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Proportional Reasoning of Adult Students in a Second Chance School: The Subconstructs of Fractions

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Abstract

One of the subjects that constitute a problem for adult students of mathematics is the understanding of fractions and proportions. These notions are interrelated with proportional reasoning which was the focal point of our study. Using a mixed methods approach, we tried to assess the proportional reasoning of adult students during problem solving. We designed a task sheet with 13 open-ended tasks, each attempted by 30 adult students, and we conducted "clinical interviews" (Ginsburg, 2009) with five participants in order to acquire more data about their ways of thinking. For the task categorisation we used a typology of fraction subconstructs (Kieren, 1980). The results indicate that the adult students' performance was relatively good despite their limited mathematical background. However, they had difficulties with the tasks that required a higher level of thinking.

Keywords: second chance school; proportional reasoning; rational numbers; fractions.

Introduction – Rationale of the study

In an era of rapid technological changes, the development of mathematical understanding in accordance with the contextual knowledge and skills for the use of mathematics in different settings is of great significance (FitzSimons, 2019). One of the most important understandings of mathematics is proportional reasoning, since it is strongly connected with other areas of science and mathematics across all levels of compulsory education. Proportional reasoning refers to the ability to use ratios in situations involving comparison of quantities and it has been described as the foundation for the understanding of algebra and the transition from informal to formal mathematical thought (Doyle, Dias, Kennis, Czarnocha, & Baker, 2016). Additionally, proportional reasoning "is one of the most commonly used applications of mathematics in everyday life" (Hilton, Hilton, Dole, & Goos, 2016, p. 194). Among the skills needed for proportional reasoning are multiplicative and relational thinking and an understanding of concepts such as rational numbers, fractions, decimals, multiplication, and division (Lamon, 2005). The comprehension of rational numbers and fractions is of especial importance since these notions are connected with real world situations (Behr et al., 1983).

While the importance of proportional reasoning is indisputable, many adults and students fail to reason proportionally (Lamon, 2007; 2012). This may have a serious impact for adults that have not completed compulsory education or lack basic mathematical knowledge, since they will not be able to operate in their daily lives. Based on the above assumptions we designed our study which aimed at the assessment of adults' proportional reasoning in the setting of a Second Chance School in Greece, which offers education to adults over 18 years of age.

Proportional reasoning presupposes the understanding of fractions. The teaching of fractions is a precondition of the teaching of rational numbers. Whilst there are many studies about children's understanding of fractions and proportional reasoning (e.g. Charalambous & Pitta-Pantazi, 2007), few studies refer to adults. The objective of our study was to examine the ways adult students use fractions in proportional reasoning problems. Our main research question was: *To what extent do students understand the different subconstructs of fractions?* In order to address this, we examined how students deploy subconstructs of fractions in solving problems involving proportions and percentages.

In the following sections, we review the research on how adults understand fractions in the context of proportional reasoning. Then we describe our methods and present the setting of our study followed by the results. In the Discussion and Conclusion, we sum up the key findings and discuss their theoretical and practical implications in the general framework of adults learning mathematics.

Literature review

There are many studies that indicate the limited understanding of students of rational numbers and fractions (e.g., Charalambous & Pitta-Pantazi, 2007, Vamvakoussi, Van Dooren, & Verschaffel, 2012). Concerning adults' understanding of fractions and rational numbers the relevant studies focus on preservice (e.g., Livy & Vale, 2011) or in-service teachers (e.g., Depaepe et al., 2015), and their results refer to a poor understanding of ratio and proportional reasoning among these students. More relevant to our research are: the studies of Baker et al. (2012) and Doyle et al. (2016), set in a community college, the study of Alatorre and Figueras (2005), in which the participants are adults of different ages and schoolings and Ahl's (2019) study, which refers to adults in a prison education program.

Baker et al. (2012) investigated adults' competency in fractions in a community college in United States. Their objective was to extend their earlier study (Baker et al., 2009) and to describe the way the understanding of different subconstructs of fractions work together and complement each another in explaining students' competency. For this purpose, the researchers used quantitative methods, and added a transcript of a small group session.

