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Examining the Processes of High School Students to Do Proof Without Words*

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Abstract

There are opinions about approaching the proof in mathematics education need to change as providing students to understand the mathematical proof rather than developing formal mathematical proof skills. In this context, proofs without words described as informal proofs can be used in proof teaching. The purpose of this study was to investigate the processes of high school students to do proofs without words. This study was designed as the case study a high school in Turkey. Consequently, activities of proof without words that allow the student to take an active role in the proof process can be presented as an alternative method in proof teaching and the proof without words can be used for teaching students the stages of proof process. It is necessary for teachers to be aware of the proof process stages and guide the students.

Keywords: Proof without words, visual proof, proof process, Boero's model

Introduction

Proof is considered as the highest level of mathematics practice (Miller, Infante, & Weber, 2018; Tall, Yevdokimov, Koichu, Whiteley, Kondratieva, & Cheng, 2011) and many mathematicians criticise the proof teaching approaches presented in the schools and the studies have tried to reveal new ways for the proof teaching and learning (Hanna, 2000; Hanna & de Villiers, 2012). One of the main research topics of the mathematics is to determine whether or not the mathematical proof is used to justify mathematical propositions by the students and to what extent they are used (Rodd, 2000). Indeed, the proof is a ritual made by many students without understanding and the studies have showed that students cannot even do and understand simple proofs (Ball, Hoyles, Jahnke & Movshovitz-Hadar, 2002; Harel & Sowder, 2007; İnam, Ugurel & Yaman, 2018).

Although the key role of the proof is to increase the mathematical understanding, the real challenge is to find the most effective ways of using proof for this purpose (Hanna, 2000). It is important

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to show the proof to the students as a way of showing the accuracy of mathematical expressions to other people and helping them to understand why the theorem is correct so that the students understand the proof and comprehend its importance (Alibert & Thomas, 1991). According to Ugurel, Morali, Karahan & Boz (2016) if the proof is ensured to be understood by students, it becomes possible to change their negative views about the proof. It is an important skill to understand a proof, but it is not necessary for the student to have a high mathematical success for understanding the proof. So we can state that, it is necessary to provide an opportunity to the students for understanding the proof.

Proof requires a series of mental activities such as research, the use of mathematical concept knowledge, ability to apply heuristic strategies to the problem, determining the hypotheses, organising logical arguments, analysing them, and organising the necessary transformations. The reason for the failure in proof teaching is the lack of recognising this complex process. Understanding the nature of the proof in mathematics signifies the realisation of these mental activities (Ball, Hoyles, Jahnke & Movshovitz-Hadar, 2002; Heinze, Cheng, Ufer, Lin, & Reiss, 2008).

It is reasonable to raise the student's awareness on the proof stages and proof process in order to support the proof learning. Therefore, the teaching of mathematical content placed under the proof problems and encouraging the students to explain during the proof process are important matters points to take into consideration during formation of the learning environments (Inam, Ugurel & Boz Yaman, 2018; Reiss, Heinze, Renkl & Gross, 2008). Indeed, when the students do not know which steps the mathematical proofs are realised in, this causes them to have difficulties in mathematical proof (Caliskan, 2012). Students generally use informal proofs (Coe & Ruthven, 1994; Reiss et al., 2002). Therefore, informal proofs appropriate for the learning statuses of the students should also be included in proof teaching. As a matter of fact, approaching the proof in mathematics education has changed as providing students to understand the mathematical proof rather than developing formal mathematical proof skills (Marrades & Gutierrez, 2000). In this context, proofs without words described as informal proofs can be used in proof teaching.

A proof without words is a proof that using visual presentations are used to present a mathematical idea, equation or theorem. Proofs without words do not contain any word other than the geometric drawings and numerical or verbal symbols. However, these proofs may contain several equations, arrows or highlights providing a visual clue to guide the reader (Gierdien, 2007). Some proof without words contain no information about the theorem but some of them contain textual expressions about the hypothesis as given in Figure 1.

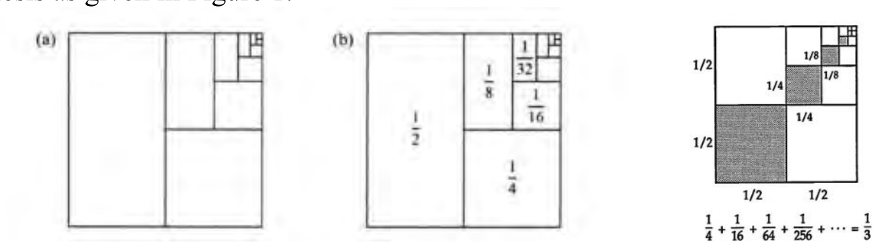


Figure 1. Examples of proof without words

Sigler, Segal, and Stupel (2016) stated that proof without words is an elegant visual proof of a theorem. Proofs without words are used in number theory, history of mathematics, trigonometry, mathematical inequalities, proof of geometric theorems (Alsina & Nelsen, 2010; Bell, 2011). Proofs without words can consist of a single shape or multiple shapes. Figure 2 shows examples of proof without words from different fields.

