

Language-Sensitive Support Of Multiplication Concepts Among at-Risk Children: A Qualitative Didactical Design Research Case Study

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In the last decades, many research projects have indicated that, especially for children with mathematical difficulties, procedural training programs seem to be very helpful in promoting learning of multiplicative routines such as the times table. However, there is less knowledge about whether and how a conceptual understanding of multiplication could also be fostered in these children. A brief review of the current research in this field serves as a basis for a single-case intervention study of two at-risk children. These children's understandings were fostered in a language-sensitive way placing emphasis on the relationship of different multiplicative representations and meaning-related vocabulary associated with this operation (e.g., 2 times 4 means there are 2 groups of 4 each). In-depth analysis of the individual learning trajectories of these two children show that at-risk children can understand such multiplicative concepts if the children start to think in groups of equal sizes.

Keywords: **at-risk children, multiplication, conceptual understanding, equal groups, derived facts strategy, CRA approach, language-sensitive support**

Introduction

During their primary school education, children should not only be able to execute the four basic arithmetic operations, they should, more than anything else, also be able to understand them in terms of content. Primary school children who have not acquired the basic skills in the four arithmetic operations are regarded as being "at-risk" for mathematical failure (Stegemann & Grünke, 2014). The lack of understanding of the basic and fundamental mathematical skills of such at-risk children has been verifiably demonstrated to make it more difficult to achieve higher learning targets (Stegemann & Grünke, 2014). Recently, focus has shifted towards a balance between procedural knowledge and conceptual understanding. Whereas procedural knowledge is defined as knowledge that focuses on procedures, skills and routines to accomplish a goal, conceptual understanding is defined as an understanding that is manifested in rich relationships to and interconnected with different mathematical ideas (Rittle-Johnson & Schneider, 2015). Both forms of knowledge are important for learning mathematics. In fact, however, most empirical studies focus predominantly on fostering procedural knowledge in at-risk children (Maccini, Mulcahy, & Wilson, 2007) because it is well known that at-risk students have problems with fact retrieval in different mathematical contents (Gersten, Jordan,

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& Flojo, 2005). Despite that, there is a particular research goal of fostering conceptual understanding among these children (Gersten et al., 2005). The few studies that exist have focused on conceptual understanding of additive concepts, subtractive concepts, or counting strategies (Canobi, 2004; LeFevre et al., 2006). While less is known on how to foster a conceptual understanding of multiplication among at-risk children, however, very few at-risk children have a conceptual understanding of multiplication (Cawley, Parmar, Foley, Salmon, & Roy, 2001; Moser Opitz, 2013). Therefore, the question arises whether at-risk children are perhaps not able to conceptually understand multiplication. Answering this question is the main purpose of the study presented in this article.

Background and Research Interest

In this section the differentiation between procedural knowledge and conceptual understanding will first be discussed in general and then will be examined more specifically for developing multiplication concepts in the field of special needs education. The results of this comparative analysis will lead to a presentation of central research gaps and research questions.

Fostering procedural knowledge versus conceptual understanding of multiplication

Wide consensus has emerged that conceptual understanding and procedural knowledge are equally important for mathematics learning (Rittle-Johnson & Schneider, 2015). However, no broad consensus exists on which type of understanding should be fostered first. It has often been speculated that conceptual understanding depends on procedural knowledge. This has led to the idea that procedural knowledge should be fostered first, hoping that conceptual understanding will develop at the same time and be based on procedural knowledge.

> [T]here is extensive evidence from a variety of mathematical domains indicating that the development of conceptual and procedural knowledge of mathematics is often iterative, with one type of knowledge supporting gains in the other knowledge, which in turn supports gains in the other type of knowledge... Both kinds of knowledge are intertwined and can strengthen each other over time. (Rittle-Johnson & Schneider, 2015, p. 1125)

The relations between the two types of knowledge, however, are not symmetrical. In many studies, fostering a conceptual understanding had a stronger influence on procedural knowledge than vice versa (for an overview, see Rittle-Johnson & Schneider, 2015). However, procedural training does not always support growth in conceptual understanding (for addition and subtraction, see Canobi, 2009). Consequently, the interconnection is not symmetrical, so both procedural knowledge and conceptual understanding must be fostered in at-risk children. But what does this mean for learning multiplication concepts?