A basic notion that emerges in a significant number of studies is the notion of a fraction subconstruct that was introduced in Kieren (1980); see the next section. Doyle et al. (2016), in the setting of a community college in United States, present a teaching experiment, in which Kieren's (1980) fraction subconstructs were used as a basis during their analysis of problem solving. The researchers used quantitative methods, such as a multiple linear regression model, in order to predict or explain students' competency in formal problem solving. They also used qualitative methods, such as the analysis of transcripts from classroom lectures during the teaching experiment, in order to describe how students used these rational number subconstructs during problem solving with ratio, quotient, proportion, and percent. Their results confirmed that the subconstruct of *part-whole* (see next section) is significantly easier for the adult students than the other subconstructs. Next in difficulty are the *equivalence* and *ratio* subconstructs, and then *operator* and *quotient*. The subconstruct of *measure* follows, and finally problem solving or proportional reasoning is the most difficult of all. As far as it concerns the students' reasoning, the analysis showed that when the problems included a visual form of the concepts of part-whole such as circles and number lines, measure acted as an accelerator for the students' reasoning, since the students reasoned correctly because they were able to relate problem information to these pictures. These picture-concepts also underpinned informal reasoning as students expressed strategies and applied processes based upon how the information in the problem was represented. However, see the limitations of this approach in one example in the Results section (e.g., one student's reasoning about Task 11).

Alatorre and Figueras (2005) describe the answers that adults without primary schooling gave to different ratio and rate comparison tasks. The research took place in a centre for adult education in

Mexico City. The participants were six quasi-illiterate adults, who. were interviewed, and the sessions were videotaped. During the interviews, the participants had to answer several questions about problems that demanded proportional reasoning. The problems that included ratio comparison tasks were “easier” for the adults when these included non-proportionality situations (equivalent fractions) than those that included non-proportionality ones (non-equivalent fractions). The results of this study have confirmed that the ability for proportional reasoning develops in the form described by the developmental model that these researchers propose.

Ahl (2019) presents an analysis of a test which was constructed for the assessment of adults’ prior knowledge of proportional reasoning. The participants were adult students of a prison education program in Sweden. The test was administered to 32 adults, and follow-up interviews are provided from three participants. The results indicate that the test could serve as an instrument for the detection of adults’ prior knowledge on proportional reasoning. In addition, the results of this research give information about the ways that adults express notions of proportional reasoning in different contexts.

Context and methodology

The setting of our study was a Second Chance School (SCS) in a city of north-western Greece. The Second Chance Schools offer education to adults over 18 years old, who have dropped out of ordinary compulsory education (Vekris & Chontolidou, 2003) and operate as a pathway for them to complete their basic secondary education (Papaioannou & Gravani, 2018). The students of SCSs generally have a low socio-economic background (Lamb et al., 2011), since among them are immigrants, refugees, people from minorities, people with health problems and people who live in poverty. These students come from settings that, in the context of another European country (Germany), are called “vulnerable” (Heilmann, 2019).

The duration of studies in these schools is two academic years and mathematics is taught for three hours per week. The mathematics course for adults, correspond to the first two years of Greek middle school (gymnasium). According to the mathematics curriculum for the SCSs (Lemonidis & Maravelakis, 2013), the structure and the content of the course include proportional reasoning, since among its outcomes we read:

- The understanding of the importance of natural numbers, fractions, decimals and percentages and their use in real problem situations.
- The conversion of one type of rational number to another, for example, the conversion of a fraction to a decimal number and vice versa.
- The development of different strategies for the use of rational numbers in real problem situation.

Concerning fractions, by the end of the course adults are expected to be able to:

- read, write and order simple fractions;
- represent fractions with sets of objects, area modes and number lines;
- convert equivalent fractions, e.g. from 2/4 to 1/2;
- compare fractions with the unit;
- add and subtract fractions with the same denominator;
- convert improper fractions to mixed numbers;
- add and subtract dissimilar fractions (fractions with different denominators) and mixed numbers;
- multiply and divide fractions and mixed numbers;
- connect the notion of the ratio with the fraction.

Our study involved 30 adult students attending the second year of the SCE, 16 men and 14 women, ranging from 25 to 65 years. Most of them were unemployed or unskilled workers. Their mathematical skills corresponded to a lower secondary level, since they had completed the first year of the SCS. Nonetheless, all were highly motivated to learn (McGregor & Mills, 2012).

For our study we designed a task sheet with 13 open-ended tasks (see the Appendix) and asked the participants to complete it in 1½ hours. The tasks were adapted either from studies in the research field on proportional reasoning or constructed by us. The latter items (three out of the 13) were based on real problems that our students referred to during the teaching of the relevant notions. During the preceding three to four weeks the students had been taught fractions and their different representations, the equivalence of fractions, percentages, addition of fractions with the same denominator and problems that required proportional reasoning.

The tasks were chosen and categorised according to the fraction subconstructs definitions of Charalambous and Pitta-Pantazi (2007) as implemented for adults by Doyle et al. (2016). Charalambous and Pitta-Pantazi (2007) based their study on the assumption that fractions constitute a multifaceted notion encompassing five interrelated subconstructs – *part-whole, ratio, operator, quotient,* and *measure* – as proposed first by Kieren (1980) and expanded by Behr et al. (1983) to a theoretical model which linked the fraction subconstructs to the operations of fractions, fraction equivalence, and problem solving. The objective of Charalambous and Pitta-Pantazi (2007) was to examine how students at the two last grades of elementary school understand different subconstructs of fractions. Doyle et al. (2016) used the fraction subconstructs for their study with adult students.

