Full research paper

INVESTIGATING THE VARIETY AND USUALNESS OF CORRECT SOLUTION PROCEDURES OF MATHEMATICAL WORD PROBLEMS

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ABSTRACT

The contribution focuses on issues related to the implementation of formative assessment methods into inquiry based teaching, by means of issues related to solving twelve multiple-step arithmetic word problems based on operations with natural and rational numbers. These word problems have multiple correct solution procedures and the presented qualitative exploratory empirical study investigates how varied and how usual might be correct solution procedures provided by diverse groups of solvers – future primary school teachers attending diverse university mathematics courses of diverse forms and/or time extent. According to written data collected from 149 solvers, six notions are introduced in the paper: majority, minority and even solution procedures, and majority, minority and mixed solvers. Issues regarding minority solvers are recognized as an important element for implementing formative assessment methods. All the six notions are illustrated in the paper by samples of solution procedures and diagrams of relative frequency. Implications are given for formative assessment within any kind of education involving multiple-step word problems, regardless of the extent of implemented inquiry.

KEYWORDS

Formative assessment, inquiry based mathematics education, open approach to mathematics, primary school teachers, solving strategies, word problems

HOW TO CITE

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Highlights

- *Open approach to mathematics is closely connected to formative assessment.*
- *Open word problems and records of their solution procedures offer an effective insight into classroom events related to formative assessment.*
- *Six notions established in the paper: majority, minority and even solution procedures, and majority, minority and mixed solvers.*
- *Ilustrations given to the variety and usualness of correct solution procedures provided by diverse groups of future primary school teachers.*

INTRODUCTION

One of the educational concerns that has attracted quite a lot of attention across countries in recent decades relates to looking for sustainable ways of implementing inquiry based education into everyday teaching and learning of various school subjects at various school levels. Alongside efforts to specify potential advantages and disadvantages of the inquiry based approach (McComas, 2002; Minner, Levy and Century, 2010; Bruder and Prescott, 2013), to conceptualize this approach within the context of other educational frameworks and in international contexts (Artigue and Blomhøj, 2013; Schoenfeld and Kilpatrick, 2013), to analyse the effects of this approach on knowledge and attitudes of pupils (Hattie, 2008; Jiang and McComas, 2015; Savelsbergh et al., 2016) and to provide teachers with enough training and didactical materials such as assignments and didactical analyses of inquiry tasks (Ulm, 2011; Baptist and Raab, 2012; Maaß and Reitz-Koncebovski, 2013), lately there also emerged efforts to analyse and conceptualize various ways of classroom assessment that would be suitable for the inquiry based approach. As summarized by Dolin and Evans (2018), one of the possible solutions to that call might be hidden in formative assessment methods (Black and Wiliams, 2009; Shavelson et el., 2008), e.g. in on-the-fly assessment and peer-assessment. Another solution to that call might be obtained by investigating the inquiry based environment through problems and tasks that are assigned to pupils during inquiry based lessons. Such tasks are usually open in the sense of an open approach to mathematics (Pehkonen, 1997; Nohda, 2000), an approach that has a lot in common with inquiry based education (Samková, 2017) as well as with formative assessment methods (Hino, 2007). One of the subcategories of open tasks consists of tasks with multiple correct solution procedures, which are the tasks that are employed in this particular contribution.

Issues reported here are a part of a larger educational research project named *Learning hyperspace for formative assessment and inquiry based teaching in science and mathematics* that is supported by the Technology Agency of the Czech Republic. The aim of the project is to create a learning hyperspace (online interactive environment) for teachers where they could learn how to implement formative assessment into their inquiry based teaching. This paper belongs to the preparatory stage of the project, where we intend to map the classroom environment related to inquiry based education from the perspective of formative assessment. In particular, the presented study focuses on mathematical inquiry provided by word problems with multiple correct solution procedures and on issues related to the assessment of such problems. In a broader context, the paper aims to contribute to the ongoing establishment of the principles of on-the-fly assessment and peer-assessment within inquiry based teaching and learning, while touching matters that go far beyond this environment – matters that are relevant also for teachers that do not intend to implement formative assessment and/or inquiry based methods to their teaching knowingly or in a systematic way. Since word problems are regularly employed in mathematics education at all school levels, the paper addresses matters which concern every teacher who has ever employed in their teaching a word problem that happened to have multiple correct solution procedures.

The reported qualitative exploratory empirical study investigates 12 word problems with multiple correct solution procedures and then focuses on particular correct solution procedures provided by individual solvers as well as on the usualness of these procedures among the group of solvers that attend together the same mathematics lessons. The mathematical content of the word problems consists of operations with natural numbers and operations with rational numbers (namely fractions). The participants of the study (i.e. the solvers of the word problems) were 149 attendants of various university programs conducted in various school years. Within the programs, all of the attendants were trained to become teachers at the primary school level.

The themes that meet behind the reported study have already been discussed at ERIE conferences and in the ERIES Journal: formative assessment (Hošpesová and Žlábková, 2016; Jahodová Berková, 2017), inquiry based education and open approach to mathematics (Samková and Tichá, 2016; Medová, Bulková and Čeretková, 2018), correct and incorrect strategies for solving word problems (Novotná and Vondrová, 2017; Samková, 2018a).

This paper has been developed as an extension of the contribution (Samková, 2019). Data analysis from that contribution was enriched by additional data from three other groups of participants that all attended university mathematics courses on the same mathematical content as the original group of participants but their courses differed in the form of teaching and/or time extent. Such an enrichment offered a wider variety of outcomes and the method of constant comparison then led to more precise specifications of notions related to usualness of correct solution procedures that had been newly established in (Samková, 2019) as well as to the establishment of another new notion. The issues discussed in (Samková, 2019) form a part of the first stage of the study presented in this paper.

The text is organized as follows: at the beginning, it presents the context of the study (inquiry based mathematics education, word problems with multiple solution procedures and formative assessment), the four groups of participants and the diagnostic instrument. Then it describes consecutively the four stages of the study (the course of data collection and data analysis, findings, emerging concerns), discusses their findings and captures further implications for our research project.

Inquiry based mathematics education and open approach to mathematics

The term of inquiry based education refers to a student-centered type of education in which students are invited to work similarly as scientists work: observe, pose questions, reason, search for information, collaborate, collect data and interpret them, discuss obtained results (Dorier and Maass, 2014). In mathematics, an appropriate inquiry based learning environment can be successfully achieved through tasks with multiple correct ways of interpreting the task assignment, multiple correct ways of solving, multiple correct results and/or multiple correct ways of interpreting the results. Such tasks are called open in the sense of open approach to mathematics (Pehkonen, 1997; Nohda, 2000).

For assessment of open mathematical problems, Nohda (2000) suggests to refer to *fluency* (how many solutions the student produced), *flexibility* (how many mathematical ideas the student employed or discovered), *originality* (to what extent are the ideas original) and *elegance* (to what extent are the explanations simple and clear). Bulková and Čeretková (2017) put their emphasis during assessment more on the practical and analytical aspects and suggest to refer to *originality*, *correctness of conclusion* (which includes exactness, clarity and coherence of information used, relevance of sources and closeness of the conclusions to the goal of the task) and *applicability of conclusion and solving process value for following studies* (to what extent the conclusion and/or solving process could be easily generalized within same, similar or distinctive contexts). In this study, I propose another aspect to take into consideration. This aspect is *usualness* (how usual among the group of solvers is the particular way of solving that the student provided). It relates to originality and remotely also to fluency and applicability, and these relations are illustrated in the paper.

