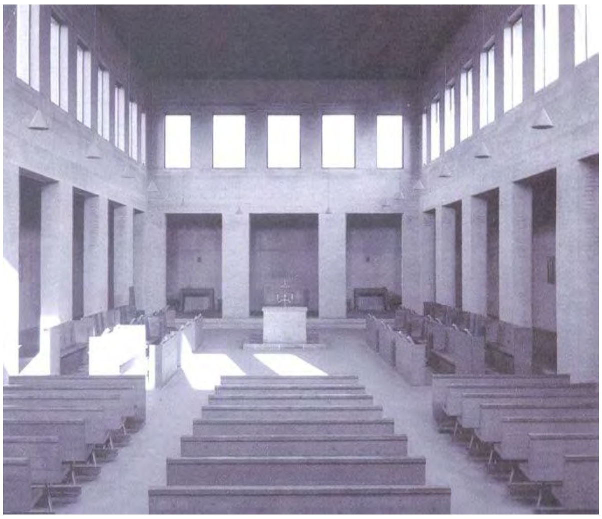

Figure 14. ***Church rebuilt after World War II. Source:*** ***(Marohnie & Strmecki, 2012).***

From studies related to the plastic number, the Italian architect Richard Padovan (1935 -) created the sequence of Padovan. This sequence is a kind of cousin of the Fibonacci sequence, and is therefore of the third order. Richard de Padovan's work and contribution has important repercussions for the field of research in Mathematics.

Definition 8. The Padovan sequence (Pa_n), has the fundamental recurrence, for every integer $n \in$ □ and the initial values $Pa_0 = Pa_1 = Pa_2 = 1$, defined by:

$$Pa_n = Pa_{n-2} + Pa_{n-3}, n \geq 3.$$

When performing algebraic manipulations in the recurrence of this sequence, we then have its characteristic equation given by $x^3 - x - 1 = 0$, in which the only real solution is given by the plastic number, and the other two roots are complex and conjugated numbers. Performing a comparison with the Fibonacci recurrence, we can see that the value of the coefficient of the generalized formula ($F_n = xF_{n-1} + yF_{n-2}, n \geq 2$) is equal to zero. Thus, one more term is added to the end of this formula, characterizing it, as a third order sequence, and its three initial numerical values are equal to 1.

There is also a geometric representation for these numbers, given by Padovan's spiral (see Figural 15). According to Vieira and Alves (2019), we can see that:

> This representation is composed by the juxtaposition of equilateral triangles respecting a characteristic construction rule. Consider the side 1 triangle highlighted

in blue as the starting triangle. The spiral is formed by adding a new equilateral triangle to the largest side of the formed polygon, starting with the blue triangle" (Vieira & Alves, 2019, p. 7-8).

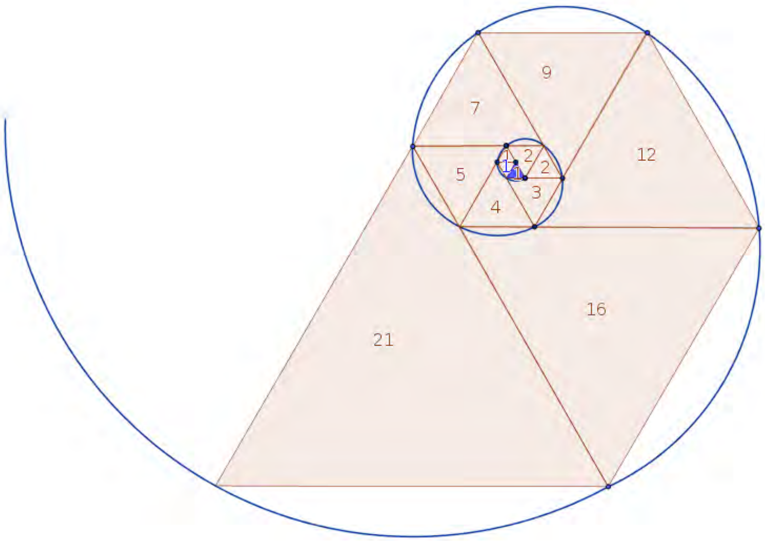

Figure 15. ***Spiral of Padovan. Source: (Vieira & Alves, 2019, p. 7-8).***

2. 9. The Perrin sequence

The sequence of Perrin, has its name attributed to the French Olivier Raoul Perrin (1841-1910), an engineer who in his spare time liked to produce scientific works. He had a greater preference and a special inclination for mathematics. Many of his memoirs, which reveal a great depth of spirit, were objects of communications much appreciated in the Academy of Sciences or in other educated societies, yielding flattering distinctions.