In particular, the *part-whole subconstruct* refers to the symbolic notation p/q to represent the partitioning of a whole entity into q equal shares and then taking p out of the q shares, the fraction that will be produced must be smaller than the unit. As Figure 1 shows for Task 6 this sub-construct refers to the partitioning of a unit into three equal pieces and then the comparison of the shaded part with the fraction 2/3; see the list of tasks in the Appendix. The point of this particular task is that the three pieces are *not* equal, in opposition with the fundamental definition of fractions, according to which a continuous quantity or a set of discrete objects are partitioned into parts of *equal* size (Lamon, 1999).

Does the shaded part of this rectangle correspond to the fraction 2/3? Justify your answer.

Figure 1: Task 6 (Charalambous & Pitta-Pantazi, 2007)

The *ratio subconstruct p/q* involves a comparison between the two quantities *p* and *q* of the numerator and denominator, respectively. Since it includes the notion of relative magnitude, it is not considered as a number but more as an index of comparison (Behr et al., 1983).

For Task 1 students had to compare lemonade mixtures incorporating in their answers the proportions of sugar and water:

> John and Maria are making lemonade. Given the following recipes whose lemonade is going to be sweeter? Justify your answer.
> John uses 2 spoons of sugar for every 5 glasses of lemonade. Maria uses 1 spoon of sugar for every 7 glasses of lemonade.
> John uses 2 spoons of sugar for every 5 glasses of lemonade. Maria uses 4 spoons of sugar for every 8 glasses of lemonade. (Doyle et al., 2016)

According to Charalambous and Pitta-Pantazi (2007), for this subconstruct the students have to realize that there is a relationship between two quantities and that when these quantities change together (via the multiplication or the division by the same number) the relationship remains constant. In this case, the understanding of equivalent fractions refers to the ratio subconstruct.

The *operator subconstruct* refers to the process of taking the fraction *p/q* of some quantity by the multiplication of *p* with this quantity and then the division of the product by *q*. In Task 2, which is based on a real situation that one of our adult students confronted, the students had to apply the given fraction to a certain number.

> Johanna's child, whose weight is 12 kilograms, is sick and needs an antibiotic. The recommended dose in ml is equivalent to 1/4 of its weight [in kilos]. How many ml of the antibiotic should Johanna give to her child? Justify your answer.

This subconstruct could be conceived as the result of two multiplicative operations since the numerator is multiplied by a given quantity, followed by the division of the result with the denominator, or vice-versa (Charalambous & Pitta-Pantazi, 2007).

The *quotient subconstruct p/q* is perceived as the amount obtained when *p* quantities are divided into *q* equal shares. In Task 3 the students had to connect a fraction with a division:

> Three pizzas are shared equally among four students. What fraction of a pizza will each student receive? (Charalambous & Pitta-Pantazi, 2007).

In this subconstruct there are not any limitations like with the part-whole subconstruct since the numerator can be smaller, equal to or bigger than the denominator and the resulting quantity can be less than, equal to or more than the unit (Charalambous & Pitta-Pantazi, 2007).

The *measure subconstruct* is connected with the placement of a fraction on the number line. For Task 11 the students had to locate three different fractions on a number line and to justify their answer.

> Locate the following numbers on the number line and justify your answer.

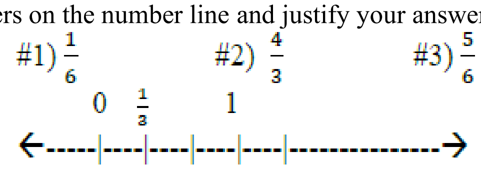


Figure 2: Task 11 (Charalambous & Pitta-Pantazi, 2007)

In this subconstruct the fraction has a duality since it corresponds to a number and it is also associated with the measure allocated to some interval (Charalambous & Pitta-Pantazi, 2007). This subconstruct is difficult for the students to understand, since they have to make a *qualitative leap* when they move from whole to fractional numbers (Lamon, 1999). This observation is affirmed also by other researchers (Vamvakoussi et al., 2012), who connect students' difficulties with rational numbers to their differences with natural numbers and the disparity between the students' deeply rooted prior knowledge about the set of natural numbers.