In this regard, it is well known that a procedural focus on multiplication facts such as the times table ends in effective training programs. These programs have in common the requirement that children automatize the multiplication tasks and results of the times table, for example, by repetition of tasks and results in written form (Skinner, McLaughlin, & Logan, 1997), by listening to an audio tape (McCallum, Skinner, & Hutchins, 2004), by increasing speed of reciting (Beveridge, Weber, Derby, & McLaughlin, 2005), or by repeated repetition of the rows, sometimes in combination with counting songs (Grünke & Stegemann, 2014). It has remained undisputed that, from a long-term perspective, procedural knowledge is an important learning target because automaticity relieves the burden on the working memory and results in increased speed when it comes to completing routine tasks (Hurst & Hurrel, 2016; Willingham, 2009; Wong & Evans, 2007).

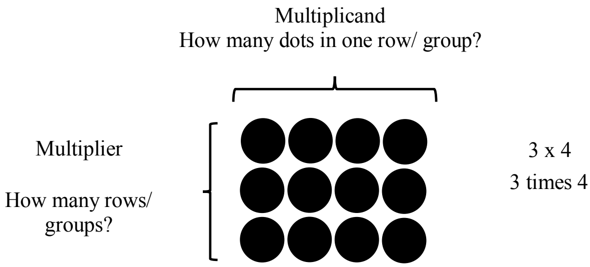

Figure 1. **Visualization of multiplier and multiplicand in the rectangular array**

However, knowledge that builds primarily on procedural counting and factual knowledge is one-sided and limited (Anghileri, 1995; Downton & Sullivan, 2017; Jacob & Willis, 2003). Besides, children also need to develop multiplication concepts (Anghileri, 1995; Downton & Sullivan, 2017). This means that on the one hand they need to be able to work flexibly with tasks and therefore to understand how they are interconnected, and on the other hand they need to be able to interconnect different representations (enactive, iconic, and symbolic) of multiplication (Anghileri, 1995; Siemon, Breed, & Virgona, 2008). For this purpose, children need to understand that cardinal quantities are combined with each other, wherein, formally speaking, the multiplier represents the frequency of the cardinal quantities and the multiplicand represents the size of the cardinal quantity (Figure 1). Research indicates that this distinction between multiplier and multiplicand while learning multiplication in primary school is fundamental for developing an understanding of the commutative and distributive properties (Downton & Sullivan, 2017; Jacob & Willis, 2003). Consequently, to multiply conceptually means to coordinate interrelated units. Understanding the grouping concepts of multiplication is a basic requirement for working flexibly with tasks as well as with different representations of multiplication (Downton & Sullivan, 2017, Hurst & Hurrel, 2016; Jacob & Willis, 2003). If children tend to concatenate the multiplication one-sidedly with repeated addition or factual knowledge, this means that the differentiation between multipliers and multiplicands does not become clear and the knowledge in relation to associative and distributive contexts is not accessible to them (Jacob & Willis, 2003). However, focusing on factual knowledge and repeated addition is the preferred strategy used by at-risk children (Gaidoschik, Deweis, & Guggenbichler, 2018; Moser Opitz, 2013; Zhang, Xin, Harris, & Ding, 2014). Nevertheless, it should also be a central learning target to obtain access to multiplicative grouping concepts, especially for these children. This is particularly the case because these multiplication concepts represent a fundamental precondition for understanding further learning content such as proportionality, algebra, fractions, decimal numbers, and percentage calculation (Anghileri, 1995; Downton & Sullivan 2017; Park & Nunes, 2000; Singh, 2000). For example, how is a multiplication idea based on repeated addition or factual knowledge to be applied in order to calculate a task such as $^1/_2$ x $^1/_4$?

Fostering multiplication concepts in the field of special needs education

In the few studies focusing on fostering conceptual understanding of multiplicative grouping in the field of special needs education research one of the two main central approaches of multiplication concepts are usually pursued: Certain studies take a strategy-training approach, which involves replacing less efficient strategies, such as the counting of all individual objects or repeated addition, with efficient strategies, such as derived facts strategies. Using flexible derived facts strategies is based on conceptual understanding of multiplicative grouping, because understanding how to derive tasks from automatised tasks needs an understanding of how the tasks are related. Other studies foster an understanding for this operation by implementing a concrete-representational-abstract approach (CRA).