Formative assessment

The type of assessment that is in the focus of this contribution is the formative one, which is, in simple terms, assessment for learning, i.e. assessment that helps students to learn. According to Black and Wiliam (2009: 8), ‘formative assessment

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can be conceptualized as consisting of five key strategies: 1. Clarifying and sharing learning intentions and criteria for success; 2. Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; 3. Providing feedback that moves learners forward; 4. Activating students as instructional resources for one another; and 5. Activating students as the owners of their own learning.' In the classroom, the formative assessment may appear in various forms referring to various key strategies or their combinations. In this contribution, I address the forms called on-the-fly assessment and peer-assessment. The on-the-fly form refers to interactions among the teacher and the students that have not been planned by the teacher in advance. These interactions take place in the classroom when the teacher recognizes an opportunity to support students in their further learning and acts on it by questioning or commenting (Shavelson et al., 2008). The peer-assessment form refers to interactions among the students themselves, where the students discuss without the help of the teacher and provide feedback to each other (Topping, 2013).

Both the above mentioned forms of formative assessment require quality feedback, which can come in the classroom in four different levels related to four different focuses (Hattie and Timperley, 2007). In relation to matters that occur in the classroom during task solving, this contribution mainly addresses two of the levels: the one that focuses on the task that the students are solving, and the one that focuses on the processes used by students to complete the task. In case of peer-assessment, the quality of the feedback is strongly connected to the extent of students' understanding of the assessed topic (Le Hebel et al., 2018); in case of on-the-fly assessment, the quality depends on teacher's ability to notice specific solutions, problems or innovative approaches and on their willingness to initiate conversations (Harrison et al., 2018).

In the Czech Republic, where the referred study is situated, the notion of formative assessment is generally known in the educational community but many of future teachers and teachers have never experienced formative assessment as learners, nor have they been trained to implement it in their own teaching (Rokos and Závodská, 2015).

MATERIALS AND METHODS

My study aims to answer three research questions:

Q1: „How varied are correct procedures that future primary school teachers use for solving open mathematical word problems?"

Q2: „How usual among the solvers are particular correct procedures for solving particular word problems?"

Q3: „How usual among the solvers are the procedures used by particular solvers?"

The design of the study is an exploratory qualitative one since the phenomena of variety or usualness have not been studied before. To explore and describe the nature of the variety and the nature of the usualness, collected data would go through qualitative analysis, using open coding and constant comparison (Miles, Huberman and Saldaña, 2014). Afterwards, additional information on the issue would be obtained through basic statistic methods and illustrated by diagrams of relative frequency.

PARTICIPANTS

The research was conducted with four consecutive groups of participants that differed in the form and extent of a mathematics course that the participants attended just before data collection. The participants were students of two diverse study programs at the Faculty of Education, University of South Bohemia in České Budějovice. Within both the study programs, the participants were trained to teach all school subjects at primary school level (pupils from 6 to 11 years of age).

The first group participants were 24 students of the second year of a five-year full time master degree program for future primary school teachers. This program is mostly frequented by students that came to university directly from the secondary school, with no experience in teaching. During the whole school year, these participants were attending a course on mathematics conducted in an inquiry based manner, which focused on content issues related to natural and rational numbers. The course was held regularly in the time extent of 3 hours a week, i.e. 81 hours altogether. At the seminars of the course, the participants often solved word problems that were open. At first, the word problems had a unique way of grasping and a unique correct answer but multiple correct ways of solving. For each of the tasks, the teacher asked the solvers to look for various correct ways of solving and record them *all* on a blackboard. Later on, they also faced word problems with multiple ways of interpreting the assignment or/and multiple correct answers. The participants solved the tasks individually, and then they altogether presented, discussed and defended their various solution procedures and answers, looked for relations among them. In such a setting, they had a lot of opportunities to observe and discuss various ways of solving open tasks.

The second group participants were 49 students attending the same study program and the same mathematics course in one of the subsequent academic years. This subsequent course was not conducted in an inquiry-based manner but focused on the same mathematics content in the same extent as the previous one. During the seminars of the course, the second group participants solved the same word problems as the first group participants. Each of the tasks was solved on the blackboard by one of the attendants but no other solution procedures were presented or widely discussed.

The third group participants were 37 students attending the same study program and the same mathematics course in a different subsequent academic year. The design of the course was the same as with the second group participants.

The fourth group of participants consisted of 39 students attending the first year of a two-year distance retraining program for kindergarten and secondary school teachers of various teaching experience and various specialisations, designed to qualify them for teaching at the primary school level. These fourth group participants attended only a ten-hour condensed mathematics course in the form of a lecture (i.e. without seminars) that covered the same mathematics content as the whole-year courses attended by the first, second and third group participants. The lecturer informed the attendants briefly about various didactical models for solving word problems but the attendants themselves did not solve any tasks on the blackboard.

Data collection and data analysis were carried out in four separate consecutive stages, each stage with one group of participants. Data from the second, third and fourth stages served as additional to data from the first stage.

Diagnostic instrument

As a diagnostic instrument in my study, I used twelve multiple-step arithmetic word problems related to mathematical content at the primary school level. All of them are listed in Table 1.

W1	Wes plays the violin. The last week before the competition, he has been training 4 hours a day. How many minutes is it?
W2	How many different ways can 44 children be divided into three and five-member teams provided the number of three-member teams is less than 10?
W3	A 21-meter straight fence consists of 13 posts on which the mesh is taut. The posts are equally spaced apart. What is the distance between adjacent posts?
W4	A lorry should transport 67 tons of sand. After 6 rides of a fully loaded car, 19 tons remain to be transported. How many rides does the lorry still have to make?
W5	Tom and Karel have 68 marbles altogether. Karel has 14 marbles more than Tom. How many marbles has Tom?
W6	Edith and Jane bought a book together. Jane contributed 120 crowns to the book, Edith 74 crowns. How many crowns does Edith have to pay to Jane to participate equally?
W7	One big dumpling can be cut into 12 slices. How many big dumplings does the family need for lunch if the father eats 2/3 of the big dumpling, the mother 1/4, the daughter eats 4 slices and the son 6 slices? How many slices are left?
W8	There are 16 girls in our class, which is 4/7 of all pupils. How many boys are there?
W9	A greengrocer came to a market for two days. On Monday he sold 3/8 of his potatoes, on Tuesday 4/5 of the rest. How much of the potatoes was not sold? How many kilograms of potatoes did the greengrocer bring to the market provided he sold 200 kilograms on Tuesday?
W10	Yesterday, a bakery driver delivered baked rolls three times. During the first drive, he delivered 2/5 of the rolls, then 2/5 of the rest. 900 of the rolls remained for the last drive. How many rolls did the driver deliver during the first drive?
W11	With a big pump, the water reservoir would have been filled in 7 days, with a small pump in 9 days. The big pump is broken and needs to be repaired, so only the small one can be used for the first three days of the filling process. Both pumps will be used from day four. When the reservoir will be filled?
W12	A breeder keeps rabbits. Currently, 1/3 of his rabbits are white, and the others are grey. The breeder plans to give 3 grey rabbits to his neighbour today and get 3 white ones for exchange. After this exchange, the proportion of white rabbits will rise to 4/9. How many rabbits does the breeder have?

Table 1: The word problems on natural numbers (W1 to W6) and on rational numbers (W7 to W12); own translation

Six of the word problems focused on natural numbers and operations with them, the other six on rational numbers and operations with them. The word problems were based on various didactical concepts: on time unit conversions, diophantine partitions, equidistant partition, equal partition with a remainder, unequal partition and equal sharing in the case of natural numbers, and on various combinations of part-whole interpretations of fractions (one/more wholes, wholes that are complements to fractional parts of other wholes, fractional changes) in the case of rational numbers.