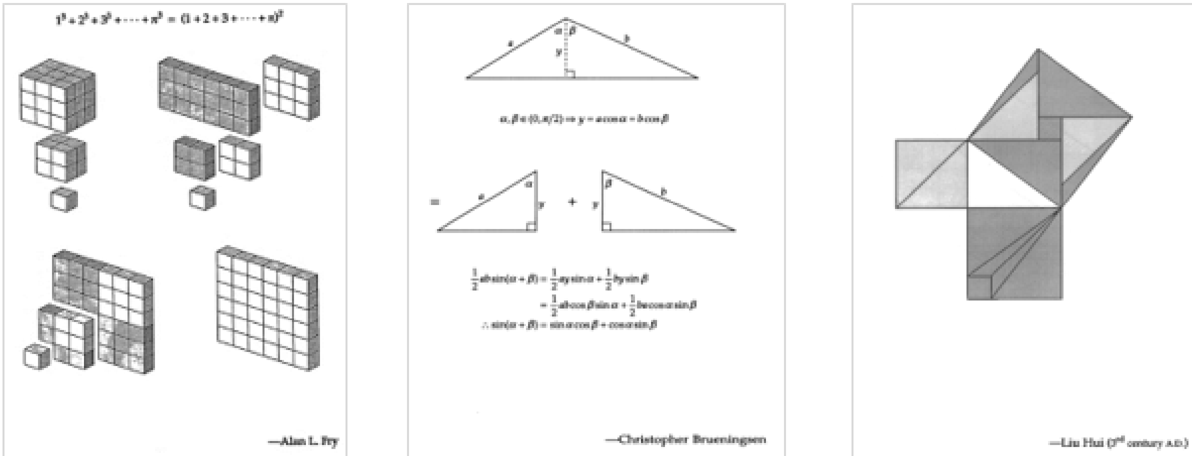


Figure 2. Examples of proof without words from different areas

There are discussions about whether a proof without words is a real proof. Brown (1997) states that since the visual presentations can also be persuasive and descriptive as traditional proofs, same level of proof can be made with the traditional proofs. Rodd (2000) expresses that proofs without words are the tool for discovering the geometric facts, they can express a hypothesis by words but they can be understood better if they are shown with an appropriate visual presentation and the proof without words provides direct information to students and this information is general, indisputable, and reliable. Proof without words are didactic tools and make formulas more transparent (Hauben, 2018). Proofs without words in pure mathematics are not regarded as formal proof methods but various arguments have shown that this situation may change in the future (Rinvold & Lorange, 2011). Nelsen (1993) mentions that proofs without words are diagrams or pictures that will help us to see why a specific mathematical expression is correct and even how we can handle while proving the accuracyof a mathematical expression. Proofs without words help the viewer to see and understand why a theorem is true (Alsina & Nelsen, 2010; Kristiyajati & Wijaya, 2018; Sigler et al, 2016). In of them, clues guide the viewer in the proof process. For example, a student trying to explain proofs without words uses nontrivial mathematical ideas in the proof. In other words, the proofs without words have an embedded invisible mathematics. As an example for this situation, students do not know the cause of $\frac{1}{2}$ in the formula $1 + 2 + 3 \ldots + n = \frac{n^2+n}{2}$related to the sum of integers. Contrary to the inductive proof, in this proof without words (Figure 3), the proof is geometrically justified by using the area of a specific triangle having height and base length of n. The visual proof seems to work considerably better than the algebraic alternatives in order to give students meaning and richness of experience when understood or objectified. It can be difficult to see the visual proof by student (Rinvold&Lorange, 2013). This difficulty is due working with a visual proof requires a continuous interplay between the semiotic system of figures and the semiotic systems involved in the statement, usually verbal texts or symbolic expressions. Another main difficulty encountered by students is due to the lack of coordination of systems of semiotic representations (Bardelle, 2009).

If there is a good picture in the proofs without words, the students see the accuracy of the mathematical expression and form the reasoning and identification chains by visualising the process. The students explain what they see in the proofs without words and thus use the visualisation process skills such as definition, estimation, description, conclusion, observation, and generalisation (Gierdien, 2007). Therefore, the students take part in the proof process and also understand why the proof is correct.

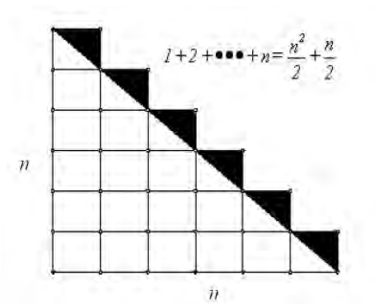


Figure 3. Proof without words about integer sums

Boero's model

The proof process and the proof output must be separated from each other. Boero (1999) defined a model to differentiate the proof process and proof output. This model is divided into different stages and it is the combination of experimental-inductive and hypothetical-deductive steps during the formation of a proof (Heinze & Reiss, 2004).

Boero (1999) states that entering the culture of theorems requires certain competencies such as producing hypotheses and proving the hypotheses produced by considering the theoretical knowledge elements. Epistemological and scientific analyses are needed to manage the coping of students, who are just new in proof, with the theorems. For this analysis, hypothesis production and mathematical proof structure are called as phases (Boero, 1999).

Boero divided the proof phases into six including ―production of a hypothesis", ―formulation of the statement according to shared textual conventions", ―exploration of the content (and limits of validity) of the hypothesis", ―selection and enchaining of coherent, theoretical arguments into a deductive chain", ―organisation of the enchained arguments into a proof that is acceptable according to current mathematical standards", and ―approaching a formal proof" (Boero, 1999).

The proof process of the mathematicians and students is quite different from each other. While mathematicians can argue richly and freely, this is not the case for the students (Boero, 1999). Since the Boero's model is related to the proof process of the experts, Heinze and Reiss (2004) made changes to adapt the Boero's model to the proof process of the students. The proof process, expressed as six phases by Boero was stated as five stages by Heinze and Reiss (2004).

Stage 1: This stage is the description of the problem status and the identification of the arguments that will support the hypothesis. Examination of a drawing during the geometry lesson plays this role. The proof process is introduced with this stage by asking the students to describe the figure in proofs without words.

Stage 2: This step is the expression of the hypothesis based on the textual rules. If the student can express the hypothesis obviously and clearly during this stage, this means that this stage is completed successfully.