Strategy training approach

Strategy training has certainly indicated to be effective in the case of at-risk students and students with learning difficulties. Woodward (2006) showed that after having received strategy training, at-risk children are able to solve multiplication problems. However, their performance still remains far behind that of their fellow pupils who do not have difficulties in doing mathematics. It has furthermore been demonstrated that at-risk students, in contrast to their peers, have problems when it comes to individually applying the strategy knowledge that they have been taught, as well as applying these abilities to other tasks (Fuchs et al., 2003; Xin & Zhang, 2009). It is particularly remarkable that predominantly at-risk children use decomposition strategies much less frequently than normal-achieving children (Zhang et al., 2014). These children persist in the strategies of repeated addition or counting. This can be traced back to the fact that the decomposition strategies are very working-memory demanding, and at-risk children are characterized by their limited working memory (Swanson & Beebe-Frankenberger, 2004; Zhang et al., 2014). However, the cause can be seen in an insufficient conceptual understanding of multiplicative grouping (Downton & Sullivan, 2017). These children do not seem to have fully understood how multiplicative tasks are related. Many low-achieving children have problems understanding distributive decompositions. For example, 7 x 8 can be decomposed in different ways: (5 x 8 + 2 x 8), (7 x 7 + 7 x 1), (7 x 5 + 7 x 3) and so on. But for many low-achieving children it remains a mystery why only one number and not both must be split (Hurst & Hurrel, 2016).

Concrete-representational-abstract (CRA) approach

The CRA instructional approach assumes that children achieve conceptual understanding of calculation operations through the use of manipulatives and drawings. Children are meant to develop a mental model that they can apply to solve symbolic problems as flexibly as possible (Flores, Hinton, & Schweck, 2014). To foster multiplication concepts, the use of rectangular arrays has been established (Cawley et al., 2001; Downton & Sullivan, 2017; Hurst & Hurrel, 2016), because they represent in equal measure three quantities; for example, three rows (the multiplier) with four in each row (the multiplicand) make a total of 12 (the product; Figure 1).

In the CRA approach, the array is central for the development of multiplication concepts (Hurst & Hurrel, 2016). However, according to Moser Opitz (2013), very few at-risk children benefit from work done with the multiplicative arrays. In this study, only about 30% of the at-risk fifth-year students surveyed could solve a problem using the manipulative approach, despite the fact that all children had learned multiplication with the help of rectangular arrays in their second year. These children often represented multipliers and multiplicands as two summands. If the children are asked to explain why the problem 3 x 4 matches the corresponding rectangular array, these children then usually point to the first line and the first column. According to these children, the other points are actually of no importance. Such a lack of understanding of the row-by-column structure is often seen in the case of at-risk children (Battista, Clements, Arnoff, Battista, & Borrow, 1998).

Language-sensitive teaching as a new approach

Strategy training and CRA approaches repeatedly indicate that the transition from repeated addition to a conceptual understand of multiplicative grouping seem to be challenging for at-risk children. However, both research directions often ignore a language-sensitive approach, which seems to be a promising approach for fostering conceptual understanding. Language serves the purpose of knowledge acquisition in mathematical learning situations when it supports the establishment of an understanding of a mathematical learning content (Prediger & Wessel, 2013). The phrases *three times four* and *four multiplied by three* are initially of no meaning to children and are not connected with any mental representation. These are the exact phrases, however, which the children are confronted with in mathematics classes. More than 20 years ago, Anghileri (1995) emphasized using expressions that highlight meaning for the individual child as well as mathematical correctness.

