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Interpretations of Slope Through Written and Verbal Interactions Between a Student and Her Tutors in Algebra 1

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Abstract

There is an ongoing need to support students' learning of linear functions, and the study of slope makes up a foundational component of this learning. We applied techniques from systemic functional linguistics to document the meanings that were established through spoken interaction between a student and her tutors during discussions of slope. We found that, while fraction notation gave the student and tutors a common reference point to discuss slope, it also masked important differences in how the student interpreted slope compared to her tutors. The findings of this analysis imply the need not only to attend to how students quantify slope, but also whether students recognize slope as an attribute of a line.

Keywords: Algebra, slope, mathematical discourse, thematic analysis, struggling learners

Interpretación de la Pendiente a Través de Interacciones Escritas y Verbales entre una Estudiante y sus Tutores en Álgebra 1

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Resumen

Existe una necesidad continua de apoyar el aprendizaje de los estudiantes sobre las funciones lineales y el estudio de la pendiente es un componente fundamental de este aprendizaje. Aplicamos técnicas de lingüística funcional sistémica para documentar los significados que se establecieron a través de la interacción oral entre una alumna y sus tutores discutiendo la pendiente. Descubrimos que, aunque la notación de fracciones le dio a la estudiante y a los tutores un punto de referencia común para discutir la pendiente, también ocultó diferencias importantes en cómo la estudiante interpretó la pendiente en comparación con sus tutores. Los resultados de este análisis implican la necesidad no solo de atender la forma en que los estudiantes cuantifican la pendiente, sino también si los estudiantes reconocen la pendiente como atributo de una línea.

Palabras clave: Álgebra, pendiente, discurso matemático, análisis temático, alumnos con dificultades

Within classroom settings, students and teachers create meaning together through interactions (Forman, McCormick, & Donato, 1997; Moschkovich, 2008; O'Halloran, 2015; Schleppegrell, 2007). In most classrooms spoken language is the primary means through which information is shared, and meaning is created through the connections that speakers establish among ideas (Lemke, 1988; 1990). Spoken interactions are supported through the use of other representations such as visual representations (Alshwaikh, 2011; Chapman, 1993; O'Halloran, 2005) and the use of symbolic notation (O'Halloran, 2003). With this study, we address a question of how students and tutors construct meanings together through their talk and their shared use of symbolic notation, in the context of slope in Algebra 1.

This work comes from a project in which a group of university pre-service teachers provided tutoring for eighth-grade students in Algebra 1 at an urban, high-needs public school (Hord, DeJarnette, & Marita, 2015; Hord, Marita, Walsh, Tomaro, & Gordon, 2016; Hord et al., 2016). Tutoring can be a valuable setting for supporting the needs of individual students, but to optimize on such opportunities, it is necessary to have detailed descriptions about how one-on-one interactions create opportunities for learning. With this case study, we document the meanings of slope established by one student, Tanisha (all names are pseudonyms), and the tutors that she worked with across several sessions. Our analysis was guided by the following research questions:

1. What connections did Tanisha and her tutors establish among ideas related to slope when working on tasks about linear functions?
2. How did the use of symbolic fraction notation shape the spoken interactions between Tanisha and her tutors?

We applied techniques from systemic functional linguistics (SFL) to examine how speakers construct meaning through their talk (Halliday & Matthiessen, 2014). We adopt the perspective that meaning is constructed in interaction through the connections that speakers establish between different words and phrases (Chapman, 1993; Herbel-Eisenmann & Otten, 2011; Lemke, 1990; O'Halloran, 2005). This study contributes to research on the teaching and learning of linear functions by describing features of student meaning making that are likely to be overlooked in the moment of interaction. We also intend to elaborate on how the shared use of written representations, which can be interpreted in multiple ways, can obscure

differences in how students and tutors talk about mathematical concepts, such as slope.

Teaching and Learning About Linear Functions and Slope

There is a compelling argument that functions are, or ought to be, the fundamental objects of study framing the algebra curriculum (Blanton, Levi, Crites, & Dougherty, 2011; Carraher, Schliemann, & Schwartz, 2008; Dubinsky & Harel, 1992; Yerushalmy, 2000). Schwartz and Yerushalmy (1992) described functions as a unifying object from which other algebraic objects (e.g., expressions and equations) can stem. They also noted that functions naturally lend themselves to being represented in multiple ways, including graphs, tables, and real-world contexts in addition to traditional symbolic notation. Because functions are used to represent many of the real-world phenomena to which algebra is applied, making functions more prominent within the curriculum has the potential to motivate students' study of algebra (Chazan, 2000; Yerushalmy, 2000). In the *Common Core State Standards for Mathematics* (National Governors Association [NGA] Center for Best Practices & Council of Chief State School Officers [CCSSO], 2010), which guide mathematics instructions in most public schools in the United States, functions constitute one of the core content domains beginning in eighth grade and extending through high school.