Besides the tasks that referred to the subconstructs of fractions, we used tasks that referred to formal proportional reasoning (Doyle et al., 2016). As reported by these researchers, in a proportion problem someone first has to find the unit rate between the given referents and then to follow a multiplicative strategy; the level at which this procedure can be accomplished constitutes formal proportional reasoning. For example, in Task 7 the student has to calculate the rate of the medicine and the water and then to convert it in an equivalent fraction in order to answer the question:

> If 0.5 ml of medicine is mixed with 2 ml of water to form a solution, what is the ratio of drug to water in the simplest terms? Justify your answer. (Doyle et al., 2016)

Although a mathematical task can serve as a window into the students' thinking (Hunting, 1997), it cannot give enough information about the cognitive processes the student followed in order to answer the task (Ginsburg, 2009). For this reason, we have decided to deploy a mixed methods approach (Sammons & Davis, 2016). In particular, we conducted "clinical interviews" with five of the participants, three women and two men, in order to have more data about their ways of thinking. The interviews were semi-structured, lasted for approximately 40 to 50 minutes and were audio taped. During the interviews we asked the participants to reason aloud about their answers to the tasks. By that method we had the opportunity to enrich our data since the participant had the opportunity to explain why s/he answered in a particular way or even to amend her/his answer to try to find the correct one.

Results

As we stated earlier, the main objective of our study was to identify the processes that adult students in an SCS use when confronted with mathematical tasks related to fraction subconstructs and to formal proportional reasoning, and to monitor the difficulties that these students experience. In general, the tasks that referred to the subconstructs of ratio (T1, T5) and part-whole (T6) were the easiest, in the sense that the majority of the participants answered them correctly; see Table 1. The most difficult tasks were those that referred to the measure subconstruct (T11, T12) or involved formal proportional reasoning (T8, T13). Table 1 below summarises the results in all tasks. The first column refers to the fraction subconstructs in general and to the formal proportional reasoning. The second column indicates the corresponding items. The strategies used by the students are outlined in the third column while the fourth column includes the number of answers correct and justified (rather than merely stated without justification).

We now present a representative response to a task for each subconstruct with relevant transcripts from the interviews.

As we stated above, the task which elicited the best performance was T6, which referred to the part-whole subconstruct. Below Helen (all names are pseudonyms) explains why she answered negatively:

> Helen: No, because the middle one is smaller than the other two parts.
> Teacher: So, what?
> Helen: I don't think that this represents the 2/3.
> Teacher: If you had to draw it how would you do it?
> Helen: I would make a similar one, but I would make three equal parts. I would divide it to three equal parts, and I would take the two of them (as she speaks, she draws the correct shape).

Helen had understood that a fraction describes a situation, in this case a rectangle, in which a quantity or a unit is divided into equal parts. She divided the rectangular correctly and she stated that the parts must be equal.

Task 1 referred to the ratio subconstruct. For this task most of the students computed fractions, used the notion of ratio, and then tried to compare them. In order to make the comparison, some of them converted the initial fractions, while others based their answers on the relationship of the numbers that constituted the initial fractions. Maria below explains how she converted the initial fractions, in part 1 of the task, in order to compare them:

> Maria: I get the largest denominator and I put it in the first fraction and the second denominator in the second fraction. I do it in reverse. And then I multiply 2×7=14 and 5×7=35. I continue the next and compare 14/35 with 5/35. Surely 14/35 is larger, since it has the bigger numerator. The sweeter lemonade is John's.

In her answer it is obvious that she tries to compare the initial ratios by converting to equivalent fractions. After this conversion, she was able to compare them and answer the question. (We may say that she compared the ordered pairs using multiplicative rules.)

Table 1: Strategies used by the students

Fraction subconstructs	Items	Strategies used by the students	Number of justified answers (Total N=30)
Part- whole	T6	Detect that the bar is not divided to equal parts.	29
Ratio	T1	Compute fractions and try to compare them by converting them to equivalent fractions.	15
		Divide the numerator with the denominator.	6
		Convert the fraction to a decimal number.	
		Find relationships with the numbers on the numerator and the denominator. Compare the fractions with a unit fraction.	3
Ratio-equivalence	T5	Convert the fractions to equivalent ones.	20
Operator	T2	Apply the fraction to the given number	20
Quotient	T3	Compute the fractions and compare them.	15
	T4	Compute the fraction and divide the numerator with the denominator.	10
Measure	T11 T12	Divide correctly the number line and locate the fractions on it.	3 3
Formal proportional reasoning	T7 T8 T9 T10 T13	Apply rational number concepts to problems that include proportions and percents.	19 8 20 18 4

A few students divided the numerator with the denominator interpreting the fractions as quotients. Others tried to find how many spoons of sugar are used for every glass of lemonade by using proportions and trying to compare magnitudes. In the next transcript, Yanna explains her way of thinking for part 2 of the task. She did not use fractions but instead she converted to proportions and tried to compare the magnitudes by using multiplicative thinking.