> The teacher's role in developing understanding will involve "negotiation of new meanings" for words and symbols to match extensions to the procedures that become appropriate for solving problems. New meanings will need to be "reconciled" with children's existing understanding and this reconciliation is part of the negotiation process that takes place between pupils and teachers in the mathematics classroom. (Anghileri, 1995, p. 10)

Thus, to negotiate the meaning of multiplication, meaningful expressions such as *three groups of four, three times the four, three rows of four columns each* should be connected with more technical terms (Anghileri, 1995). Through this "basic meaning-related vocabulary" (Prediger & Wessel, 2013; Pöhler & Prediger, 2015), the differentiation between multipliers and multiplicands can be made clearer to children through language in addition to row-by-column structures in rectangular arrays. Furthermore, this language may help children to understand how to derive tasks from each other. If a child knows the result of 7 x 10 and interprets it as *seven tens*, this task can easily be used to derive, for example, 7 x 9 (as *seven nines*).

Thus, in terms of fostering a conceptual understanding of division in children with special learning needs, a CRA-based strategy training combined with a language-sensitive approach has seemed to be promising for initiating a conceptual understanding of division as “sharing” and “grouping” (Götze, 2018, 2019). Prediger and Wessel (2013) showed that seventh-grade low-achieving second-language learners’ understanding of fractions can be significantly increased by means of strongly connecting and interconnecting concrete, iconic and abstract representations while at the same time focusing on basic meaning-related vocabulary. Whether or not students benefit from such support seems to depend primarily on whether this meaning-related vocabulary becomes the students’ language of thinking (Pöhler & Prediger, 2015). This means that in the course of such fostering, understanding depends primarily on if and to what extent the students are able to internalize these meaning-related verbalizations and how they can contribute to the forming of mental models. However, whether this basic meaning-related vocabulary can help support the development of at-risk students’ multiplication concepts has not yet been studied.

Research interest and question

Having procedural multiplication knowledge is important for retrieval rapidity. Many studies have indicated that drill and practice teaching seem to be very supportive for developing procedural knowledge. Thus, while having a conceptual understanding of multiplicative grouping is important for future learning targets such as fractions and percentages, how to foster understanding of such a grouping concept among at-risk children is less clear. Can these children really reach this level of understanding? A number of studies up to the present have confirmed that the development of a conceptual understanding of multiplicative grouping among at-risk students often represents a major obstacle in the mathematical learning process. Even if strategy training or support following the CRA approach results in learning effects, these nevertheless often culminate in weak multiplication concepts. After interventions, most at-risk children still use repeated addition strategies but not grouping concepts (Downton & Sullivan, 2017; Gaidoschik et al., 2018; Moser Opitz, 2013; Zhang et al., 2014); however, in other contexts, verbalizations that focus on understanding the content have indicated to be very helpful in promoting learning (Götze, 2018, 2019; Pöhler & Prediger, 2015; Prediger & Wessel, 2013). It is less known, however, if and how this language-sensitive design principle works for developing multiplication concepts among at-risk children. Thus, the purpose of the present study is to contribute to the small body of existing literature by examining whether a language-sensitive intervention can help at-risk children to develop multiplication concepts. Therefore, the following research question is addressed:

> Can language-sensitive support help at-risk students gain a conceptual understanding of multiplicative grouping and, if so, how?

Methods

The study presented in this article pursues the goal of both investigating whether at-risk children can develop a conceptual understanding of multiplicative grouping through language-sensitive support and analyzing first indications of potential success factors for such language-sensitive support. Therefore, it is necessary to make a qualitative in-depth analysis. Such an approach requires a special research design with well-selected participants, instruments that permit an in-depth analysis, and qualitative methods of analysis.

Research design

The study is structured as single-case didactical design research (Pöhler & Prediger, 2015). Such a particular qualitative research project combines four different work areas: (a) specification and structuring of the learning goals, (b) development of a teaching-learning arrangement, (c) systematic evaluation of the fostering, and development of a local theory by means of comparing hypothetical learning goals and the actual learning successes of the children and (d) further development of the teaching-learning arrangement. "The research outcomes consist of empirical insights and contributions to local theories on learning and teaching processes of the treated topic" (Pöhler & Prediger, 2015, p. 1705): in this case, fostering understanding of multiplication concepts among at-risk children.