THE FIRST STAGE

Data collection and data analysis

During the first stage, I collected written records of solution procedures that the first group participants submitted as parts of two standard written tests. For all of these participants, the two tests served as parts of the course assessment, i.e. they were compulsory and came after the related topics had been discussed at lectures and properly practised at course seminars. For the purpose of each of the tests, the participants were divided into two almost equally sized subgroups and each of the subgroups got different assignments of the test. One of the assignments on natural numbers comprised of word problems W1, W3 and W6, and the other one of W2, W4 and W5. One of the assignments on rational numbers comprised of word problems W7, W10 and W12, and the other one of W8, W9 and W11. In such an arrangement, each of the first group participants got to solve six of the word problems included in this study (three on natural numbers and three on fractions), and each of the twelve word problems included in this study was assigned to approximately half of the first group participants as well.

Since all of the word problems fit into the primary school curriculum and the participants were trained in the course to become primary school teachers, they were not allowed to employ tools beyond primary school curriculum in their solution procedures, i.e. they were not allowed to use

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unknowns or equations. For the same reason, the solvers were allowed to use only natural numbers in their solution procedures to the natural number tasks W1 to W6. In particular, they had to employ centimetres to solve the task W3 correctly.

During data analysis related to the first research question, I registered various correct solution procedures that appeared in data related to particular word problems and the nature of their differences. I considered as same the procedures that consisted of the same constituent steps (calculations, employed concepts) provided in the same order.

During data analysis related to the second research question, I ascertained the usualness of each correct solution procedure based on the relative frequency of the solution procedure among the group of all participants.

During data analysis related to the third research question, I observed whether there was any tendency in the usualness for individual participants across all word problems.

RESULTS OF THE FIRST STAGE

From the perspective of individual word problems – variety

The initial analysis of data from the first stage of data collection revealed four word problems that were solved successfully by all of the first group participants (W1, W2, W6 and W9) and eight word problems with one or more unsuccessful participants who did not provide a solution to the problem or provided an incorrect one. The least successful were the participants at the tasks W11 where 5 of 11 failed and W12 where 8 of 13 failed.

The subsequent analysis focused in more detail on the correct solution procedures that were provided by the successful solvers. It revealed 2 to 5 different correct solution procedures provided by the participants to *each* of the word problems. The word problems with the highest number of different correct solution procedures were W9, W11 and W12, while the problems with the lowest number of different correct solution procedures were W1 and W5. Samples of correct solution procedures are presented in Table 2.

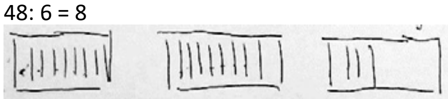
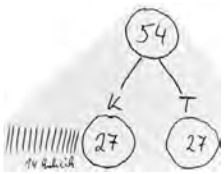
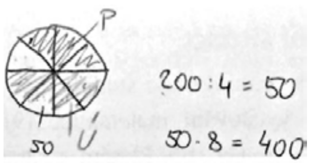
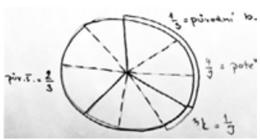
W1	4 h = 240 min 240 · 7 = **1680**	7 · 4 = 28... hours 28 · 60 = **1680**... minutes	
W4	67 – 19 = 48 48: 6 = 8 19: 8 = 2 + remainder 3 two rides full, another one with just 3 tons → **3** rides	67 – 19 = 48 48: 6 = 8 8 · 2 = 16 8 · 3 = 24 16 < 19 < 24 **3** rides	67 – 19 = 48 48: 6 = 8  **3** rides
W5	68 – 14 = 54 54: 2 = **27**		68: 2 = 34 14: 2 = 7 34 – 7 = **27**
W6	120 + 74 = 194 194: 2 = 97 97 – 74 = **23**	120 + 74 = 194 194: 2 = 97 120 – 97 = **23**	120 – 74 = 46 46: 2 = **23**
W7	2/3 of 12 = 8 1/4 of 12 = 3 8 + 3 + 4 + 6 = 21 12 · 2 = 24 **2** dumplings 24 – 21 = **3** slices	2/3 of 12 = 8 1/4 of 12 = 3 8 + 3 + 4 + 6 = 21 21: 12 = 1 9/12 **2** dumplings 12 – 9 = **3** slices left	father + daughter = 1 whole dumpling mother + son = 3 + 6 = 9 slices from the **second** dumpling → **3** slices left
W9	4/5 of 5/8 is 200 kg 5/5 is 250 kg... the rest from Monday 5/8 = 250 kg 8/8 = **400 kg** altogether not sold... **50 kg**	8/8 – 3/8 = 5/8 5/8 ·1/5 = **1/8**... not sold 200: 4 = 50 50 · 5 = 250 250 = 5/8 of the potatoes 250: 5 = 50 50 · 8 = **400 kg** brought	
W12	1/3 + 3... 4/9 1/3 = 3/9 3... 1/9 4/9... 12 5/9... 15 12 + 15 = **27**	white 1/3 = 3/9 4/9 – 3/9 = 1/9 grey 2/3 = 6/9 6/9 – 5/9 = 1/9 1/9 = 3 rabbits altogether... 3 · 9 = **27**	 1/9 = 3 rabbits 9/9 = 3 · 9 = **27**

Table 2: Various correct solution procedures to the tasks W1, W4, W5, W6, W7, W9 and W12 from the first stage; translation of texts in embedded pictures: kuliček = marbles, P = Monday (abbr.), Ú = Tuesday (abbr.), pův. š. = grey before (abbr.), původní b. = white before (abbr.), poté b. = white after (abbr.), k = rabbit (abbr.)

Detailed analysis of the provided procedures drew my attention to frequently occurring misplacement of active and passive factors in multiplications. Some of the occurrences also appeared in Table 2: the second solution procedure belonging to the task W4 says 8 · 2 and 8 · 3 instead of 2 · 8 and 3 · 8, the second solution procedure belonging to the task W9 says 50 · 5 instead of 5 · 50 and 50 · 8 instead of 8 · 50, the second and third solution procedures belonging to the task W12 say 3 · 9 instead of 9 · 3. Since the participants had not yet attended courses on didactic of mathematics, I did not consider these procedures as incorrect. For the purpose of data analysis, namely for the purpose of decision on sameness of solution procedures provided by various solvers, I considered the solution procedures with misplacements as having the order of factors in multiplications swapped to the proper one.

The solution procedures that were not same differed in various aspects: used different models for the situation described in the word problem, used the same model but employed different relations found in it, choose a different order of relations found in the same model, grasped the assignment of the word problem differently. Some samples from those shown in Table 2 are commented below.

For the task W1, all the solvers used the same model but employed two different orders of solution steps depending on two different placements of unit conversions within the solution procedure: at the beginning, or at the end.

For the task W4, the solvers based their solutions on three different calculation models: division with a remainder, comparison to multiples and one-to-one distribution provided by an illustrative picture.

For the task W5, the solvers used two different models of unequal partition to represent the situation of the task: the sum-of-parts model and the division-into-parts model (MacGregor and Stacey, 1998). One of the solvers with the sum-of-parts model also accompanied her solution by an illustrative picture.

For the task W6, there appeared three different solution procedures, the first and second ones were based on the same model but used different relations from the model in the last step of the procedure (Edith's perspective vs Jane's perspective). The third solution procedure used symmetry and offered an original perspective on the situation.

For the task W9, there appeared two different ways of grasping the first question in the assignment. In the Czech language, the original wording of the question has two common meanings: „How many of the potatoes..." as well as „How much of the potatoes...". Majority of the solvers addressed the first meaning but some of them addressed the second one (e.g. in the second sample related to W9 in Table 2).