Linear functions, which are the focus of functions learning through Algebra 1, become prominent in eighth grade and build from students' work with ratios and proportional relationships as early as sixth grade (NGA Center for Best Practices & CCSSO, 2010). Linear functions constitute a broad topic in secondary mathematics curriculum, with emphases spanning connections to modeling, algebra, and geometry. Students working with symbolic representations of the form $y = mx + b$ have often used what Schoenfeld, Smith, and Arcavi (1993) described as the "3 slot schema" to identify the parameters of a linear function through the placement of numbers and variables. However, conceptual understanding of linear functions requires making connections between different representations of linear relationships (Smith, Arcavi, & Schoenfeld, 1989).

A *covariational* approach to teaching about linear functions prioritizes student understanding of a linear function as representing a constant multiplicative relationship between two quantities (Carlson, 1998; Carlson,

Jacobs, Coe, Larsen, & Hsu, 2002; Castillo-Garsow, 2012; Confrey & Smith, 1995; Johnson, 2015a; 2015b; Thompson, 1993). Taking a covariational perspective, the concept of slope is particularly salient to students' learning about linear functions because slope is the ratio that describes the multiplicative relationship. A ratio requires forming a "complex composite unit" from two other composite units (Lamon, 1995, p. 169). For example, a ratio such 75:2 can be considered a complex composite unit, which students must act on as a single entity (e.g., calculating equivalent ratios, or comparing to other ratios) to engage in the type of reasoning necessary to describe rates of change. Ratio reasoning is a critical element of learning about linear functions because students apply ratio reasoning to represent constant covariation between quantities (Ellis, 2007a; 2007b; Harel, Behr, Lesh, & Post, 1994; Johnson, 2015a; 2015b).

In practice, students have many ways of reasoning about slope that are different from the notion of a continuous rate of change. Students might interpret slope as a pair of differences (Lobato, Ellis, & Muñoz, 2003) or as a description of horizontal or vertical movement along a linear graph (Zahner, 2015). Even when teachers emphasize slope as a single quantity to describe rate of change, students often persist in thinking of slope in two distinct pieces. Moreover, students do not necessarily know that their interpretations of slope differ from the teacher's.

When students are asked to make observations related to slope based on symbolic, tabular, or graphical representations, they draw upon a variety of resources for comparing representations. In some cases, when given a graph, students interpret slope to represent the scale of either the x - or the y -axis (Earnest, 2015; Lobato et al., 2003), or they attend to the changes in one variable without coordinating with the other variable (Carlson et al., 2002; Confrey & Smith, 1995; Lobato et al., 2003). On some tasks, students can compare the slopes of lines by comparing two points (e.g., (1, 2) on one graph versus (1, 5) on the other graph), although this strategy only works when the two lines share the same y -intercept (Earnest, 2015). Although this type of attention to *discrete variation*—focusing on individual points rather than continuous relationships (Castillo-Garsow, 2012)—is not generalizable, there are some questions and problems that students can address by focusing on individual points.

The coordination of slope and y -intercept, while attending to different representations of linear functions, can create contradictions in students'

work, thus leading to opportunities for learning. Moschkovich (1996) described several cases in which students needed to negotiate their use of informal phrases such as “steeper,” “less steep,” “moves up,” and “moves down.” Early in their learning about linear graphs, students were unclear about whether steepness of a line referred to its slant or to its global height on a coordinate plane. Through discussions of their intended meanings, some students showed growth in their understandings of the concepts of slope and y -intercept. Comparing linear functions only according to the steepness of a graph can become problematic, however, particularly if graphs are created with different scales (Earnest, 2015). Understanding of slope requires connections between the steepness of a line and the value of m in an equation in the form $y = mx + b$. While existing research offers insights into the ways that students create meaning through their talk, there are still open questions about how students and instructors (in our case, students and tutors) co-construct meanings through their interactions. In this study we describe how one student established meaning related to linear functions through interactions with her tutors, and how those interactions were shaped by their shared use of symbolic fraction notation.

Analytical Framework

We draw on a social semiotic framework to inform this study. This perspective emphasizes the importance of our selection of representations to communicate meaning through social activity (Kress & van Leeuwen, 2006; Morgan, 2006; O'Halloran, 2015). Interactions in mathematics classrooms are multi-semiotic; communication requires a variety of representation systems including speech, visual representations, symbolic notation, and gesturing (Alshwaikh, 2011; Arzarello & Edwards, 2005; Chapman, 1993; Dimmel & Herbst, 2015; O'Halloran, 2003; 2005; Radford, 2009). When individuals communicate, they make choices in their selection of these different representations to build meaning around a particular topic. Because much of the work that goes on in a classroom on a daily basis is communicated through talk, the use of spoken language can be considered one of the primary means through which academic subjects such as mathematics are taught and learned (Lemke, 1988). We use spoken language as our primary means of analysis while accounting for the ways in which other representations support individuals' communication through speech.