In 1876, this sequence was mentioned implicitly by Édouard Lucas, known for creating the mathematical game tower of Hanoi, the sequence of Lucas and the numbers of Lucas. Lucas, observed that if p it is a prime number, then p it divides Pe_p, being a consequence of Fermat's theorem (Adams & Shanks, 1982). However, in 1899, R. Perrin defined the sequence of Perrin, as being a sequence of third order, and having great importance, in particular, for the Theory of Graphs. Used to discover the coordinates of taxis in urban networks in a confidential manner, these numbers have application in the area of Computing and in several other areas (Sugumaran & Rajesh, 2017).

Definition 9. The Perrin sequence (Pe_n), has the fundamental recurrence, for every integer $n \in \mathbb{N}$ and the initial values $Pe_0 = 3, Pe_1 = 0, Pe_2 = 2$, defined by:

$$Pe_n = Pe_{n-2} + Pe_{n-3}, n \geq 3.$$

This numerical sequence differs from Padovan's sequence only in relation to its initial terms, thus presenting the same recurrence formula. In view of Definition 9, when performing some algebraic manipulations, obtain the characteristic equation of these numbers, in an identical way to the characteristic Padovan equation. Thus, its roots are identical to the roots of Padovan's numbers, having a relationship with the plastic number $\psi = 1.324718... \cong \frac{4}{3}$ (approximate value of 1.32).

A 2D geometric representation for this sequence is given through the Perrin spiral, similarly to the Padovan spiral, having the initial equilateral triangles of sides 2,3,2 and 5, advancing the Perrin sequence. Thus, according to the largest side of the new formed polygon, a new equilateral triangle with this new value will be inserted, forming the Perrin spiral, with its respective sequence terms, as shown in Figure 16. Originating in the Padovan spiral , this representation of the Perrin sequence, undergoes a small change in its initial values (Knuth, 2011).

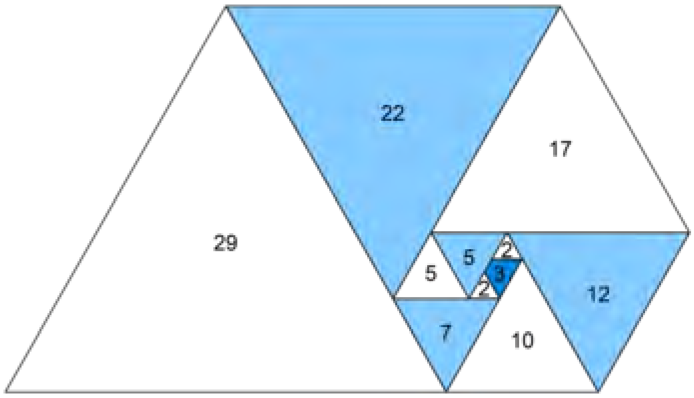

Figure 16. ***Spiral of Perrin. Source: (Knuth, 2011).***

Our last case to be analyzed and discussed reveals aspects and cultural traits characteristic of Hindu Mathematics. In fact, we observed that most authors of History of Mathematics books adopted in Brazil point to an essentially Western mathematical culture. Thus, the Narayanna sequence is an important example and one that provides a corresponding interest in certain conceptual numerical entities.

2. 10. The Narayana sequence

The Narayana sequence was introduced by the Indian mathematician Narayana Pandita (1340 - 1400) (see photo 17), author of the work Ganita Kaumudi (Puttaswamy, 2012, p. 541) and of an algebraic treatise called Bijaganita Vatamsa. His texts were the most important Sanskrit mathematics treaties after the Bhaskara II theorems (Sarma, 1972).

Plofker (2009, p.208), comments that the study of permutations and combinations is very comprehensive. On the other hand, Plofker (2009, pp. 208-209) still reports that "Narayana seems to have been one of the first authors to introduce magic squares as a topic in Indian arithmetic, although they were certainly known". Between the 12th and 16th centuries, some new treatises on mathematics were composed, such as: Ganitakaumudi (Moonlight of Computation) complemented by Narayana in 1356; the Bijaganitavatamsa (Garland of Algebra), algebra work and; introduced, in a primordial way, the magic squares in mathematics.

The Narayana sequence, better known as the Narayana cows, is derived from the problem of the herd of cows and calves, proposed by Narayana in the 14th century, through the problem: a cow gives birth to a calf every year. In turn, the calf gives birth to another calf when it is three years old. What is the number of progenies produced by a cow during twenty years ?. Thus, its solution is similar to the problem of the pairs of rabbits in the Fibonacci sequence.