For the task W12, there were only five successful solvers and each of them provided different solution procedure.

Some of the correct solution procedures were accompanied by or based on pictures (schemes, pie diagrams, segment diagrams, etc.), samples of these pictures are presented in Table 2. However, the majority of the solvers did not use any illustrations.

From the perspective of individual word problems – usualness

Further analysis of first stage data revealed two diverse types among the twelve observed word problems: five of the word problems were with several most frequent correct solution procedures evenly used by the successful solvers (W5, W9, W10, W11 and W12), and seven of the word problems were with the most frequent correct solution procedure used by majority of the successful solvers (W1, W2, W3, W4, W6, W7 and W8). For the tasks W1, W4, W6 and W7, the most frequent correct solution procedures are the first ones given in Table 2. For detailed diagrams of the relative frequency of individual solution procedures among the first stage participants see Figure 1. The diagrams have been composed as follows: the sectors related to incorrect solution procedures are shaded, the sectors related to correct solution procedures are unshaded, the labels that are written in italics belong to word problems with different correct solution procedures evenly used by the solvers, the labels in bold roman belong to word problems with majority and minority correct solution procedures.

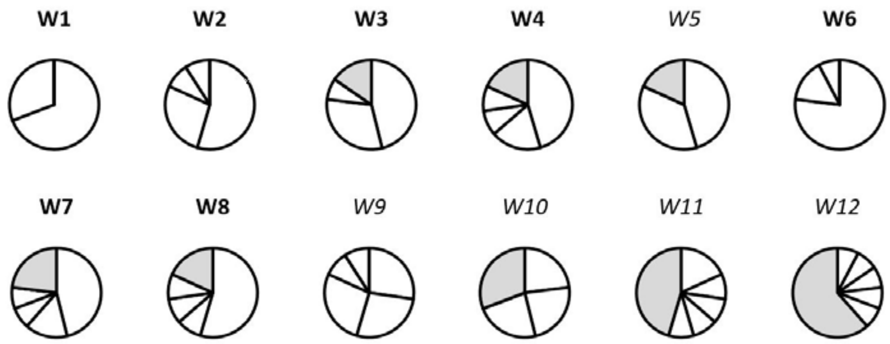


Figure 1: The diagrams of the relative frequency of individual solution procedures among the first group of participants that got to solve particular word problems, *n*=13 (for W1, W3, W6, W7, W10, W12), *n*=11 (for W2, W4, W5, W8, W9, W11), 2014-2015 (source: own calculation)

The task W5 met both the above characteristics since it got solved by 11 participants, 2 of them provided incorrect solution procedures, 5 of them provided the first correct solution procedure and 4 of them provided the second correct solution procedure. Here the 5 solvers with the first solution procedure form the majority of successful solvers. But with odd number of successful solvers (5 + 4 = 9) and only two different correct solution procedures, the two alternatives with numbers of corresponding solvers differing only by 1 can also be considered as evenly distributed. Moreover, there is no possibility to avoid majority with odd number divided into two integer parts. Similar situation appeared also with the task W11, where the most frequent correct solution procedure was provided by 2 solvers, and the others four by 1 solver each. In order to deal in general with this ambiguity, I decided to categorize the word problems with two most frequent correct solution procedures with numbers of solvers differing only by 1 as word problems with correct solution procedures evenly used by solvers.

The correct solution procedures used evenly by the solvers will be called *even solution procedures*, the correct solution procedures used by the majority of successful solvers will be called *majority solution procedures* and the correct solution procedures used by the minority of successful solvers will be called *minority solution procedures*.

From the perspective of individual solvers

From the perspective of individual solvers, I analysed in more details data related to the seven word problems with majority and minority solution procedures. Data analysis revealed four diverse groups of participants: those who used the majority solution procedures in all relevant cases (3 participants), those who used the majority solution procedures in all but one (10 participants), those who used the minority solution procedures in all relevant cases (7 participants) and those who used the minority solution procedures in all but one (4 participants). The participants from the first two groups might be together characterized as tending to use the majority solution procedures, the participants from the other two groups as tending to use the minority solution procedures. With such specifications of the term *tending*, we may state that 3 + 10 = 13 of the participants tended to use the majority solution procedures (we may call them *majority solvers*) and 7 + 4 = 11 tended to use the minority solution procedures (*minority solvers*).

Emerging concerns

After the first round of data analysis, the above mentioned findings naturally emerged a concern on how the dataset would be enriched when addressing the three research questions with a different group of participants – for instance with participants attending the same study program and the same mathematics course but not in the inquiry based manner. To address more directly the particular issues from the first stage related to tasks with majority and minority procedures, I decided to focus in detail on the test on natural numbers where all tasks but one were of this type. This choice led to the second stage of the study. To address more directly the particular issues from the first stage related to tasks with even procedures, I decided to focus in detail on the test on rational numbers where all but two tasks were of this type. This choice led to the third stage of the study.

THE SECOND STAGE

Data collection and data analysis

During the second stage, I collected written records of solution procedures that the second group participants submitted as a part of a standard written test on natural numbers. As in the first stage, the test served as a part of the course assessment and came after the topic of natural numbers had been discussed at lectures and properly practised at course seminars. Again, the participants were divided into two almost equally sized subgroups and each of the subgroups got a different assignment of the test. The assignments were taken from the first stage: one of them comprised of word problems W1, W3 and W6, and the other one of W2, W4 and W5. The method of data analysis was the same as in the first stage.

RESULTS OF THE SECOND STAGE

From the perspective of individual word problems – variety

In the second stage, each of the word problems had several unsuccessful participants who did not provide a solution to the problem or provided an incorrect one. The least successful were the participants at the tasks W4 and W5 where 7 of 24 failed and at the task W3 where 7 of 25 failed. The participants provided 2 different correct solution procedures to the word problems W2, W3, W5 and W6, and 3 different correct solution procedures to the word problems W1 and W4. Only two of the correct solution procedures from the second stage had not appeared during the first stage, both of them with different authors but similarly employing an additive model for solving a multiplicative situation for the tasks W1, W4 (for a sample see Table 3).

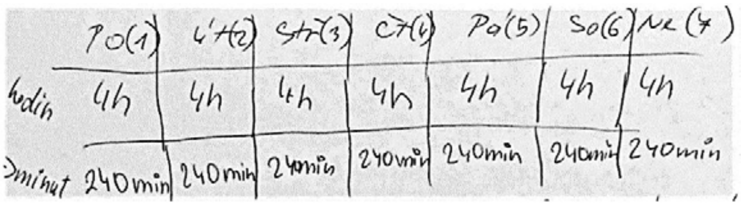
W1	4 h = 60 + 60 + 60 + 60 = 240 min 7 days = 240 + 240 + 240 + 240 + 240 + 240 + 240 = **1680** min 

Table 3: One of the newly emerged correct solution procedures from the second stage; translations of texts in the embedded picture: Po = Monday (abbr.), Út = Tuesday (abbr.), St = Wednesday (abbr.), etc.

The number of provided correct solution procedures was smaller in the second stage than in the first stage, although the number of participants was twice as large.

From the perspective of individual word problems – usualness

Similar as in the first stage, the correct solution procedures from the second stage were either even, majority, or minority ones. Considering the characteristic of the task as an information about the type of procedures used by its solvers (even, majority/minority) and about the order of the procedures by their relative frequencies, collected data showed that only the characteristics of the tasks W4 and W6 stayed the same in the second stage as in the first stage. For the tasks W1, W2 and W3, the majority solution procedures from the first stage were not the same as the majority solution procedures from the second stage. For the task W5, the frequency of the first correct solution procedure increased in the second stage in such a way that it changed from even to majority. That means that all of the six tasks had majority and minority procedures in the second stage. See Figure 2 for detailed diagrams of the relative frequency of individual solution procedures among the second stage participants. The sectors related to incorrect solution procedures are shaded and the sectors related to correct solution procedures are unshaded. The order of particular solution procedures around the diagrams in Figure 2 is the same as in Figure 1.