> Yanna: It is 2 spoons of sugar in 5 glasses of lemonade and 4 spoons of sugar for every 8 glasses of lemonade, so the 8 glasses have 4 tablespoons of sugar and the 5 glasses… Because in the 8 glasses it is about 1 spoon in every 2 glasses while here at 5 comes less than half of a spoon for every 1 glass of lemonade. Isn't it?

Whilst the initial ratio was equal to 2/5, Yanna tried to convert it to the equivalent unit fraction, which would be easier for her to make the comparison.

For Task 2, which referred to the operator sub-construct, the students had to apply a fraction to a given quantity. Most of the students managed to calculate the correct result by multiplying the fraction with the given number, as we can see in Figure 3. The student's written answer reads: "one of the four is equal with three because we divide the 12 kilograms and we take one of these" (translated from the Greek).

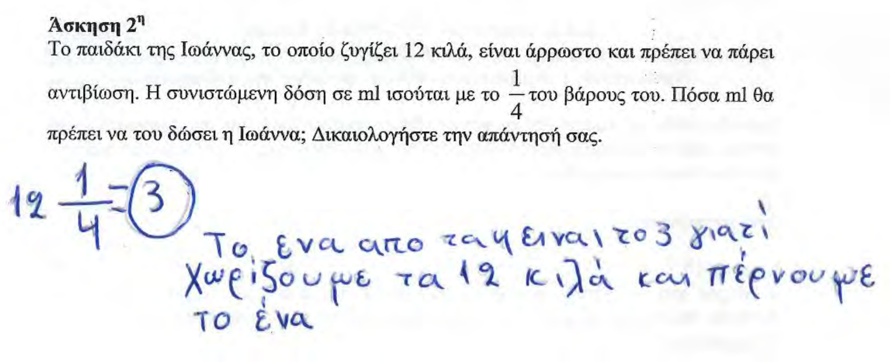


Figure 3: Task 2 sample solution

Some of the students divided the given number with the denominator and found the result.

> Helen: 3 yes, the 1/4 is the 3.
> Teacher: How did you think it? Why is the 1/4 equal to the 3?
> Helen: Because 3 times 4 is 12. I split the 12 to four pieces. So, she will have 3 ml.
> Teacher: Why did you decide to divide it to four pieces?
> Helen: Because it says that she wants the 1/4 of the child's weight.

In Helen's answer, it is evident that she interpreted the fraction as the partition of the given number to four equal units.

Task 3 referred to the quotient sub-construct:

> Three pizzas are shared equally among four students. What fraction of a pizza will each student receive?

This was an easy task since all students were able to answer it correctly, although only 15 justified their answer (see Table 1). Many of them designed three rectangles representing the three pizzas (Figure 4) and divided them to four pieces. These students used the visual representations of the fractions. In Figure 4 a student writes "we cut the pies in four pieces and every child takes three pieces", although he makes a formal mistake since he writes that 3/4 is equal to 4/3.

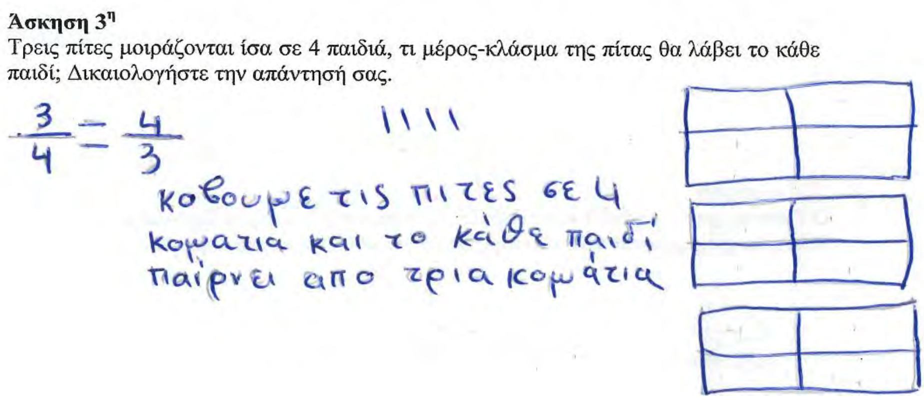


Figure 4: Task 3 sample solution

The most difficult tasks (T11, T12 see Appendix) for the students were those that referred to the measure subconstruct, since only three of the students answered them correctly. These students based their answers on the comparison of the fractions and the division of the number line. For T11, all of them used additive reasoning since they wrote that three parts, or six parts, make the whole but only one used multiplicative thinking since he highlighted the fact that 1/3 is the double of 1/6.