Thematic analysis is a method within the theory of SFL (Halliday & Matthiessen, 2014) that focuses on the ways ideas are connected to one another in a text (Lemke, 1990; see also Chapman, 1993; DeJarnette & González, 2016; Herbel-Eisenmann & Otten, 2011; O'Halloran, 2005; Webel & DeLeeuw, 2016). The primary assumption guiding thematic analysis is that meaning is given to words and phrases through the ways in which they are connected to other words and phrases. Thematic analysis can be accomplished by identifying the semantic relationships among words or phrases in a text. Semantic relationships refer to the ways that words or phrases are connected, and there are many semantic relations that can connect phrases across contexts (Lemke, 1990). For example, numbers can be used to quantify objects; certain objects might constitute sub-categories of broader categories; phrases can be used to describe attributes of objects (see, e.g., DeJarnette & González, 2016; Herbel-Eisenmann & Otten, 2011 for descriptions of semantic relations that surface in mathematical conversations).

Most relevant to the present study are the semantic relations that one might use to construct meaning around slope. Lemke (1990) described the semantic relation of “process” to describe an action or operation. For example, if a teacher asked students to define slope and a student responded, “slope is over and up,” this would invoke a process relation towards slope. Alternatively, there is an “attribute/carrier” semantic relation through which an object is described with a particular attribute. If a student were to say, “slope is the steepness of a line,” this would invoke an attribute/carrier semantic relation. Finally, a “quantifier” semantic relation assigns a numerical value to an object or process. If a student were to say, “the slope is 4 over 3,” this could potentially quantify the process of counting over and up, or it could quantify the steepness of a particular line.

Data and Methods

We conducted a case study (Stake, 1995) focusing on a single student, Tanisha, who participated in a mathematics tutoring program throughout the spring of 2015. Analyzing a single case allowed us to document semantic relationships that are rarely made explicit in the moment of interaction (Lemke, 1990), and that can serve as a road map for future instructor-student interactions. We applied thematic analysis from SFL to explore how Tanisha

and her tutors constructed meanings of slope in interaction. In the following subsections we describe the setting of the tutoring program, including how we selected Tanisha as a focal student for the study, as well as our procedures for data collection and analysis.

Setting of the Study

We conducted this study in an urban public school in the midwestern United States serving grades 7–12. At the time of the study, all students at this school took Algebra 1 during their eighth-grade year. In response to a high number of students with learning disabilities and students who were at risk of not passing Algebra 1, the third author of this paper established a tutoring program at the school in collaboration with the eighth-grade mathematics teacher and the special education teacher. Beginning in December 2014, and running through the end of the school year, pre-service teachers went to the school on a weekly basis to work individually or in small groups with students on their current classwork and homework. Tutors were recruited from undergraduate courses for pre-service teachers working towards certification in middle childhood mathematics (grades 4-9) or special education (grades K-12). The third author selected tutors who had strong mathematics content knowledge, who had demonstrated capability for working with struggling learners through class discussions and assignments, and who were interested in gaining additional field experience in mathematics. Tutors were trained on strategies for supporting struggling learners in mathematics and frequently met with the third author to discuss the successes and challenges of tutoring the students.

During the tutoring sessions, the tutors worked with the Algebra 1 students on their current classwork and assignments. The mathematics classes at the school frequently used Assessment and Learning in Knowledge Spaces (ALEKS; McGraw-Hill, 2019), a web-based assessment and learning system that adapts questions to a student's progress. Each student had an individual account in ALEKS, with a range of topics that students could access and complete. The tutors typically helped students work through teacher-assigned problem sets or to complete different sections of work in ALEKS.

Tanisha, the focal student of our case study, was a 14-year-old African American student. Tanisha was struggling to maintain a passing grade in

Algebra 1 and participated in tutoring throughout the spring of 2015. We selected Tanisha as our case for a combination of reasons. Tanisha's performance in Algebra 1 was fairly typical of students participating in the tutoring project, namely in that she struggled with much of the content of the course. This suggested to us that an in-depth analysis of Tanisha's strengths and challenges might provide insight that could inform work with other struggling students. Additionally, Tanisha had good attendance and was usually very talkative during her tutoring sessions. Thus, we had a thorough set of data to use, across multiple days and types of tasks, to analyze how Tanisha and her tutors constructed meaning related to slope in their talk.