Figure 2: The diagrams of the relative frequency of individual solution procedures among the second group of participants that got to solve particular word problems, *n*=25 (for W1, W3, W6), *n*=24 (for W2, W4, W5), 2018-2019 (source: own calculation)

The case of the most frequent correct solution procedure of the task W1 is interesting from the point of view of the dynamic nature of the task and the sequential events that form the situation described by the task. As Thevenot and Oakhill (2006) showed in their research, while solving multiple-step dynamic arithmetic word problems, the order of individual calculations is usually determined by the order of events described in the assignment of the word problem. For W1, such usual order corresponds to the second solution procedure given in Table 2, i.e. to the less frequent solution procedure in the first stage and to the most frequent solution procedure in the second stage.

From the perspective of individual solvers

From the perspective of individual solvers, there were five diverse groups of participants: those who used the majority solution procedures in all relevant cases (27 participants), those who used the majority solution procedures in all but one (9 participants), those who used the minority solution procedures in all relevant cases (3 participants), those who used the minority solution procedures in all but one (5 participants) and those who used the majority and minority procedures equally (5 participants). That means that 27 + 9 = 36 of the second stage participants were majority solvers and 3 + 5 = 8 were minority solvers. The remaining 5 participants who equally provided majority and minority procedures will be called *mixed solvers*.

Back to the first stage

In order to align better the first and second stages from the perspective of individual solvers, we have to return to the first stage and to the reasoning about the task W5 that dealt with the question whether the task should be characterized as a task with evenly used procedures or as a task with majority and minority procedures. In the first stage, a certain argument led to the decision for the first option. However, since the task W5 met both the characterizations, it is possible to consider now the second option and distinguish majority and minority solution procedures of this task (the more frequent solution procedure for the first stage participants was the first one in Table 2). In that new setting, all six tasks W1 to W6 are assigned majority and minority solution procedures in both stages.

With this adjustment and while taking into account just tasks W1 to W6 from the first stage, we can obtain adjusted information about majority and minority solvers of word problems on natural numbers from the first stage. These solvers can be divided into four diverse groups: those who used the majority solution procedures in all relevant cases (9 participants), those who used the majority solution procedures in all but one (4 participants), those who used the minority solution procedures in all relevant cases (3 participants), and those who used the minority solution procedures in all but one (8 participants). That means that 9 + 4 = 13 of the first stage participants were majority solvers of tasks W1 to W6, and 3 + 8 = 11 were minority solvers of tasks W1 to W6. No mixed solvers.

The ratio of majority solvers of tasks W1 to W6 is much bigger with the second group of participants than with the first group of participants and the ratio of minority solvers is much smaller – see Figure 3 for detailed diagrams.

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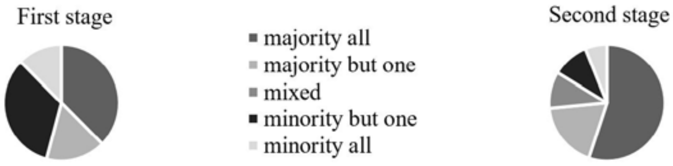

Figure 3: The diagrams of the relative frequency of various types of the tendency to use majority or minority procedures when solving word problems on natural numbers in the first stage (left, *n*=24, 2014-15) and in the second stage (right, *n*=49, 2018-19) (source: own calculation)

THE THIRD STAGE

Data collection and data analysis

During the third stage, I collected written records of solution procedures that the third group participants submitted as a part of a standard written test on rational numbers. As in the first stage, the test served as a part of the course assessment and came after the topic of rational numbers had been discussed at lectures and properly practised at course seminars. Again, the participants were divided into two almost equally sized subgroups and each of the subgroups got a different assignment of the test. The assignments were taken from the first stage: one of them comprised of word problems W7, W10 and W12, and the other one of W8, W9 and W11. The method of data analysis was the same as in the previous stages.

RESULTS OF THE THIRD STAGE

From the perspective of individual word problems – variety

In the third stage, each of the word problems had several unsuccessful participants who did not provide a solution to the problem or provided an incorrect one. The least successful were the participants at the tasks W11 where 17 of 20 failed and at the task W12 where 10 of 17 failed. The participants provided 3 to 6 different correct solution procedures to each of the word problems, the most to the word problem W9. Nine of the correct solution procedures from the third stage had not appeared during the first stage, see Table 4 for two of them.

W7	1 dumpling... 12 slices... 12/12 1 slice... 1/12 1/3 + 1/4 + 1/3 + 1/2 = 21/12 = 1 9/12 2 – 1 9/12 = 3/12 **2** dumplings **3** slices left	father 2/3 = 8/12 mother 1/4 = 3/12 daughter 4 slices son 6 slices father + mother 8 + 3 = 11 slices daughter + son 10 slices first dumpling 1 slice left second dumpling 2 slices left **2** dumplings, **3** slices left

Table 4: Two of the newly emerged correct solution procedures from the third stage

Although the third group participants were less successful in their solving than the first group participants, the variety of correct solutions to the tasks W7 to W12 can be considered as similarly wide.

From the perspective of individual word problems – usualness

Similar as in the first stage, the correct solution procedures from the third stage were either even, majority, or minority ones. Only the first solution procedure for the task W10 increased in the third stage in such a way that it changed from even to majority. The rest of the tasks did not change in their characteristics even though the task W9 had got a completely different composition of correct solution procedures: two of the three most frequent solution procedures from the first stage did not appear in the third stage at all. See Figure 4 for detailed diagrams of the relative frequency of individual solution procedures among the third stage participants. The sectors related to incorrect solution procedures are shaded in the diagrams, the sectors related to correct solution procedures are unshaded, the labels written in italics belong to word problems with evenly used correct solution procedures and the labels in bold roman belong to word problems with majority and minority correct solution procedures. The order of particular solution procedures around the diagrams in Figure 4 is the same as in Figure 1.

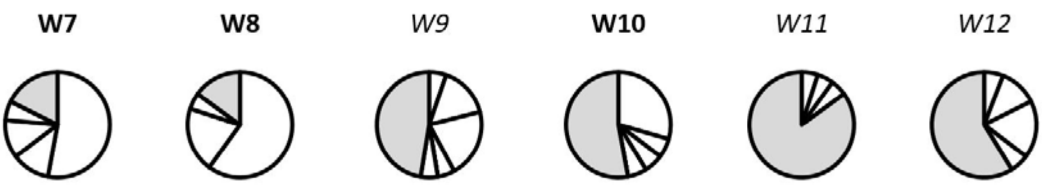

Figure 4: The diagrams of the relative frequency of individual solution procedures among the third group of participants that got to solve particular word problems, *n*=17 (for W7, W10, W12), *n*=20 (for W8, W9, W11), 2016-2017 (source: own calculation)

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Emerging concerns

The groups of participants from the first, second and third stages were all rather homogenous: the participants of each group were of similar age, had come to university directly from the secondary school and had spent the two years of their university studies at common lessons (lectures and seminars) on a daily basis, including the 81 hours of the mathematics course. So that another concern emerged on how the dataset would be enriched with a less homogenous group of participants – for instance with participants attending the first year of a distance retraining study program where the attendants have diverse age, had finished their secondary school studies different times ago and are spending together just two days a month. Such participants do not have seminars on mathematics, just a condensed 10-hour lecture. Since these participants are not divided into subgroups for tests and have just one common assessment test for both the topics of natural and rational numbers, I had to choose two tasks on natural numbers and two tasks on rational numbers to include into the test. I decided to choose the task W5 that had interesting findings in previous stages, and accompanied it by three tasks with majority and minority procedures: W6, W7, W8. This choice led to the fourth stage of the study.