> Jim: Here we have a ruler… We counted, and we found that for the fraction 1/6 we cut the ruler to 6 pieces and we took one of the six. Here the four, the five of the six and the four of the three. Isn't that so? I have divided one piece into three pieces, and I moved further, I said, plus 1. One more than the whole unit, namely 4 out of three.
> Teacher: For the fraction 4/3 you said one more than?
> Jim: One more than the unit, one more than the 3/3.
> Teacher: So, you understood every line once equal with the 1/6 and once with the 1/3.
> Jim: Yes.
> Teacher: Why did you place the 5/6 there?
> Jim: Because I divided this piece to 6 equal parts, and I chose the five of the six.

In Jim's answer it is obvious that he used additive reasoning since he added the 1/6 in order to construct the fraction 5/6. He had understood what a fraction constitutes but it was difficult for him to explain the relation between the 1/3 and the 1/6. However, he understood that a fraction bigger than the unit has a bigger numerator than denominator and he understood how to convert the unit to a fraction with equal numerator and denominator.

As we mentioned before only one student used multiplicative thinking in his answer to T11. Below we present a transcript of the interview with him.

> Teacher: How did you think it through?
> George: Here is the 0 and we move right and left. It gives us the 1, the 1/3 and we are asked to find the 1/6.
> Teacher: Why did you place there the 1/6?
> George: We start and we count from the 0…. The 1/3 is here…. When we move to the left is minus and when we move to the right is the plus. The 1/6 if we start and count 1, 2, 3, 4, 5, 6 …
> Teacher: Why did you write that "the 1/3 is the double of 1/6"?
> George: Because if we have 6… the 1/3 is the double of 1/6. We want to find the 1/6, we have the 0 and we search for the … In order to find the 1/3 is two times the 1/6.

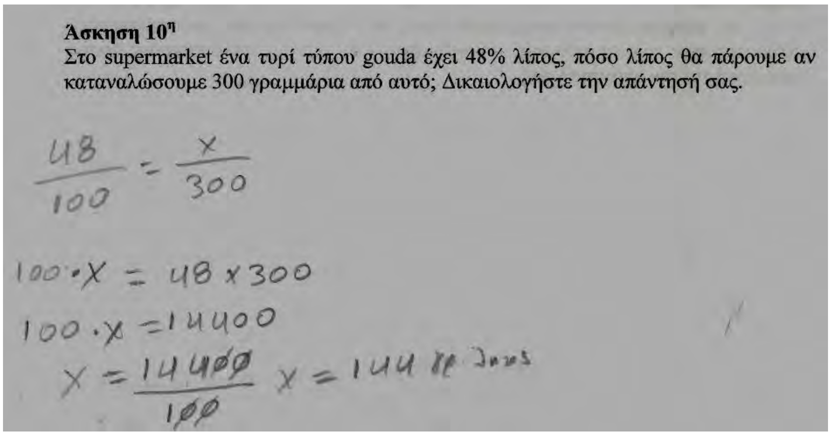
George in his answer used multiplicative thinking since he realised that 1/3 is the double of 1/6. In order to explain his answer, he drew six lines to denote the 1/6 and three for the 1/3. He was able to assign the 1/3 to the correct interval and to detect the relationship between the 1/3 and the 1/6 on the number line. So, he came to the right conclusion using relational thinking. In the next transcript George compares correctly the 4/3 with the 1 using additive thinking.

> Teacher: Why did you place the 4/3 after the 1?

George: Because we have 1, 2, 3 and 4 thirds. I add the 1/3 until I reach the 4/3. Every piece corresponds to 1/3.

Besides the tasks that referred to fraction subconstructs, we used tasks that included *formal proportional reasoning*. Some of these were relatively easy for the students (T7, T9 and T10) while others were rather difficult (T8, T13); see Table 1. Most students answered correctly Task 10, maybe because it described a situation they confronted in their everyday life. They multiplied the percent with the given number, or they used the equivalent fractions (Figure 5):

In the supermarket Gouda cheese has 48% fat, how much fat will we get if we consume 300 grams of it? Justify your answer.


Άσκηση 10η
Στο supermarket ένα τυρί τύπου gouda έχει 48% λίπος, πόσο λίπος θα πάρουμε αν καταναλώσουμε 300 γραμμάρια από αυτό; Δικαιολογήστε την απάντησή σας.

$\frac{48}{100} = \frac{x}{300}$

$100 \cdot x = 48 \times 300$

$100 \cdot x = 14400$

$x = \frac{14400}{100}$ $x = 144$

Figure 5: Task 10 sample solution

Maria: The 100 grams have 48, we want to find about the 300 so we will multiply the 48 with the 3 which is equal with the 144.
Teacher: Why did you say that in the 100 grams we have 48? How did you get this?
Maria: Because it says 48%.
Teacher: Why did you multiply with the 3?
Maria: It is three times the 100.
Teacher: Did you use as a base a certain number?
Maria: Yes, the 100.