Stage 3: The purpose of this stage is to plan roughly the possible proof strategies by determining the appropriate arguments for the validity of the hypothesis. This stage is divided into four subcategories as ―hypothesis reference, investigation of hypothesis, collection of more information, and the formation of a proof idea". The observation of at least three of these subcategories shows that this stage is completed successfully.

Stage 4: This stage includes the elements of a previous stage. The third stage ends with the rough planning of the proof by determining the arguments. In this stage, the hypotheses in the draft must be combined within a deductive chain. If both the student and the teacher contribute to the process at this stage, it can be asserted that this stage is completed successfully.

Stage 5: This stage is the final stage of the proof process in school mathematics. In this stage, there is a retrospective general evaluation. If all the claim steps are expressed and there is an abstract of the process, this stage is completely successfully. During the process of proof without words, the teacher may ask for a general evaluation from the student to complete this stage successfully.

The students cannot comprehend that the proof is an operation from the given to reach the requested within a series of consecutive logical steps and cannot transfer this into the proof process (Ugurel & Morali, 2010; Urhan & Bulbul, 2016). However, a successful proof teaching should include the whole proof process but most of the students lose control over the proof process (Heinze & Reiss, 2004). Therefore, the fact that the teachers who plan and control the proof process ensure that the stages of the proof process are completed successfully is important for the proof teaching. Research on proof without words in mathematics education is not much. Also, the proof without words can be used for teaching proof process stages to the students. Sothe purpose of this study is to investigate the processes of high school students to do proofs without words. For this, the study was conducted to find the answers to the following question:

Research Questions

- How is the high school students' processes of performing proofs without words?
- What are the stages of proof without words?
- What are the proof stages that high school students ignore during the proof process?
- What are the relations between the stages of proof without words?

Research Method

Design

This study was conducted as the case study with the qualitative research methods since it is aimed to investigate whether or not the students who did the proofs without words experienced the stages of proof process in the mentioned proof model and the group to be investigated is in the same environment and situation, it can be asserted that this study can be examined within the holistic single case studies. In the study, the students' skill of doing the proof without words was examined within the theoretical framework of the model investigating the proof process. In this sense, based on the purpose of this study, it was considered appropriate to investigate the skill within the descriptive case study from the classifications made by Yin (2003). In addition, this study can be included into the exploratory case study in terms of adapting the existing model into a new topic. Exploratory case studies can be used to explain the situations that have not yet been clarified and to evaluate any case from different perspectives. Such studies have a draft quality for further studies (p. 15).

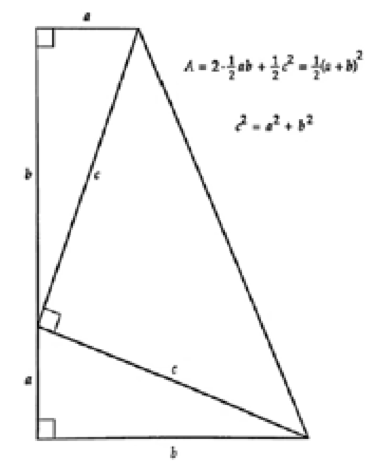
Participants

This study was conducted in a high school located in Turkey during the 2016-2017 academic year. The sample of the study consisted of 25 students who were studying in the 9th grade. The reason for conducting the study with 9th graders was the content of the topic. The study was conducted with 9th grade students in order for them, who just started to learn the proof subject, to meet with proofs without words as an alternative proof method and to gain the skills for the structure of the proof by reaching the proof step by step via the activities of proof without words. The sample was selected randomly from two classes in the school. In order to get in-depth information after the activities, four students were included in the study to do individual activities of proof without words by considering their ability to express themselves, their interest in the lesson and participation to the activities in the class, their success scores and exam scores.

Instruments and data collection procedures

In order to reveal the stages of process of proof without words, the data were obtained from the video records of the activities conducted for a semester and the individual activities whis was conducted with four students.

Activities consisted of a total of sixteen activities related to eight topics. Activities were composed of the proofs without words related to the sum of internal angles of the triangle, triangle inequality, Pythagorean Theorem, Viviani's theorem, Euclid's theorem, Sine law, areas, and Cosine law. These activities were prepared based on the steps of the proof model of Heinze and Reiss (2004). One of the activities related to the proof without words of the Pythagorean theorem was presented in Figure 4.



- What does the figure mean to you? Discuss. (explanation of the problem situation)
- Write the lengths of the triangles' side given to you. Use these three triangles to form a vertical trapezoid.
- Discuss the area of the trapezoid and the total area of the three triangles.
- Find the areas ıf the triangle and the vertical trapezoid.
- Discuss the equation.
- Evaluate the activity.

Figure 4. Activity of Pythagorean theorem

The students followed the instructions, completed the stages of the model, and reached to the proof without words. While researchers worked only as an observer in some activities in the study, they conducted the activities with the teacher of the lesson in some activities. The teacher has successfully completed the course related to the proofs without words during his period of undergraduate education. So he has sufficient equipment about proofs without words.

It was observed that the students participated more actively in the lessons conducted together by the teacher and the researchers. This is thought to be associated with that the researchers guided the students better in the proof process since she has more knowledge about stages of the proof process and the teacher guided students better because of knowing about which subject the students in the class have knowledge.

After the eight week, four students were interviewed in order to examine deeply the process of doing proofs without words. Proofs without words were presented to the students and they were asked to explain the proof without words. The aim was to reveal the stages that students neglected in the process of proof and how the stages affected each other.