In figure 17 we bring an image of the eminent Indian mathematician Narayana Pandita (1340 - 1400).

Figure 17. ***Narayana Pandita (1340-1400). Source: https://www.mathsone.com/blog-details/blog/89-Narayana-Pandit.***

Definition 10. The Narayana (N_n), has the fundamental recurrence, for every interger $n \in \mathbb{N}$ and the initial values $N_0 = N_1 = N_2 = 1$, defined by:

$$N_n = N_{n-1} + N_{n-3}, n \geq 3.$$

Similar to the previous cases, the characteristic equation of this sequence, is obtained by means of algebraic manipulations performed in the equation of Definition 10, resulting in $x^3 - x^2 - 1 = 0$. Thus, we have three roots, being a real solution and two complex, having the real value as being the proportion of super-gold (approximate value of 1.46).

A 2D representation of this sequence is given through the super-gold rectangle, in which a rectangle is constructed with lateral lengths in proportion to the super-gold number, that is, the length of the longest side is divided by the length of the shortest side, thus obtaining the super-gold number, as shown in Figure 18. When removing a square from one side of the rectangle, whose side length is equal to the length of the shortest side of the rectangle, the resulting rectangle will be in proportion $\psi^2 : 1$. The rectangle is then divided into proportions of lateral length $\psi : 1$ and $1 : \psi$, forming its super-gold proportions, with areas $\psi^2 : 1$ (Crilly, 1994).

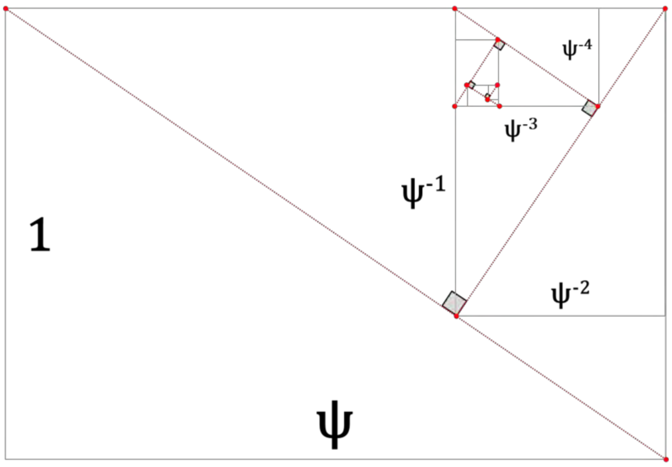

Figure 18. **Super-gold rectangle.** ***Source: (Crilly, 1994).***

Comparing this sequence with the Fibonacci numbers, using its generalized form ($F_n = xF_{n-1} + yF_{n-2}, n \geq 2$), we have then that the coefficient has a zero value, and another variable is added. In addition, the initial terms are the same as for Fibonacci, inserting one more initial value, as it is of a third order. This sequence also has a melody in relation to Narayana cows authored by Tom

Johnson, which appears in the work of Waldschmidt (2009), in which he conducts a study on the construction of the musical notes of this melody, presenting the terms of the Narayana sequence.

3. Conclusion

In this work, the historical and mathematical aspects of the Fibonacci, Lucas, Pell, Jacobsthal, Leonardo, Oresme, Mersenne, Padovan, Perrin and Narayana sequences were studied. The information and aspects discussed in the previous sections serve to confirm the importance of spreading a broad mathematical culture in Brazil, with interest also to students, on the notion of recurring sequences.

Initially, brief historical investigations were carried out, inherent to the mathematicians who created their respective sequences, as well as their contributions made to the world of mathematics. Starting from the Fibonacci sequence, since this is the precursor to the world of sequences, it was then possible to establish a generalized formula for its recurrence, thus showing the origin of other sequences, derived from this mathematical equation.

Continuing the mathematical process, the characteristic equations of each sequence were presented, also observing their respective relations with the number of others, silver, bronze, and others. The variant formula of Binet, the matrix form and other mathematical aspects, were not presented in this research, since several numerical sequences were discussed, thus making the work a little extensive.

It is worth noting the progress of the research provided by the participation and cooperation of Portuguese experts and teachers involved in high-level mathematical research, however, we seek to carry out a didactic transposition process in Brazil capable of bringing such mathematical knowledge closer to mathematics teachers who work in the field of teaching mathematics in Brazil.

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