In Maria's answer it is obvious that she had understood the method for calculating the percent from a given number. Also, she had understood that the base-whole of the percent is the 100 which she used as a reference point for the calculation.

Task 8 was a bit difficult for the students.

Task 8 (formal proportional reasoning): Which of the following fractions is the closest to 1?
a) 2/3 b) 3/4 c) 4/5 d) 5/6

Most of them answered that 5/6 is the closest to 1, but they did not correctly justify their answer. In the next transcript, Yanna tries to explain her thinking. We may say that she understands the fraction intuitively but cannot express it with mathematical notions. For this reason, she was classified as producing a "justified answer" in Table 1.

Yanna: 5/6 is closer to 1, then 4/5, then 3/4 and then 2/3.
Teacher: The answer you gave is correct, but how did you think about it?
Yanna: Because we make a pie and divide it into 6 pieces if I eat the 5 I get the most. At 4/5 you will get less. The other one who gets the 3/4 will get less and the other who gets the 2/3 he takes even less. So, when I get the most, I'm closer.
Teacher: Closer, to where?
Yanna: To 1.
Teacher: What does 1 represent for you?
Yanna: 1 is the whole.

Few students justified their answer by using either the equivalence of fractions for their comparison, or the subconstruct of quotient. One student represented visually the fractions and based her answer on the fact that the piece left from the 5/6 seemed to her to be the smallest one. Figure 6 presents her answer, where she states that “the closest is the 5/6”.

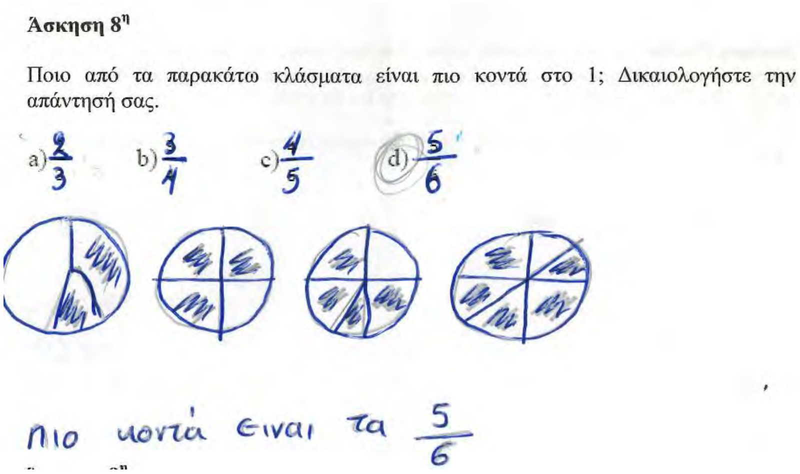


Figure 6: Task 8 sample solution

However, the reader will notice that relying on these diagrams is not convincing, since the relevant parts of the last circle, for example, are not equal. We could ascribe this fact to the difficulty that these students had to express their thoughts with the mathematical language.

Discussion

Our study focused on proportional reasoning during problem solving in the setting of a Second Chance School. The participants of our study were adult students who had been away from school mathematics for many years and returned in order to complete their secondary education. Since proportional reasoning is inextricably connected with the rational numbers, we sought answers to a research question about the range of students’ understandings of the different subconstructs of fractions. In order to achieve this, we examined how students deploy subconstructs of fractions in solving problems involving proportions and percentages.

Although at the beginning of the lessons the adult students expressed their anxiety and fear concerning the notion of fractions, at the end they seemed able to incorporate an understanding of the properties of fractions in order to reason, at least to some extent, proportionally. We could claim that our students overcame their previous experience of mathematics which is a significant goal of learning mathematics as an adult (Wedege & Evans, 2006). Concerning their performance, most of them easily solved the tasks that required the formulation of a fraction either from a sentence or from a pictorial representation. They were also able to construct ratios based on the data of the task and to convert them to equivalent ones. The tasks that corresponded to the subconstructs of *ratio*, *operator*, *quotient*, and *part-whole* were relatively easy for most of the students since they managed to establish relationships between quantities, and then those relationships (ratios) became the new elements with which they operated.

At the same time, most of them had difficulty with the tasks that referred to the *measure* subconstruct, since these tasks require both an understanding of the splitting and iteration processes, a finding that is similar to that often found among younger students (Charalambous & Pitta-Pantazi, 2007). Very few students answered these and in their answers they used additive thinking instead of multiplicative reasoning.