The individual activities were selected according to the proof without words properties defined by Davis (1993). He states that the proofs without words can be used to demonstrate all results of heuristically seen plane and solid geometry, theorems of higher mathematics having geometric or visual basis, and graphic display of pure or applied math results. Four questions of the study were composed of the proofs without words being examples to the theorems of higher mathematics based on visual presentation, five questions were composed of the proofs without words being examples to the plane and solid geometry, three questions were composed of the proofs without words being examples to the graphical display of pure or applied mathematics.

Data analysis procedures

The data were obtained from the observations, video records during the class activities and the individual activities. By transforming separately the interview and video records into text documents, the data set of the study was obtained.

In fact, the framework to be used to analyse is a framework adapted by Heinze ve Reiss's model from the Boero (1999)'s model whis is used for the proof process of students. In this study, this framework was adapted to the process of the students to do proof without words and it was tried to reach new categories and differences in the process via content analysis. Table 1 shows the codes related to the models used in the data analysis of the study.

The textualised form of the lessons recorded with a camera was reviewed by different experts. Since the basic stages of the process were revealed with the model, the data were analysed in order to obtain results about which stages took place, which stages did not or which different stages took place.

Table 1. Codes Related to the Model Used for the Data Analysis

Boero's model	Heinze and Reiss's model
• Production of a hypothesis ➢ Exploration of the problem ➢ Identification of regularities ➢ Identification of conditions under which such regularities take place ➢ Identification of arguments for the plausibility of the produced hypothesis	• Exploration of the problem ➢ Identification of arguments that give support for the plausibility of the hypothesis ➢ Investigation of the drawing
• Formulation of the statement according to shared textual conventions	• Formulation of the hypothesis according to the shared textual convention
• Exploration of the content of the hypothesis ➢ Heuristic, semantic elaborations about the links between hypotheses and thesis ➢ Identification of appropriate arguments for validation	• Identification of appropriate arguments for the validation of the hypothesis and rough planning of a possible proof strategy ➢ The reference to the hypotheses ➢ The investigation of the hypotheses ➢ Collection of further information ➢ Generation of a proof idea
• Selection and enchaining of coherent and theoretical arguments into a deductive chain	• Combination of the elements of the previous stage ➢ Combination of the arguments into a deductive chain
• Organisation of the enchained arguments into a proof that is acceptable according to mathematical standards ➢ Production of a text for publication	• Overviewing about the proof process
• Approaching a formal proof	• This stage does not occur at all mathematics lessons

In the study, many details were described and tried to be transferred to the reader. The researchers described the environment and the participants in detail and included the direct citations from the participant interviews and research documents. In the study, multiple data sources and methods were utilised (method triangulation). In the data presentations, graphs, tables and qualitative analysis were utilised (analysis triangulation). The analyses were carried out by different researchers and the agreed points were presented as the result (researcher triangulation).

In individual activities, how many students would be interviewed was not determined. The researchers, who thought that the data reached a certain satisfaction, decided to end the interviews after four students. The data, analyses, and interpretations were examined by experts and the research process was also controlled by the people other than the researchers. The research strategy, the data collection, and the data analyses were mentioned in detail so that the transparency was provided. Figure 5 indicates the process of the study.

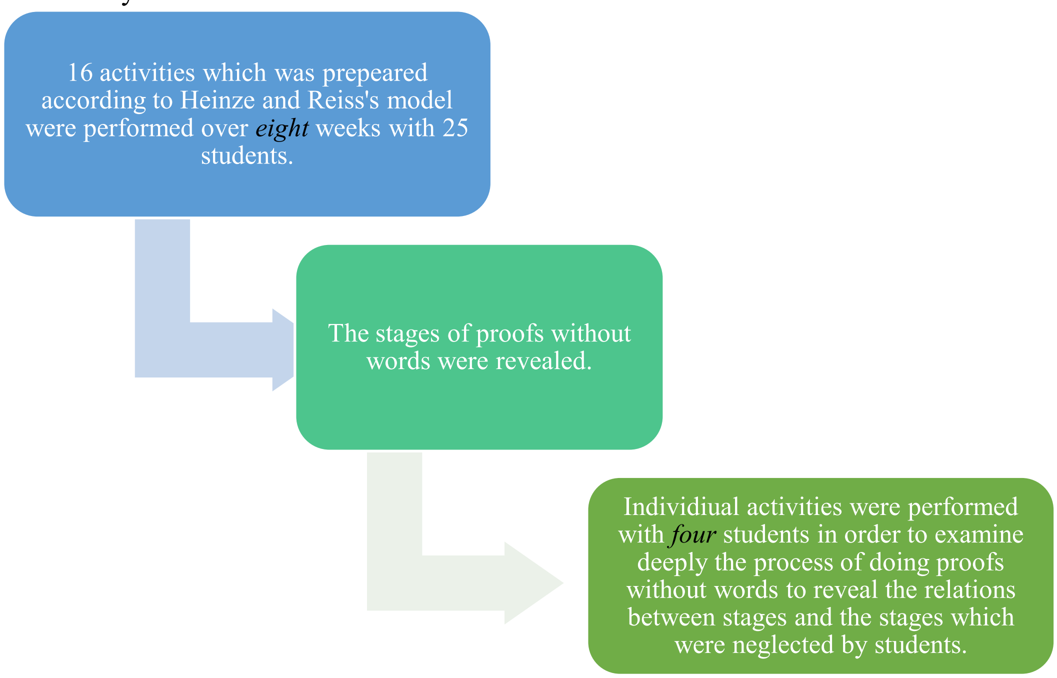


Figure 5. The process of the study

Findings

The findings are presented in two parts which are obtained from in-class activities and from individual activities. Class activities reveal the stages of proofs without words, individual activities reveal the stages that students neglected in the process of proof and how the stages affected each other.