THE FOURTH STAGE

Data collection and data analysis

During the fourth stage, I collected written records of solution procedures that the fourth group participants submitted as a part of a standard written test on both natural and rational numbers. The test served as a part of the course assessment and came a month after the condensed lecture. All of the participants got the same assignment of the test which included word problems W5, W6, W7 and W8. The method of data analysis was the same as in the previous stages.

RESULTS OF THE FOURTH STAGE

From the perspective of individual word problems – variety

In the fourth stage, each of the word problems had several unsuccessful participants who did not provide a solution to the problem or provided an incorrect one. The least successful were the participants at the task W6 where 16 of 39 failed and at the task W8 where 15 of 39 failed. The participants provided 3 different correct solution procedures to the task W6, 5 to the tasks W5 and W8, and 6 to the task W7. Five of the correct solution procedures from the fourth stage had not appeared during the previous stages, see Table 5 for two of them.

W5	68 + 14 = 82 82: 2 = 41 41 – 14 = **27**	68 + 14 = 82 82: 2 = 41 68 – 41 = **27**

Table 5: Two of the newly emerged correct solution procedures from the fourth stage

The fourth group participants were less successful in their solving than the previous groups participants, and the variety of correct solutions was wider in the fourth stage than in the first stage but the number of solvers to each of the tasks was three times bigger than in the fourth stage. What is important is the fact that the fourth stage participants provided five correct solution procedures that had not appeared during any of the previous stages.

From the perspective of individual word problems – usualness

Similar as in the first stage, the correct solution procedures from the fourth stage were either even, majority, or minority ones. But only the characteristics of the tasks W6 and W7 stayed the same. For the task W5, the first correct solution procedure again increased from even to majority. For the task W8, the second correct solution procedure increased in such a way that it differed only by 1 from the first correct solution procedure, i.e. the task W8 changed its characteristics to the task with correct solution procedures evenly used by solvers. See Figure 5 for detailed diagrams of relative frequency of individual solution procedures among the fourth stage participants. The sectors related to incorrect solution procedures are shaded in the diagrams, the sectors related to correct solution procedures are unshaded, the labels written in italics belong to word problems with evenly used correct solution procedures and the labels in bold roman belong to word problems with majority and minority correct solution procedures. The order of particular solution procedures around the diagrams in Figure 5 is the same as in Figure 1.

Figure 5: The diagrams of the relative frequency of individual solution procedures among the fourth group of participants, *n*=39, 2019 (source: own calculation)

From the perspective of individual solvers

From the perspective of individual solvers, there were six diverse groups of participants: those who used the majority solution procedures in all relevant cases (20 participants), those who used the majority solution procedures in all but one (3 participants), those who used the minority solution procedures in all relevant cases (5 participants), those who used the minority solution procedures in all but one (2 participants), those who used the majority and minority procedures equally (7 participants) and those who were not successful with any of the tasks with majority and minority procedures (2 participants). That means that 20 + 3 = 23 of the fourth stage participants were majority solvers, 5 + 2 = 7 were minority solvers and 7 were mixed solvers.

DISCUSSION

The results of this study enriched the puzzle on the topic of *„Which aspects may affect the implementation of formative assessment in inquiry based mathematics education"* by another piece of knowledge. In an inquiry based mathematics environment, students are naturally exposed to various solution strategies which may significantly affect their own choice of solution procedures. To illustrate this aspect, I chose a group of students from the Faculty of Education (i.e. future primary school teachers) attending a one-year inquiry based mathematics course, and explored the variety of correct solution procedures provided by them to twelve word problems that had multiple correct ways of solving. To get a better insight to the problematics, I enriched the research by additional data from three other groups of participants that differed from the initial group in the form of the study program (distance retraining study instead of full-time initial teacher training), in the form of the mathematics course (non-inquiry instead of inquiry based) and/or in its time extent (condensed instead of standard).

Regarding the origin of data and the timing of data collection

A thorough observation of diagrams presented in Figures 1 to 5 can reveal that the timing of data collection stages is not consecutive in the presented research, and that there is a gap between the stages. An explanation of the causes of such a disharmony follows.

Data provided by the initial group of respondents come from the academic year 2014/15 when they were initially collected within a long-term research project that focused on possible advantages and disadvantages of employing inquiry based mathematics education in preservice primary school teacher training. The project concentrated mainly on issues related to designing and performing a one-year compulsory mathematical course fully in an inquiry based manner, and on various facets of mathematics such as reasoning, generalization or open approach (Samková and Tichá, 2016; Samková, 2017). Project data were of various nature, including written records of solutions to twelve word problems. These written records had initially been collected and analysed as a means of distinguishing levels of mathematical performance of the participants but at last the levels were not included in the final stages of data analysis. As a by-product, the initial stage of data analysis related to the written records drew my attention to various solution procedures used by project participants when solving the twelve word problems, so that I decided to revisit this part of data later to explore further its potential.

The time for such an exploration came with the research project discussed in this paper, especially with the research questions focusing on the variety and usualness of correct solution procedures. Proceeding from the initial dataset of written records of solutions to the twelve word problems, I conducted a completely new data analysis on the variety and usualness, and this analysis established the first stage of the new study presented in this paper. The course of data analysis naturally led to the need to enrich the study with other groups of participants. To preserve as many common features as possible, I searched for future primary school teachers that would undergo a mathematical course covering the same mathematics content as the whole-year course attended by the initial group of participants, including standard written tests that would form a part of the course assessment and cover all or some of the initial twelve word problems. Given the necessity to be able to influence assignments of official course assessment tests, the options were limited. One such group was available in my archive of written tests from the academic year 2016/17, with data to all the six word problems on rational numbers. This group underwent a whole-year course within the same program of initial teacher training that covered the same content and the same tasks solved at seminars as the initial course, just not in the inquiry based manner. I chose this group to address further the issues related to tasks on rational numbers. Two other suitable groups started to attend their mathematical courses at the beginning of the year 2019 when the new study started: one of them within the same program and in the same version as the group from the archive, the other within a different program and in a condensed version. The group with the same version of the course naturally became the group to address further the issues related to tasks on natural numbers, while the group with the different version became the group to address generally the issues related to diverse participants. The non-consecutive timing of stages then emerged as a result of the decision to order the description of the two stages focusing further on issues related to rational and natural numbers accordingly to the difficulty of the mathematical topics in the focus, i.e. natural numbers first. The two stages were independent, therefore, the change in order did not affect the findings.

Three of the word problems from Table 1 (W8, W9, W12) appeared also in a study presented in (Samková, 2018a), again just as a means of distinguishing levels of mathematical performance of the participants. Variety or usualness of their solution procedures were not studied there. Data for that former study were collected during the academic year 2017/18, i.e. there is no overlapping with data used in the current study.