The students also found difficult some tasks that required *formal proportional reasoning*. In particular, this observation refers to tasks which included the finding of a ratio through its implementation in a given quantity or the placement of unlike fractions in ascending order. At the same time, the students' performance was very good in tasks related to situations of their everyday life. Since proportional reasoning is an ability which is known to take a long time to develop, we may claim that the participants' performance was relatively satisfactory. According to Boyer and Levine (2012), the necessary skills for proportional reasoning are multiplicative and relational thinking, supplemented by a highly developed understanding of foundational concepts such as fractions, decimals, multiplication, and division. Obviously, the students in our study had not developed entirely the above skills; that is why they did not rely solely on their mathematical knowledge, but they also drew on their out-of-school experience when they returned to school (Evans, 2000).

Conclusion

Proportional reasoning has such a critical role in a student's development of mathematical thinking that it has been characterized as a cornerstone of higher mathematics and the capstone of elementary concepts (Lesh, Post, & Behr, 1988). Additionally, it has been characterised as a "pervasive activity that transcends topical barriers in adult life" (Ahl et al., 1992, p. 81), therefore there is no doubt that it should be included in the mathematics courses that are taught to adults who return to school. The results of our study have shown that realistic contexts are quite helpful, especially because they assist adults to overcome their anxiety by relating the given situation to their everyday lives. Thus, adult educators should have this in mind, at the same time considering other factors, such as the participants' backgrounds (including socioeconomic factors), in order to design relevant and interesting contexts. From a mathematical point of view, we have seen these adults' difficulties with particular subconstructs, such as the measure subconstruct and with mathematical processes such as formal proportional reasoning. These difficulties are probably related to their prior learning experiences.

Summing up, our study has shown that achieving a balance between applicability to everyday life and mathematical richness should be a main goal for mathematics educators at all levels, including adult education.

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Appendix: Task Sheet

Ratio

Task 1 (Doyle et al., 2016)

John and Maria are making lemonade. Given the following recipes whose lemonade is going to be sweeter? Justify your answer.

1) John uses 2 spoons of sugar for every 5 glasses of lemonade. Maria uses 1 spoon of sugar for every 7 glasses of lemonade.
2) John uses 2 spoons of sugar for every 5 glasses of lemonade Maria uses 4 spoons of sugar for every 8 glasses of lemonade.

Operator

Task 2 (constructed by the authors)

Johanna's child, whose weight is 12 kilograms, is sick and needs an antibiotic. The recommended dose in ml is equivalent to 1/4 of its weight. How many ml of the antibiotic should Johanna give to her child? Justify your answer.

Quotient

Task 3 (Charalambous & Pitta-Pantazi, 2007)

Three pizzas are shared equally among four students. What fraction of a pizza will each student receive?

Task 4 (Charalambous & Pitta-Pantazi, 2007)

If 3 pizzas are shared evenly among seven girls, while 1 pizza is shared evenly among three boys, who gets more pizza, a girl or boy?

Ratio - Equivalence

Task 5 (Charalambous & Pitta-Pantazi, 2007)

Can you fill in the gaps and justify your answer?

$\frac{2}{3}=\frac{;}{12}$ $\qquad$ $\frac{25}{40}=\frac{5}{;}$ $\qquad$ $\frac{7}{9}=\frac{42}{;}$

Part-whole

Task 6 (Charalambous & Pitta-Pantazi, 2007)

Does the shaded part of this rectangle correspond to the fraction 2/3? Justify your answer.

Measure

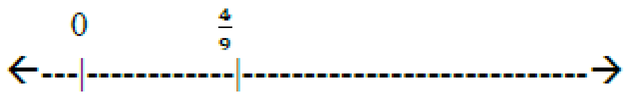
Task 11 (Charalambous & Pitta-Pantazi, 2007)

Locate the following numbers on the number line and justify your answer.

#1) $\frac{1}{6}$ #2) $\frac{4}{3}$ #3) $\frac{5}{6}$

Task 12 (Charalambous & Pitta-Pantazi, 2007)

Locate the 1 in the number line and justify your answer.



Formal proportional reasoning

Task 7 (Doyle et al., 2016)

If 0.5 ml of medicine are mixed with 2 ml of water to form a solution, what is the ratio of drug to water in the simplest terms? Justify your answer.

Task 8 (Doyle et al., 2016)

Which of the following fractions is the closest to 1?

a) 2/3 b) 3/4 c) 4/5 d) 5/6

Task 9 (constructed by the authors)

Mary used exactly 15 paint tins to paint 18 chairs. How many chairs she could paint with 25 cans? Justify your answer.

Task 10 (constructed by the authors)

In the supermarket Gouda cheese has 48% fat, how much fat will we get if we consume 300 grams of it? Justify your answer.

Task 13 (constructed by the authors)

Please fill in the gaps and justify your answer.

The … (a fraction) of 200 equals 150.

The … (a fraction) of 200 equals to … (a number larger than 100).

The … (a fraction) of 200 equals to … (a number smaller than 100).