Findings of class activities

The stages determined in accordance with the model used in the study were expressed as “*checking the figure, expressing the hypothesis obviously and clearly, determining the appropriate arguments for the validity of the hypothesis, making necessary operations, and summarising the given*”. There are sub-stages of each stage and they were stated as the first, second, third, fourth, and fifth stages.

The sub-stages of the first stage were determined as “*explaining the figure correctly, and textual sharing together and focusing on the textual sharing, and inadequate explanation for the figure*”, the sub-stages of the second stage as “*realising the hypothesis, failure to realise the hypothesis*”, the sub-stage of the third stage as “*justification of what had been done*”, the sub-stages of the fourth stage as “*completing the operation, doing operation incompletely, and failure to complete the operation*” and the sub-states of the fifth stage were determined as "*understanding which theorem is correct and why, explaining the figure and textual sharing holistically*". Figure 6 shows the stages appearing as a finding of the analysis of the study according to the mentioned model.

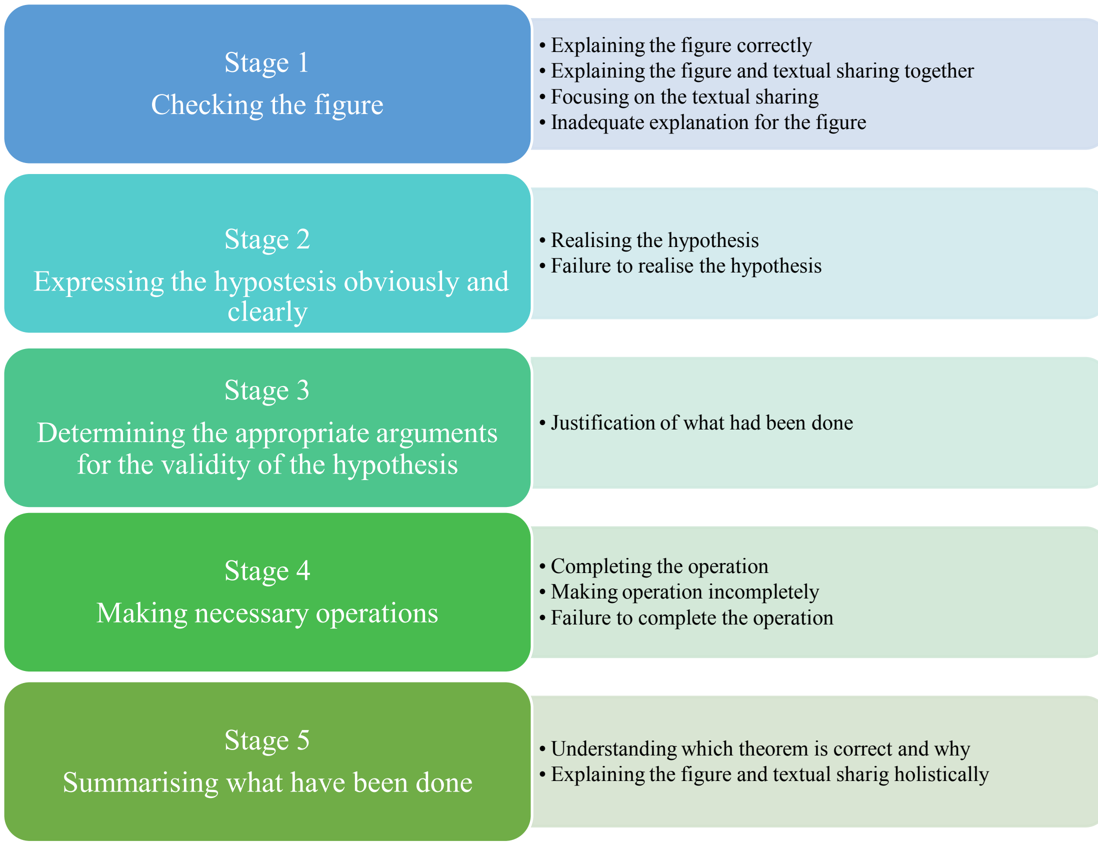


Figure 6. The stages of proof without words

First stage. When the students met the activities of proof without words, the students were asked to explain the given visual. At this stage, if the students can explain the visual adequately, this means that the first stage is completed successfully. In the proofs without words, there are textual statements about the hypothesis along with the figure. When the process was examined, it was observed that some of the students explained only the figure, some explained the figure and the textual statement together, and some explained only the textual statement. The finding of the first stage of the proof process which named examining the figure were shown below by including the citations.

It was observed that the students were not able to give sufficient explanation about the figure related to the proof without words at the beginning of the activities. For example, in the activity of proof without words concerning the triangle inequality, the students were asked to make an explanation about the figure; however, it was observed that they couldn't make a sufficient explanation and could only say the textual statement.

S1: The figure indicates the sides of a triangle. It says that $a + b$*is greater than c. The arrows show the side c. I think it states that the longest side is c, followed by b and a, respectively.* **(Focusing on the textual sharing)**

*S2: Only a+b> c.***(Focusing on the textual sharing)**

S3: Represents a triangle. The arrows have the forward and backward directions. **(Inadequate explanation for the figure)**

S4: Angle A faces the side a, Angle B faces the side b, it is just like naming. **(Inadequate explanation for the figure)**

The students' explanations about the figure are not sufficient. In the following lessons, it was observed that they could provide adequate explanations about the figure.

S5: A=$2.\frac{1}{2}ab+\frac{1}{2}c^2=\frac{1}{2}(a+b)^2$*, I guess this area refers to the big triangle therefore it is multiplied by 2. There is also* $\frac{1}{2}c^2$

R:What is $\frac{1}{2}c^2$*?*

S5:The right triangle in the middle . $2.\frac{1}{2}ab$ *the area of two triangles . The sum of those is equal to the trapezoid covering all of them.* **(Explaining the figure and textual sharing together)**

The citation given above belongs to the Pythagorean activity. As is seen, S5 could made a correct explanation about the figure and explain the figure in relation to the textual statement in the proof without words.