Regarding the method of data analysis

The presented study is of an exploratory qualitative design, since the phenomena of variety or usualness have not been studied before. The purpose of the study is to explore and describe the nature of the variety and usualness, not just to quantify them. That is why the research questions begin with

„How varied" and „How usual", and not with „How many" or „How often". In accordance with the qualitative design, the course of data collection and data analysis was not fully given at the beginning of the study but was flexible – affected on the fly by the responses in various stages of the study. This particular process is described in detail at the end of the first and third stage result sections, in subsections called Emerging concerns. The emerging concerns covering the first stage were addressed in the course of the second and third stages, and the emerging concerns covering the first, second and third stages were addressed in the course of the fourth stage. The study is complete from the exploratory perspective since all the emerged concerns had been covered. The qualitative design has been also applied in the wording of the results and discussion sections, i.e. in how the findings have been interpreted. No generalization has been provided in these sections but descriptions of individual experiences and their relations: e.g. descriptions of the nature of the variety revealed by the study (different models used, different order of steps, etc.), new notions established on the basis of the study (majority procedure, majority solver, etc.), descriptions of phenomena that differed across different groups of participants (procedures used, most frequent procedures, etc.).

To get a broader overview of the issue, the qualitative results were quantitatively enriched through basic statistical methods, by using diagrams of relative frequency. These diagrams helped illustrate the qualitative findings. Any other level of statistic would provide the same quantitative results, since the twelve word problems in the study are independent (purposefully based on different didactical concepts) and there are no commonalities among the units of meaning related to different word problems.

Regarding the individual research questions

As an answer to the first research question *„How varied are correct procedures that future primary school teachers use for solving open mathematical word problems?"* I may say that the correct procedures provided by the initial group of solvers were of a really wide variety. With 11 or 13 solvers solving each of the word problems, at least two different correct solution procedures appeared to each of the twelve word problems; three of the word problems got five different correct solution procedures. The procedures differed in used models, information used from a common model, the order of steps or interpretation of the assignment. With the additional groups of solvers, sometimes the variety was similarly wide (in case of the non-inquiry full-time group solving tasks on rational numbers), sometimes was clearly smaller (in case of the non-inquiry full-time group solving tasks on natural numbers).

As an answer to the second research question *„How usual among the solvers are particular correct procedures for solving particular word problems?"* I established three new notions in the context of correct solution procedures provided by the initial group of solvers: majority, minority and even solution procedures. Five of the twelve observed word problems were with correct solution procedures evenly used by the solvers (even procedures), and seven word problems were with one correct solution procedure used by majority of the successful solvers (majority procedure) and the others by minority of the successful solvers (minority procedures). With the additional groups of solvers, some of the procedures changed their order by the relative frequency among the groups (e.g. the most frequent correct solution procedures for the tasks W1, W2, W3 were no longer the most frequent ones) or their characteristics (e.g. the most frequent correct solution procedures for the tasks W5 and W10 changed from even to majority solution procedure, the most frequent correct solution procedure for the task W8 changed from majority to even). The analysis of the additional data also contributed to the particularization of the term "even procedures".

As an answer to the third research question *„How usual among the solvers are the procedures used by particular solvers?"* I established other two new notions in the context of the initial group of solvers: majority and minority solvers. With the initial group of solvers, half of the group were majority solvers (those who tended to use majority solution procedures) and the other half were minority solvers (those who tended to use minority solution procedures). With the additional groups of solvers, there appeared less minority solvers within the groups (both in the non-inquiry full-time and distant cases) but still the minority solvers formed about 1/6 of the groups. With the distant group of solvers, also a new notion emerged referring to mixed solvers (those who used majority and minority solution procedures evenly).

Regarding the aims of the research – formative assessment

The findings about majority and minority procedures and majority and minority solvers are important for on-the-fly assessment as well as for peer-assessment.

In case of peer-assessment, the classmates who are majority solvers might not fully understand the solution procedures produced by minority solvers, and vice versa. Even the minority solvers might not understand each other when their solution procedures are based on completely different models or completely different ways of interpreting the assignment.

In case of on-the-fly assessment, the teacher might not be able to notice some specific or innovative solution procedures produced by minority solvers. The necessity of noticing, understanding and proper interpreting diverse solution procedures comes into play not only when performing inquiry based education but also in non-inquiry cases: within the group of solvers that attended the inquiry based course, the minority solvers accounted for almost half of the group, within the other (non-inquiry) groups, the minority solvers accounted for about one sixth of each of the groups.

The above mentioned circumstances raise a question important for future implementation of formative assessment into mathematics teaching: whether and how it is possible to enhance noticing, understanding and proper interpreting of different types of solution procedures. Such a question concerns noticing performed by students as well as teachers. While the topic of noticing of students appears rarely in research and mostly focuses on students noticing what a teacher is doing (Hohensee, 2016), the topic of noticing of teachers has been lately broadly discussed (Schack, Fisher and Wilhelm, 2017).

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Taking into account the work of Naylor and Keogh (2007), an approach directed towards enhancing noticing, understanding and proper interpreting of different types of solution procedures by students could be based on Concept Cartoons – an educational tool that had already proved its usefulness in formative assessment in science classroom discussions. In the particular case of open mathematical word problems, the Concept Cartoons pictures may help visibly introduce into the classroom not only the majority but also the minority solution procedures and elicit discussions on them. In that context, Naylor and Keogh (2007) also pointed out that learning of students often depend on getting students to let go of their existing ideas while providing them with access to more productive ideas, and they introduced Concept Cartoons as a tool that enabled such processes by letting students to get to reflect carefully on their own ideas and to take alternative possibilities seriously.

Taking into account the work of van Es and Sherin (2008), an approach directed towards enhancing noticing, understanding and proper interpreting of different types of solution procedures by teachers should be based on changing what the teachers notice in a lesson and how they understand and interpret the noticed phenomena. Such enhancement is often promoted by watching, analysing and discussing video recordings of mathematical lessons or interviews with children (Schack, Fisher and Wilhelm, 2017; Simpson and Vondrová, 2019) or by supporting questioning practices (Spangler and Hallman-Thrasher, 2014; Milewski and Strickland, 2016).

As shown in my previous research (Samková, 2018b), this teacher oriented approach could also be based on Concept Cartoons: Concept Cartoons may help teachers get acquainted with various solution procedures that more or less probably might appear in the classroom and get trained in proper responses on them (e.g. in making decisions on correctness of procedures provided by students, discovering mistakes and their causes, posing indicative questions, anticipating students' reasoning, etc.). In that sense, Concept Cartoons may be considered as an artificially designed representation of the constituent part of school practice that is related to formative assessment (Grossman et al., 2009), and thus they may serve as a mediating tool between teaching practice and teacher education in the topic of formative assessment methods (Herbst and Chazan, 2011). Similar effect could be provided e.g. by simulated teaching experience (Webel, Conner and Zhao, 2018) or by multiple solution method and designed student responses (Evans and Swan, 2014; Evans and Ayalon, 2016). The use of Concept Cartoons that indirectly mix together content-centered and student-centered approaches might also help overcome the unwanted weak relation between content-related noticing and anticipation of other alternatives or continuations that was reported for primary school teachers by Hoth et al. (2016) as well as teacher's narrow focus on their own ideas instead on students' reasoning (Visnovska and Cobb, 2015).

To illustrate better the potential of Concept Cartoons in relation to formative assessment and the referred study, I have prepared a Concept Cartoon on the word problem W5 (see Figure 6), inspired by the most frequent incorrect solution procedure (presented in the figure by Peter) and by three correct solution procedures gained within the study and listed in Tables 2, 5.