Data Collection

We audio recorded all of the tutoring sessions, in addition to making copies of student work and taking field notes during the sessions. We have records of seven different tutoring sessions with Tanisha, with three different tutors, ranging from March-May of 2015. One of those tutors, Emily, was an undergraduate in the Special Education program; the second tutor, Sarah, was a graduate student working towards initial licensure in Special Education. In addition to prerequisite coursework, Emily and Sarah had both taken one semester each of upper level mathematics, mathematics methods for pre-service teachers, and practicum (tutoring in English/Language Arts). Although the tutoring program was staffed by pre-service teachers, there was one day that Hord (the third author of this paper, and a former special education teacher and math teacher) filled in as a tutor.

For this study, we were interested in Tanisha's work on tasks specifically related to describing or calculating the slope of a linear function. As such, we produced timelines of all of the tutoring sessions, segmented according to the different tasks that Tanisha and her tutors discussed. We selected only segments of the timelines in which conversations related to slope surfaced. In all, we identified five segments, across three tutoring sessions, in which Tanisha worked on tasks related to slope with her tutors (Table 1). Three of those segments involved work on "real-world" tasks; two of the segments involved work on more abstract tasks. The length of the segments ranged from around one minute to over 14 minutes. Using the audio records, as well as copies of any written work Tanisha produced and field notes of the tutor and an observer, we produced transcripts of each of the segments for analysis.

Table 1.
Dates of Data Collection and a Summary of the Slope Tasks Tanisha and her Tutors Discussed.

Date	Session #	Segment #	Summary of tasks	Time Spent
3/19	1	1	Given the graph of a line, determine its slope.	1:20
5/1	2	6	Assuming an individual earns \$95 dollars per week as a lifeguard, and deposits 10% of that into her bank account, write an equation to represent how much would be in the bank account after a given number of weeks. She started with \$60 in her bank account.	4:00
		7	If a school group pays \$120 to rent a carnival booth and charges \$1.50 per visitor to the booth, write an equation to represent how much the group has earned after a given number of visitors.	3:40
5/8	3	2	Determine the slope of a line, and make predictions about how to make the line steeper, or to shift upwards.	14:30
		3	An airplane weighs 2,178 tons with 12 gallons of fuel and 2,360 tons with 40 gallons of fuel. Determine how much the plane would weigh with 54 gallons of fuel.	8:10

Analysis

After transcribing the interactions between Tanisha and her tutors, we enumerated the semantic relationships (Halliday & Matthiessen, 2014; Lemke, 1990) between terms and phrases related to slope. The first step in this analysis was to identify the key phrases, which are defined as the nouns and noun phrases that Tanisha and her tutors used to discuss linear functions (DeJarnette & González, 2016; Herbel-Eisenmann & Otten, 2011). To identify these key phrases, we made passes through the transcripts to note the specific language used by Tanisha and her tutors to discuss the tasks at hand.

As we identified key phrases, we examined the transcripts to describe the

semantic relations that speakers invoked to connect these phrases. We looked in particular to document whether Tanisha and her tutors invoked semantic relations of process, object/attribute, or quantifier. We also documented whether Tanisha and her tutors used the same semantic relations. We illustrate our analysis with an excerpt from the conversation between Tanisha and Sarah during session 1 (Table 2). In this excerpt, Tanisha was working on finding the slope of a line whose graph had been provided by counting over and up from one point on the graph to another. Sarah encouraged Tanisha to locate two points on the graph to calculate slope, but Tanisha was focused on determining the horizontal and, respectively, vertical distances between one point and the next. In the far right column of Table 2, we note the semantic relationships that are most relevant to interpreting Sarah and Tanisha's conversation about slope.

Table 2.

Tanisha and Sarah's Discussion of Slope in Session 1, Segment 1.

Turn #	Speaker	Turn	Semantic Relationships
4	Tanisha	Um, so for this one my slope is one, one. One over one.	Quantifier ("one, one")
5	Sarah	Why don't you do the math and make sure it works? The points were what, ten, thirty [i.e., (10, 30)], and -	
6	Tanisha	Over one and, wait, yeah. It's, no, it's not one. It's one and one.	Process ("over one") Quantifier ("one and one")
7	Sarah	Ten, thirty, and twenty, forty. [Referring to the ordered pairs, (10, 30) and (20, 40) marked on the line; talking while doing a calculation to determine slope.]	
8	Tanisha	Oh I'm sorry. [Apologizing for working ahead of Sarah.]	
9	Sarah	No you're fine. So forty minus thirty, twenty minus ten. Ten, oh yeah. So you got one over one?	Quantifier ("one over one")
10	Tanisha	Mm hmm.	
11	Sarah	Okay. So, what's that?	
12	Tanisha	So, y equals one x plus twenty [i.e., $1x + 20$].	Quantifier ("one")

Note: We define a turn of speech as a segment of continuous speech followed by a break to allow another person to speak. Brackets, [], included with the transcript represent our notes to

help the reader interpret the dialogue. “Turn #” indicates where these turns of speech surfaced within the selected segment. In the right column of the table we summarize the semantic relationships that are relevant to the analysis, including key phrases or aggregate phrases that are related.