As in the proof without words example of the students showing Euclidean relation, they could only make a correct and adequate explanation about the figure without establishing an association with the textual statement, as well.

S6: One cut and removed the CDB triangle. Then, he joined it with the height of the other. He also joined the remaining side with its corner. The next step is hard. Ok, he found the height of CAE triangle.
R: What did he do for the height here?
S6: Rotated.
R: Rotated or moved?
S6: Yes, He moved it. He then removed the cut piece. He equalised them. **(Explaining the figure correctly)**
S7: He found the height of the triangle in the large figure. He made a square for that height. In other words, he made a square as much as the length of that height. This is the figure. **(Explaining the figure correctly)**

When the models about the stages of the proof process and the lesson process made with the proofs without words were considered, the stage of ―examining the figure" was seen to have four sub-stages. These stages were determined as ―***explaining the figure correctly***", ―***explaining the figure and the textual sharing together***", ―***focusing on the textual sharing***", and ―***inadequate explanation about the figure***". Explaining the figure and textual sharing together in the first stage was important for the stage of realising the hypothesis and explaining the figure and textual share holistically in the next stages. As a matter of fact, most of the students, who only explained the figure correctly and cannot make the association of the figure with the textual statement, could not use any statement about the hypothesis.

Second stage. In the proofs without words, the hypothesis is usually present in written besides the proof without words as mentioned in the introduction. Students could sometimes derive the hypothesis from the figure and sometimes from the textual statement. In some cases, the students could not say anything about the hypothesis.

It was tried to reveal the findings of this stage by involving the citations from the activity of triangle inequality about the stage of expressing the hypothesis obviously and clearly stated as the second stage of the proof process.

R: What do you see in the figure?
S3: There are a trapezoid and three right triangles.
S4: There is an isosceles right triangle
S5: There are also congruent triangles, here.

R: Yes. All right, what does this written equation show?
*S4: It is like Pythagorean Theorem.***(Realising the hypothesis)**

The students' realisation of the hypothesis was seen important in terms of having awareness about the hypothesis in the mathematical operations they would do later. In the activities, the students reached to the proof without words by following the directions given by the teacher. Giving no information about the hypothesis caused the student to focus only on the operation. As a result, the students could not see what they were proving. Students examining the figure and textual statements together could realise the hypothesis more easily.

Third stage. The third stage of this study was the stage where the students justified the operations and decided which operation they would do.This stage was defined as determining the arguments appropriate for the validity of the hypothesis. This stage was important for the students to continue by completing the operations especially in the next stages. The student was very active at this stage. The teacher constantly directed the students to access the information they would use to reach the result by themselves.

It was tried below to reveal the findings of this stage by including the citations about the stage of determining the arguments appropriate for the validity of the hypothesis stated as the third stage of the proof process.

R: Why are these angles equal?
S2: These two are parallel; the interior angle of this is equal to its exterior angle.
R: We are calling something specific to that rule. What is it?
S2: It is Z rule. This angle and that angle are equal. **(Justification of what had been done)**

As it is seen from the direct citation, the student used the information about the previously learned topics. The student was asked to justify why the given angles are equal in the activity of proof without words related to the sum of the interior angles of the triangle.

This stage is important especially for the student to understand not only that the theorem is correct but also why it is correct. This is because the students are encouraged to explain what was done. And they have to think why they made each operation.

Fourth stage. The fourth stage of this study was determined as conducting the necessary operations. As also noted in the previous stage, the operations determined in the previous stage were performed during this stage. Although the students determined correctly which operation they would do, they couldn't complete the operations since they had poor operation knowledge and this in turn reduced their motivations.

The findings of this stage were revealed by including the citations of doing the necessary operations stated as the fourth stage of the proof process as well as the images from the students' papers.

S1: The area of the large square is the sum of areas of four right triangles and the area of the small square inside.
R: What is the area of the right triangles?
S1: $\frac{a.b}{2}$*. Since there are four triangles,* $4.\frac{a.b}{2} = 2.a.b$

S1 : The area of the inner square would be$(b-a)^2$ *since its side is* $b-a$*. The area of the large square is c. Then it would be* $c^2 = (b-a)^2 + 4.\frac{ab}{2}$. **(Failure to complete the operation)**

The student could not complete the operations after this part. All the students in the class had difficulties in the identity.

R: Compare the area of the trapezoid with the sum of the areas of three right triangles.
As a class: They are equal.
R: Why?
As a class: We used the same shapes.
R: You know the formula for area of the triangle. Can you write the areas of these three triangles.
Students: That of two is $\frac{a.b}{2}$ *and the other one is* $\frac{c^2}{2}$

R: What did we do with these three triangles?
As a class: Trapezoid
R: What was the area of the trapezoid?
As a class: ……

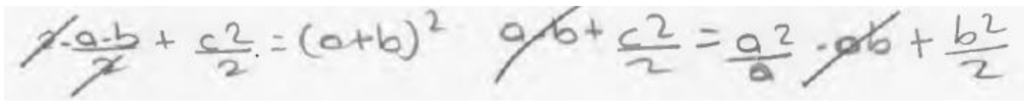


Figure 7. A student's paper for Pythagoras activity

When the student's paper in the Figure 7was examined, the student made a mistake about the formula for area of the trapezoid and the expansion of the identity. Since the students could not remember the area of a right trapezoid in this activity, this topic was mentioned. Then, the students were ensured to do the necessary simplifications and to find the Pythagorean theorem. (**Completing the operation)**

Fifthstage. The fifth stage was also defined as the stage where the student summarised what have been done. As stated in the findings and interpretations of the second stage, especially the stage of realising the hypothesis is important for the stage of summarising what have been done which is the final stage of the proof process. The fact that the student summarises what have been done by making a retrospective evaluation is important for the awareness about which theorem is correct and why.