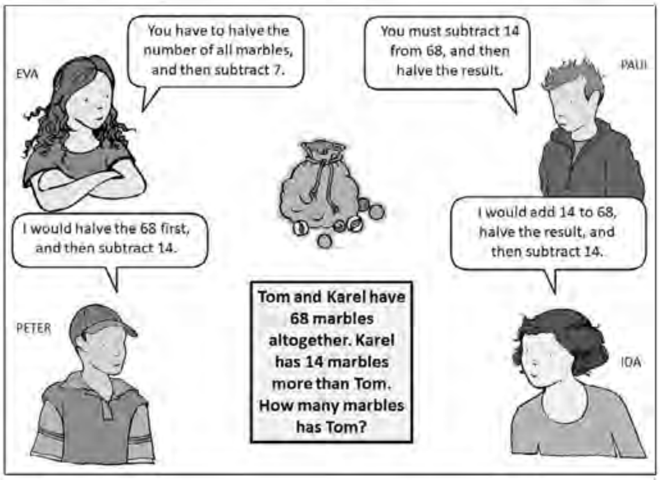

Figure 6: A newly created Concept Cartoon displaying various solution procedures of the word problem W5; (source of the template of children with empty bubbles: Dabell, Keogh and Naylor, 2008: 2.10, source of the central picture: Clipart Library, 2016)

Regarding the aims of the research – inquiry based mathematics education

The findings about majority and minority procedures and majority and minority solvers are important also from the perspective of inquiry based mathematics education regardless of the extent of formative assessment that appears in the classroom. They also confirm the potential of employing open approach to mathematics as a perspective for observing and investigating inquiry based mathematics education.

The presented study illustrates the issue of inquiry through multiple solution procedures to word tasks that belong to elementary mathematics and mostly have a unique way of interpreting the task assignment, a unique solution and a unique way of interpreting the solution. Such tasks belong to the least difficult inquiry tasks (for a detailed typology of inquiry tasks see Samková, 2017), and the reality that would appear while employing inquiry tasks that are more complicated and more difficult would reflect the situation presented in this paper and magnify it: the more the inquiry task would be open, the more instances of different interpretations, procedures and results might appear in the classroom.

Regarding other implications for school practice

The results of this study also contribute more generally to various attributes of school practice: teacher lesson planning and conducting, student learning, student attitudes.

The fact of the existence of majority and minority solvers might help teachers in preparing for and performing their lessons. Especially in case of primary school teachers who all-day work with the same group of pupils in the time span of two or three school years, the guidelines in the form of the possibility to label majority solvers and minority solvers within the group and then better anticipate the nature of their ideas and solution procedures might be really helpful. The findings may also be useful for teachers when implementing in the classroom tasks that are called multiple solution tasks – tasks that contain in their assignment an explicit requirement to solve the task in multiple ways (e.g. Levav-Waynberg and Leikin, 2012). Each of the tasks W1 to W12 might be assigned in this way and similar analysis on usualness of combinations of individual solution procedures provided by solvers might be conducted as well.

The description of the inquiry based mathematics course where the research study took place may serve as a suggestion of how inquiry or open problems can be successfully presented to students: gradually, by starting with word problems that have a unique way of interpreting the assignment and a unique correct answer but multiple correct ways of solving, by repeatedly recording all possible solution procedures on a blackboard, discussing and defending them, and later on by incorporating word problems with multiple ways of grasping or/and multiple correct answers. When the students in focus are future teachers, such an arrangement allows them to get acquainted with the breadth of possible opinions and solution strategies that they would meet in future during their own teaching practice. Such an approach also helps easily implement both inquiry and formative assessment into preservice teacher training, and so address the usual objection about the lack of opportunities for formative assessment in preservice courses (Harlen, 2003).

The above mentioned requirement for the unique way of interpreting the assignment of a word problem also means that the practical situation that is hidden behind the problem is free of unfounded assumptions (e.g. on non-existing proportionality; for tasks rich in unfounded assumptions see Verschaffel, Greer and de Corte, 2000). The existence of unfounded assumptions is desired in advanced work with open problems (especially when aiming towards practically based modelling tasks) but really complicates the initial process of getting to know the environment of open problems as well as the related diagnostic process. Thus none of the word problems that served as diagnostic in this study could be provided by plausible unfounded assumptions.

The idea of open approach to mathematics is sometimes mistakenly considered as just a token of a certain not so common attitude to mathematics. For instance, Schoenfeld (2016) reports that the opinion that each mathematical task has a unique solution and a unique solution procedure belongs to one of the most frequent misconceptions about mathematics. Such a misconception is in clear contradiction with the topic of this paper as well as with any attempts to implement practically based or modelling tasks into mathematic lessons (Kaiser et al., 2011). I personally believe that the awareness and knowledge of various solution procedures is an integral part of mathematics knowledge that can (and should) be presented to students from their early years of schooling and its importance grows with the growing difficulty of the solved problems. When solving more complex tasks (e.g. W12), one cannot have a unique solution procedure prepared for each of these tasks. The solver has to be able to think about various contexts associated with the task and about their relations, probe various strategies of solving the task and wait which of them would lead to the required results. That problematics was clearly illustrated by the two of the most difficult word problems in the study (W11 and W12) since almost each of the successful solvers in the initial group of solvers used for solving their own solution procedure. In that sense, the aspect of usualness reported in this paper is closely related to the applicability aspect for assessing open tasks that was discussed and employed by Bulková and Čeretková (2017).

The systematic work with open problems and the existence of awareness and knowledge of unusual solution procedures of the tasks also contribute to the development of mathematical creativity and divergent thinking (Hino, 2007; Kwon, Park and Park, 2006) and have an influence on affective attributes of learning such as building the persistence of students on complex challenging tasks (Clarke et al., 2014).

As for my own professional practice, I have already incorporated some of the results of the study into my courses on mathematics and didactic of mathematics for future primary and secondary teachers, in a similar way as described in the previous paragraphs. Regardless of the amount of inquiry established for the given course, I always incorporate mathematical tasks with multiple correct solution procedures. When solving these tasks at course seminars, we try to record at least several of the procedures on a blackboard and discuss them. I also attempt to seek possible majority or minority solvers among my long-term students, to be able to provide a more apposite response to them when they face difficulties within the course seminars. Such activities take more time and effort from me as the educator since they require more detailed planning and organizing of the seminars (e.g. setting up more detailed didactical analysis of the planned tasks, taking field notes) but such an approach allows including more students to classroom discussions and seems to get students more aware of the variety of the discourse.

CONCLUSION

The presented study focused on word problems with multiple correct solution procedures and on the nature of variety and usualness of solution procedures provided to these word problems by diverse groups of participants – future primary school teachers. The initial aim of the study had originated in an effort to illustrate various attributes of inquiry based mathematics education that relate to on-the-fly assessment and peer-assessment. The results drew attention to two newly established notions, to two distinctive groups of solvers labelled as majority and minority solvers, and discussed their role in both the types of formative assessment. Such results might be utilized when particularizing the theory of formative assessment in relation to actual classroom events.

The more general aim of the study directed toward guidelines that would help particularize the form and content of a learning hyperspace that we plan to create. This hyperspace is intended for teachers, to help them implement formative assessment into their inquiry based teaching. Implications given by the findings of the study for the process of creating the hyperspace resulted in a decision to promote in teachers their awareness of the variability and usualness of correct solution procedures that might be obtained in the classroom for multiple solution word problems, and thus involve the Concept Cartoons environment into the learning hyperspace. With the help of the hyperspace, we would observe how teachers respond to more or less usual solution procedures and whether they would be able to reveal probable pupil ideas that might have led to the procedures. We would also provide the teachers with examples of good practice related to the issue.

The findings of the study indirectly address not only teachers that intend to implement formative assessment and inquiry based education into their teaching but also all other teachers, e.g. by giving them implications for classroom work with multiple-step arithmetic word problems: at least two different solution procedures appeared to each of the word problems in the study, regardless of the form and time extent of the mathematics course that the group of solvers had been attending. Also, the minority solvers appeared noticeably in all of the observed groups.

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