We identified two semantic relationships in turn 4. First, we noted a quantifier relationship between the term, slope, and the pair of numbers, 1 and 1. Because Tanisha seemed to be treating 1 and 1 separately as measures of the horizontal and, respectively, vertical distances between the two points, we described the phrase “one, one” as a pair of numbers rather than as a single number. With her next statement, “one over one,” Tanisha seemed to quantify slope as the fraction, $1/1$. At the moment of turn 4, it was not explicit from Tanisha’s talk whether she drew any connection between the pair of numbers, 1 and 1, and the fraction $1/1$. In turn 5, Sarah located a point on the graph at the ordered pair, (10, 30), but Tanisha seemed to disregard this comment in favor of her “over and up” approach towards calculating the slope. In turn 6, Tanisha made an explicit distinction between slope as a single number and slope as a pair of numbers. When she said, “it’s not one,” Tanisha noted that the number, 1, was not a quantifier of the slope of the line. Instead, she noted, slope was quantified as the pair of numbers, one and one. Although it had been ambiguous from turn 4 whether Tanisha was using “one over one” and “one and one” synonymously, it became clear in turn 6 that she was not. Tanisha clearly quantified the slope of the line through a pair of numbers, rather than a single number, in turn 6. After Tanisha had proposed the slope to be “one and one,” Sarah confirmed her answer but rephrased the slope as the fraction, “one over one” in turn 9. Tanisha passively agreed with Sarah’s rephrasing (turn 10), and only then did she describe the slope as a single value (turn 12).

The analysis exemplified above helped illuminate how Tanisha and her tutors co-constructed meaning around the topic of linear functions, and especially how Tanisha’s constructions sometimes differed from her tutors. The first and second authors met regularly to identify the semantic relationships between key terms and phrases in the transcript. When a speaker’s use of a particular term was ambiguous, we referred to copies of written work and field notes of the session to better understand the speaker’s meaning. We shared the transcripts with the third author on a regular basis to check the validity of our analysis. In the end, we came to a consensus on

all of our identifications of the semantic relationships between phrases in the text.

Findings

We organize our findings to illustrate distinct meanings that Tanisha and, respectively, her tutors, used to talk about slope. We found that Tanisha most often invoked process relations when describing slope, leading to the quantification of slope through pairs of numbers; her tutors implied attribute relations through the quantification of slope as a single quantity. However, their shared use of fraction notation to quantify the process or, respectively, the attribute, obscured this distinction. We present sections of our analysis to illustrate commonalities and contrasts in how Tanisha and her tutors talked about slope across the tutoring sessions.

As was illustrated in Table 2, Sarah translated Tanisha's pair of numbers, "one and one" into the fraction "one over one," implying that the pair of numbers represented a single rational value. Following Sarah's lead, Tanisha then described the slope of the function as a single value (the integer, 1). From the joint work that Tanisha and Sarah completed in Table 2, it may seem that Tanisha and Sarah jointly constructed a sequence of semantic relationships connecting the process of determining slope to an attribute describing a line. However, based on other interactions Tanisha had with her tutors, Tanisha and her tutors interpreted fractional quantities in two different ways, although their shared written representations obscured the difference.

During session 3, Tanisha had been given a graph of the line $y=x+3$ (she was not given the equation) and was asked to write the equation for the line. Hord asked Tanisha to consider how she would determine the slope of the given line (Table 3). Tanisha instead suggested an equation for the line, and the pair continued to discuss how she had determined the slope.

In turn 31, Tanisha described the slope of the graph as "negative four over four" and wrote the fraction as $-\frac{4}{4}$, with the negative sign next to the fraction rather than attached to either digit. As Hord pressed Tanisha to describe more about how she was using that phrase, it became clearer that she was using the phrase "negative four over four" to quantify the process of counting down and to the left (turn 42). As the conversation in Table 3 progressed, Hord continued to press Tanisha on her negative value of slope for the line, suggesting that any line increasing from left to right would have a positive

slope. Hord’s characterization of the direction of a line implied a connection between the phrase, “slope,” and the steepness of a line. Again the fractional representation that Tanisha used—in this case, $-4/4$ —served to mask the difference in interpretations of slope: as a process or as a composite quantity.

Table 3.

Tanisha and Hord’s Discussion of Slope in Session 3, Segment 2.