By including the citations related to the stage of summarising what have been done stated as the fifth stage of the proof process, the findings of this stage were revealed below.
T: What do you see after this folding process?
S1: Three angles are next to each other and sum of them is 180°.
T: What did we obtain?
Students: A line.
R: All right, what have we seen with this activity?
Students: When the interior angles of a triangle form a line when they are folded.
R: What have we showed with this folding process?
Students: The sum of the interior angles of the triangle is 180°. **(Summarising what have been done)**

The quatation given was the activity of proof without words showing that the sum of the interior angles of the triangle is 180°. The fact that the teacher asked the students about what had been done at the end of the activity was important in terms of the awareness about the operations done. In the activities of proof without words, particularly the students were asked to summarise what had been done at the end of the activity. A citation about the activity of the proof without words related to a triangle inequality was given below.
T:Is there anyone who can summarise what we have done so far? What did we discover?
S1: We looked at the sides.
S2: We examined the association between the sides and angles.
R: We examined the association between the side lengths. What did we recognise at first?
Students: The sum of two sides must be greater than the other side.
T: Their difference?
Students: Small.
R: Then, what are the conditions for drawing triangles?

S3: It must be smaller than the sum of two sides and greater than their difference. **(Summarising what have been done)**

At the end of the activities, student's summarisation provides a holistic explanation of the figure and textual sharing and an understanding for which theorem is correct and why. In the individual activities of proof without words performed with students, summarisation of the things done enabled the students to realise the details they had not recognise until that moment.

Findings of individual activities

In this section it is presented that the findings of the individual activities of proof without words which is conducted with four students. In the dialogues it was shown as researcher (R), Can (C), Yunus (Y), Mehtap (M), Ferit (F).

In the question in which proof without words of algebraic inequality was given, all the students were able to explain the figure correctly. However, three of them could explain the figure and textual sharing together; one could only explain the figure. The student who explain only the figure did not realize the hypothesis. In the individual activities of proof without words, Can was able to express the hypothesis. He examined the figure and textual statement together and then he tried to explain what was done in the proof without words and reached more easily to the result. However, Yunus often did not examine the figure and textual statement together and could not explain the figure sufficiently and could not find the result in case that he only directed to the figure. It was thought that especially the stage of realising the hypothesis was important for the summarising stage which is the final stage of the proof process. The following is a dialogue that an example of this situation:

Y: The area of the rectangle was found. $\frac{a}{b} \cdot bd = ad$. *The area of another rectangle was found* $\frac{c}{d} \cdot bd = bc$. *The first area is bigger than the second area.*

R: And how did it say that?

Y: This area is cut. Therefore, it is smaller than.

As it can be seen from the citation, the student explained only the visual, he did not make any connection with the given textual expression and figure. Also, he did not use any expressions about hypothesis.

A citation about a student who solved the question by examining the visual and the textual expression together. Firstly, the student read and explained the textual sharing. After reading the textual sharing, he saw inequality in the second part of the proposition.

C: The area is a.d. Because one side is $\frac{a}{b}$ *and the other side is b.d, so if we multiply, the area is a.d. similarly one side is* $\frac{c}{d}$ *and the other side is b.d, so if we multiply, the area is b.c. The area of first rectangle is smaller than the second.*

R: Why?

C: Because one sides of two rectangles are same, the area of rectangle is smaller because its other side is smaller than. That figure shows this.

Also,Mehtapsolved the question by examining the visual and the textual expression together and she could explain hypothesis:

M:$a.d$ *is smaller than* $b.c$

A: What are this?

M: Area. If $\frac{a}{b} \leq \frac{c}{d}$ *we can say* $a.d \leq b.c$.

If the students cannot express clearly the hypothesis, they focus only the operations so they forget what they prove. As a matter of fact, proofs without words provide understanding the student why formulas are correct. However, it is thought that the failure to draw attention to the hypothesis prevent this. Therefore, it is important to direct the student in the process of proving. The following is a citation that an example of this situation:

R. What do you see in the figure? Can you explain?

Y: There is a triangle in a circle and an altitude.
A: Why this length $\sqrt{a.b}$ *?*
Y: Because of Euclid theorem.
R: What is Euclid theorem?
Y: In a right triangle the square of the altitude is multiplication of separated lines. So it is a.b
R: What do you see in the other figure?
Y: It is radius and $\frac{a+b}{2}$

The student explained the figure correctly**,** he justified the operations. But he could not say anything about hypothesis. So researcher guided the student to notice hypothesis:
R: What can you say about the relation of these lengths?
Y: The radius is equal or longer than altitude. And this figure indicates this. Indeed, this length (which means the radius) is longer than the other length.

The proof without words of the Thales theorem was first presented to the students with a single figure. And the students were asked to explain this figure. Students could not give sufficient explanation about this proof. Here is the citations about this:
F: There is a rectangle. This rectangle is separated. Is this a formula? Or is this an area?
R: If it is area, why these areas are equal?
F: One's side is longer, but its' altitude is shorter. The others altitude is longer, but the side length is shorter.
R: But does this guarantee the equality of the area?
F: I do not know.

Because of students could not give sufficient explanation about this proof, the proof without words of the same theorem was presented gradually. The students could explain the proof which was given gradually. They could realize hypothesis. When the students' proof processes of Thales theorem were examined, it was seen that the students could easily understand and explain the proof which was given as gradually. As mentioned before, the activities of proofs without words were planned in stages. The students reached the proof with the instruction on the worksheets and the guidance of the teacher. In the activities, students approached the proof step by step, they understood better the construction stages of the proof. Especially they enjoyed because they reached the results of the proof themselves.

The stage of determining the appropriate arguments for the validity of the hypothesis is important for the students to understand not only that the theorem is correct but also why it is correct. This is because the students are encouraged to explain what has been done and they have think why they do each operation. In this way, students can use the subjects they have learned in the past.

In the process of proof, students neglected the stage of summarize. In this case, students focused only on the process; so, they could not realize hypothesis and what they prove. As a matter of fact, the dialogue between the researcher and the student during the examination of the proof of the Thales theorem is an example of this matter.
R: You explained figure correctly. Can you explain what indicate this proof without words?
M: Is equality of the areas?
R: Is it about this triangle?
M: It can be. I don't know.
R: What is the equality$\frac{a}{a'} = \frac{b}{b'}$*?*
M: We equalized and simplified the fields. It means.

Discussion and Conclusion

According to Bell (2011), using the proof without words is effective on understanding the proof process by students. In this study, proofs without words were used for understanding the proof process by the students. In the light of the findings the stage of proof without words were determined as –*checking* *the figure, expressing the hypothesis obviously and clearly, determining the appropriate arguments for the validity of the hypothesis, making necessary operations, and summarising the given".*

The relationship between the stages and the stages which is neglected by students are presented in this section.In the stage of examination of the figure that the students who examined the figure and textual sharings together could more easily realise the hypothesis, the students who realised the hypothesis had higher awareness about the hypothesis in the next operations compared to the other students. Doruk (2016) stated that having a successful argumentation process before starting the proof increased the success in proof, in other words, when considering the result indicating that the process of producing hypotheses before starting the proof facilitated the proof process, it can be asserted that emphasising the stage of realising the hypothesis in the proof process is important.

It was determined that having no information about the hypothesis caused the students to focus only on the operations they did. As a result of this situation, they did not realise what they were proving. Demircioglu and Polat (2016) suggested that the having no explanation about the figures prevented pre-service teachers understanding of the hypothesis. In fact, Doyle, Kutler, Miller, & Schueller (2014) stated that when sentences and titles allow establishing an association between the theorem shown in the proofs without words having short textual statements and logical ordering, the proof standards are met better. So some explanation helps to see why the figure proves the theorem (Miller, 2012). Also, Sigler et al (2016) some clues, such as equations in proof without words guide the students in the proof process.

The stage of determining the appropriate arguments for the validity of the hypothesis stated importantence for the students to understand not only that the theorem is correct but also why it is correct. This is because the students are encouraged at each stages of the proof to explain what has been done. Also they must think why they are doing each operation. Therefore, when the perspective of students was examined in the study by Strausova and Hasek (2012), proofs without words developed the discussion ability of the students to argue the solutions and helped them to gain the ability of applying the information obtained from a wide perspective. Because, the student who tries to understand the problem of proof without words has to use the features of other topics of mathematics. In addition, when considering the result of the study by Karras (2012) indicating that the pre-service teachers having good geometric knowledge level can do the proof without words better. So it can be asserted that the proofs without words require the use of information learnt.

The stage of doing the necessary operations was the stage in which the operations determined in the previous stage were performed. Although the students determined correctly which operation they would do, they couldn't complete the operations since they had weak operation knowledge. Motivation of the students in completing the proof decreased due to their poor operation knowledge. In fact, this result was compatible with the results of the studies revealing the reasons for failure of the students in doing a proof (Bardelle, 2009; Demircioglu & Polat, 2016; Moore, 1994; Reiss et al., 2002).

The stage of the summarising what have been done is defined by Heinze and Reiss (2004) as the stage where the retrospective general evaluation is made about the process and the final stage in school mathematics. In this study, this stage was also defined as the stage in which the students summarised the operations done holistically. The stage of realising the hypothesis is important for summarising what have been done, which is the final stage of the proof process. Students' summarisation of what have been done by making a retrospective evaluation is important for their awareness on which theorem is correct and why. At the end of the activities, student's summarisation provided the holistic explanation of the figure and textural sharing and understanding which theorem is correct and why. In the individual activities of proof without words performed with the students that summarising what have been done led the students to realise the details they had not recognised until that moment. As a result of the experimental study by Heinze and Reiss (2004), the proof stages of the Boero's model are important and necessary for proof teaching; however, these stages are ignored by the teachers. In particular, this is confirmed by the fact that the third stage determining roughly the arguments and proof strategies appropriate for the validity of hypothesis was passed quickly in the classes and was not emphasised much. The result of this study indicated that this stage affected the following stages and thus affected the whole of the proof process. Similarly, it was concluded in the proof teaching made according to the model developed by Ozturk

(2016). that He tried to perform proof teaching by passing the stages a mathematician encountered while performing the proof and this process had a positive effect on proof skill. Therefore, ensuring the students complete the proof process stages had a positive effect on the proof teaching.

Consequently, it can be asserted that informal proofs gain importance in order to gain proof process skills of students. Therefore, the activities of proof without words among the informal proofs that will allow the student to play an active role in the proof process can be presented as an alternative method in proof teaching and the proof without words can be used for teaching the studentsthe stages of the proof without words. For this, it is necessary especially for teachers to be aware of the stages of proof process, direct the students, and ensure them to complete the stages ignored by them through various instructions. This study revealed the proof processes of high school students and the stages of proof without words. Also in this study it was investigated that the relationship between the stages and the stages which is neglected by high school students. So for the further studies, it is important to reveal proof processes of the students who are at various educational level.

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Conflict of Interest

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