Turn #	Speaker	Turn	Semantic Relationships
30	Hord	So what is the equation for slope?	Quantifier
31	Tanisha	Four, negative four, four, and then, plus three, y equals negative four over four plus three. [Tanisha writes, $y = -\frac{4}{4} + 3$.]	Quantifier (“negative four, four”)
32	Hord	Negative four?	
33-40		[Tanisha and Hord discuss the need for a variable, x in the equation, and Tanisha revises her written work to $y = -\frac{4}{4}x + 3$.]	
41	Hord	Okay, so where’s the negative four come from? Where’s negative? How, how do you get these two numbers? [Pointing to the two 4’s in the equation.]	
42	Tanisha	Um, I went down four and then went to the left four.	Process (“went down”, “went to the left”)
43	Hord	Oh, so you counted down.	
44	Tanisha	Mm hmm.	

We share another excerpt from Tanisha’s work with Hord during session 3 to highlight a contrast in how the two quantified slope. Near the end of their work together, Hord had expanded upon their discussion of the original problem and sketched some additional lines as an opportunity to talk with Tanisha about slope. In the last example, Hord sketched a line with slope $5/2$; together, Tanisha and Hord selected two points and counted horizontally and vertically between those two points to determine the slope. The excerpt in Table 4 began when Hord translated the fraction $5/2$ (which they had been pronouncing as “five over two”), into the decimal 2.5.

Table 4.
Tanisha and Hord's Discussion of Slope in Session 3, Segment 2.

Turn #	Speaker	Turn	Semantic Relationships
118	Hord	So then I've got five over two, which uh, what is that, two and a half? Two point five, or two and a half.	Quantifier ("five over two", "two and a half", and "two point five")
119	Tanisha	Why'd you make it a decimal?	
120	Hord	You want two and a half instead of two point five?	Quantifier ("two and a half", and "two point five")
121	Tanisha	Well like, you can reduce five over two down? Oh you made it a -	Quantifier ("five over two")
122	Hord	I made it a mixed number.	
123	Tanisha	Mm hmm.	

In the above exchange, Hord quantified the slope as a single number (as "two and a half" or "two point five", in turns 118 and 120). Tanisha also quantified slope (as "five over two" in turn 121), but she quantified it as a pair of numbers and resisted the idea of turning that pair into a single value. An analogous interaction occurred between Tanisha and Emily in session 4, when the two were working on a task to represent the weight of an airplane as a function of how much fuel it held. The problem did not provide a graphical representation, but instead offered the following scenario:

Suppose that the weight in pounds of an airplane is a linear function of the total amount of fuel, in gallons, in its tank. With 12 gallons of fuel in its tank, the plane has a weight of 2178 pounds. With 40 gallons of fuel in its tank, the plane has a weight of 2360 pounds.

Tanisha and Emily turned the information provided in the problem set-up into a set of ordered pairs, (12, 2178) and (40, 2360). The conversation in Table 5 began with Emily asking Tanisha how she might use this information to determine the slope, and Tanisha used the typical slope formula,

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Table 5.
Tanisha and Emily's Discussion of Slope in Session 4, Segment 3.

Turn #	Speaker	Turn	Semantic Relationships
25	Emily	Now do you think you can find the slope?	
26	Tanisha	Cause, cause, oh yes, I can!	
27	Emily	Perfect.	
28	Tanisha	<i>Y</i> minus <i>y</i> , minus, wait, <i>y</i> one. So, <i>y</i> sub one, <i>y</i> sub two, <i>x</i> sub one minus <i>x</i> sub two [writing <i>y</i> ₁ - <i>y</i> ₂ and <i>x</i> ₁ - <i>x</i> ₂ as in Figure 3]. So I'll do 2178 minus 2360. [Tanisha calculates 2178-2360.]	Process
29-34		[Tanisha and Emily talk through calculations and Tanisha records the value -182/-28 for slope.]	
35	Emily	So now what should you do to make that a single number?	Quantifier ("single number")
36	Tanisha	But that's my slope.	Quantifier ("that", i.e., 182, 28)
37	Emily	Yes it is. But what can you do to make it easier on yourself?	
38	Tanisha	I could make it a decimal?	Quantifier ("decimal")
39	Emily	Yeah.	

Turns 25-28 illustrate Tanisha's use of a process to calculate a pair of values that she represented as a fraction, in response to Emily's request to determine the slope. Notably, the process that Tanisha described in turn 28—using the slope formula to calculate a pair of numbers—was distinct from processes she had used on the graph-based tasks, in which she counted horizontally and vertically between two points on the graph. Although the use of slope formula is equivalent to the counting process, it is not clear from our data whether Tanisha recognized the equivalence of these processes. Nonetheless, the outcomes of these two processes—a pair of numbers—were the same.

After Tanisha calculated her two numbers, Emily encouraged Tanisha to

simplify $-182/-28$ into a single number, but Tanisha responded that the pair of numbers itself represented the slope. The only other case across the data when Tanisha simplified a fraction into a single value was when she did so at Sarah's lead (Table 2). In session 3, Tanisha explicitly questioned the choice to simplify a fraction into a decimal number (Table 4). Although Tanisha complied and represented the slope as 6.5, she did so only in response to Emily's suggestion that it might make her work easier. From Emily's perspective, however, Tanisha's use of fraction notation most likely implied a single rational number, and so she would not likely have inferred the distinction between her meaning of slope and Tanisha's.

The use of a single rational quantity in reference to slope implies the quantification of an attribute of a linear function, often referred to via phrases like 'rate of change' or 'steepness of a line.' Importantly, however, Tanisha did not adopt such phrases in her talk, although the tutors periodically did. The only instance in which Tanisha discussed steepness in reference to a line came from the following task, focusing on the meaning of the constant parameter:

The given graph represents the equation $y=x+3$. How would the graph change if the constant was changed from 3 to 5?

The question was a multiple-choice question, and two of the possible answers were, "the line would shift up two units" and "the line would be steeper." Although the question, as written, was intended to target the change in y -intercept, Tanisha and Hord's discussion offers insight into her understanding of slope, as illustrated by Table 6 below.

Although the term "slope" never surfaced in the above exchange, the notion of steepness was relevant because "the line would be steeper" was one option included with the multiple-choice question about how the line would change. Tanisha grappled with the two options, "the line would shift up two units" and "the line would be steeper" (turn 5, turn 7, turn 13). Tanisha's comment in turn 13 suggested that she was using steepness synonymously with shifting up. In turn 3, when Tanisha used her own words to describe the change in the line (and not the language provided in the multiple-choice options), she noted that the line would "go up" without specifying a shift or a change in steepness. When Tanisha described counting from three to five along the y -axis (turn 9, turn 13), that counting process is consistent with her process for calculating slope on other graph-based tasks.

Table 6.

Tanisha and Hord's Discussion of a Changing Constant Value in Session 3, Segment 2.

Turn #	Speaker	Turn	Semantic Relationships
2	Hord	So if this three right here in y equals x plus three [$y=x+3$] was changed to five, what would that do to the line? Let me write this down so I can keep up.	
3	Tanisha	I think it would go up.	Process ("go up")
4	Hord	It would go up?	
5	Tanisha	Yeah. Shifts up two units.	Process ("shifts up")
6	Hord	Do you think it would shift up two?	
7	Tanisha	Yeah, the line would be steeper. No wait. Yeah it would shift up two units, so it would be a [indicating that she should select choice a from the available answers].	Attribute ("steeper") Process ("shift up")
8	Hord	Okay. Why - How do you know it would shift up two?	
9	Tanisha	Because it's at three, and then you go up to four, and then five. That's two units right there.	Process ("go up")
10-12		[Hord sketches a graph to check whether $y=x+5$ would be the correct function, and then asks Tanisha how she had known the correct answer.]	
13	Tanisha	Well, I was gonna say the line would be steeper. But that's not really an algebraic answer. So I put, 'cause I know that, you know, going from three to five, you're going up two units, and that's a , right there.	Synonym ("steeper" and "shifts up")

The discussion in Table 6 was the only time that the language of steepness surfaced in our data, and it contextualizes many of Tanisha's other comments related to slope. Tanisha established a synonymous relationship between steepness and shifting up, but nowhere in the data did Tanisha use steepness or slope to describe the slant, orientation, or rate of change of a line. Tanisha's construction of slope referred to a process of counting between two points on a graph without any connection to how those values might describe

an attribute of the graph. It seems that Tanisha never used slope to describe the rate of change of a line because she did not perceive rate of change as a feature of a line.

Discussion

We first situate the findings of this study within existing research on the teaching and learning of slope. Then, we discuss some implications of this work for research and practice.

Connections to Existing Research Findings

Based upon prior research, it is clear that a conception of a ratio as a single, composed unit is essential for students to make connections between slope quantities and the constant, continuous change of linear functions (see, e.g., Carlson, 1998; Carlson et al., 2002; Castillo-Garsow, 2012; Confrey & Smith, 1995; Johnson, 2015a; 2015b; Thompson, 1993). This is not to say, however, that students always—or even often—make these connections (Earnest, 2015; Lobato et al., 2003; Zahner, 2015). It is clear that students need more, and more meaningful, experiences to develop ratio reasoning and apply that reasoning to linear functions. The findings of this study support this argument but also suggest another dimension to the needs of students learning about slope. Arguments for a single, composite unit to represent rate of change presuppose, to some degree, that students recognize rate of change as an attribute of a line. In Tanisha's case, she did not verbalize any knowledge of rate of change, steepness, or slant, except to suggest that the steepness of a line may be synonymous with its global location on the graph. For a student like Tanisha, the importance of focusing on attributes of graphs—separately from and prior to their quantification—should not be overlooked, even in an algebra course.

The use of written, symbolic representations of fractions also became particularly salient in this study in how they shaped the verbal interactions. A ratio represented as a/b can be meaningfully interpreted either as a comparison of two distinct quantities or as a single value (Lobato, Ellis, & Zbiek, 2010). The latter interpretation is necessary for sophisticated understanding of slope, is a typical interpretation in more advanced mathematics courses, and is how Tanisha's tutors were using fraction

representations in the problems and in Tanisha's written work. The former interpretation is more aligned with how students use ratios in earlier grades and is more consistent with the process construction of slope that Tanisha showed. It is often taken for granted that a shared visual representation can serve as an anchor to clarify spoken communication, however this was not the case here. Although studies using thematic analysis have illustrated ways in which ideas become connected through verbal text (DeJarnette & González, 2016; Herbel-Eisenmann & Otten, 2011; Lemke, 1990; Webel & DeLeeuw, 2016), there is a need for more research to articulate the role of visual and symbolic representations in supporting, or inhibiting, those connections.

It is possible that shifting instruction to emphasize the use of real-world contexts, to use technology for graphing, and to deemphasize calculations of slope can support students' development of conceptual understanding of linear functions (Bardini, Pierce, & Stacey, 2004; Chazan, 2000; Yerushalmy, 2000). However, it is important to recognize that, even in cases where a teacher works to maintain strong conceptual focus, students do not always adopt this focus. Part of this can be explained by the constraints under which teachers and students work. Teachers' interactions with students are shaped in part by institutional demands such as curriculum alignment and student achievement on standardized assessments (Zahner, 2015). In this setting, Tanisha's tutors were tasked with helping Tanisha complete her classwork and homework. They needed to make constant decisions about which ideas to pursue based on the time available, the amount of work Tanisha was expected to complete, and the degree to which they could infer her understanding given the pace of their interactions. In light of these constraints, we consider some of the implications of this work for research and practice.

Implications for Research and Practice

Our analysis of tutoring sessions uncovered nuances in Tanisha's understanding of linear functions that would be unlikely to surface without attention to not only the student's talk, but more importantly to how the student and tutors talked with one another. One-on-one tutoring has shown promise for supporting struggling learners' mathematics learning and dispositions towards mathematics (Hord et al., 2016; Hunt & Tzur, 2017). In

particular, it is clear that students who struggle in typical classroom settings can make great strides in problem solving and conceptual understanding when teachers or tutors are responsive to their individual strengths and needs. Future research should build upon this potential and continue to explore how teacher-student (or, tutor-student) interactions create opportunities for learning. We have found that tutoring contexts have been most productive when focus is shifted away from accomplishing a pre-determined number of tasks and towards finding space to expand upon the questions and unexpected ideas proposed by students.

In typical classroom settings, our analysis of Tanisha's understanding of slope could also be useful as a resource for targeted assessments of students' knowledge. Based on what we learned from Tanisha, we suggest two questions in particular that a teacher might assess through a targeted diagnostic interview: *How does a student use phrases like steepness and shift in relation to slope? Does the student translate between fraction (e.g., $\frac{1}{2}$) and decimal (e.g., 0.5) representations of slope?* The first of these questions can help identify whether a student recognizes steepness as an attribute of a line, while the second can help establish whether a student has moved beyond an "over and up" counting process. By having questions like this in mind, a teacher can begin to uncover whether a student recognizes slope as an attribute of a line and uses connections between representations meaningfully (Adu-Gyamfi & Bossé, 2014). With this knowledge, teachers can draw upon existing resources that aim to develop students' understanding of linear functions through attention to multiple representations of constant covariation (e.g., Carlson, O'Bryan, Oehrtman, Moore, & Tallman, 2015; Kaput Center, 2016; Swan, 1985).

Conclusion

The focus of this study is on one student, but there are multiple ways that the findings are applicable across broader contexts. The in-depth analysis that is feasible with one student reveals nuances that are not likely to be noticed through larger-scale analyses. By identifying these nuances, we can put language to features of students' understanding that can be applied more broadly in research or practice. Additionally, given the difficulties that so many students experience learning about linear functions (Bush & Karp, 2013; Huntley, Marcus, Kahan, & Miller, 2007; Knuth, 2000; Lobato et al.,

2003), more information about the work of struggling learners can be used to anticipate and respond to others. Finally, small-group and one-on-one interactions with students are valued as a means to ensure that all students receive the necessary supports to be successful in mathematics (National Council of Teachers of Mathematics, 2000). Descriptions of students' work in these settings can serve as important examples for educators.

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