Participants

This article gives an insight in the learning trajectories of two third graders, Ilay and Leon, from two different German primary schools. Each of the two at-risk boys worked together with a normal-achieving classmate. The classification of the children into *at-risk* and *normal-achieving* was based on their previous academic performance at school (both boys received additional support in mathematics) as well as their performance on national comparative tests, on which both boys had underperformed. Although these children had already learned multiplication in their second year in school, they had many difficulties in solving multiplication tasks, as can be seen below. A cooperative pair setting was selected because other studies have indicated that the discursive negotiation of meaning among heterogeneous children was beneficial for conceptual learning (Götze, 2018, 2019; Zhang et al., 2014). Children's learning was fostered during four 30-minute units. These units were provided by preservice teachers who had been intensively trained beforehand. These teachers and children did not know each other previously. It should be noted that in Germany the rows of times tables are determined by the second factor, meaning that, for example, 1 x 2, 2 x 2, 3 x 2 and so on are second-row tasks, while 1 x 5, 2 x 5, 3 x 5 and so on are in the fifth row.

Instruments

As has already been touched upon above, a qualitative didactical design research study begins with a specification and structuring of the learning goals. Based on the review of the current research, the following central learning and diagnosis steps and according learning goals were applied in order to foster a conceptual understanding of multiplicative grouping (based on Pöhler & Prediger, 2015):

Step 1: Informal thinking starting from students' resources. In order to determine the learning conditions, during the first support unit, the children are asked to arrange rectangular arrays to a corresponding term and name the correct matching terms. They are also asked to explain the connection between the terms and the rectangular arrays. In order to check their strategy knowledge, the children are asked to recognize, continue, and explain patterns in the multiplicative problems (Figure 2).

5 x 4 =	7 x 5 =
6 x 4 =	7 x 6 =
7 x 4 =	7 x 7 =
8 x 4 =	7 x 8 =

Figure 2. **Pattern tasks for diagnosing the use of derived facts strategies**

Step 2: Focusing first on meaning-related vocabulary for understanding multiplication. The central learning goal of this step is that the children see and express (possibly with the teacher's help) the multiplicative grouping concept in the rectangular arrays as well as in the terms. In order to accomplish this, a classic CRA approach combined with meaning-related vocabulary is initially taken. Therefore, a multiplicative understanding is addressed directly: *three times four* means that there are three fours or there are three groups of four each. Consequently, the children should become sensitized to another way of understanding.

Step 3: Independent use of meaning-related vocabulary. The central learning goal of the third step is that the children argue multiplicatively and express independently the connection of tasks and rectangular arrays in a multiplicative way. Subsequently, a conceptual understanding of multiplicative grouping is deepened further through more game-based assignments (e.g., pairs card games). The children are required to name the matching terms for orally described rectangular arrays. For this, they have to interpret the group number (the multiplier) and the group size (multiplicand).

Step 4: Use of meaning-related vocabulary in order to initiate derived facts strategies. The central learning goal of this step is that the children express conceptually how multiplicative tasks can be distributively combined, partitioned, and derived. In order to accomplish this, the children are given rectangular arrays, which they can distributively combine in sometimes different ways. They need to think about which arrays could be laid next to each other and which overall term would match with the constructed array. In order to check their multiplicative strategy knowledge, the children are once again asked to recognize, continue, and explain multiplicative patterns (Figure 2).

So far, particularly in the second and third support unit, the principal focus is on supporting conceptual understanding of multiplicative grouping with the help of meaning-related verbalizations. In the last unit, the children can show whether the new way of understanding of multiplication helps them to solve multiplication problems.

Method of analysis

Overall, especially in Steps 2 through 4, the focus is on fostering conceptual understanding of multiplicative grouping in different situations or games and in terms of language reception and language production. Thus, the understanding is intensified from step to step. Even though this "hypothetical learning trajectory" (Clements & Samara, 2004, p. 82) consists of four steps, each of these four steps must not be realized in one unit each. The aim of such a "hypothetical learning trajectory" is that "researchers build a cognitive model of students' learning that is sufficiently explicit to describe the processes involved in the construction of the goal mathematics across several qualitatively distinct structural levels of increasing sophistication, complexity, abstraction, power, and generality" (Clements & Samara, 2004, p. 83). Consequently, the individual learning trajectory of a child could be characterized by stagnations or jumps, both forward and backward. The challenge of a qualitative analysis is to recognize and interpret these individual learning trajectories with all stagnations and jumps in terms of conceptual learning (Clements & Samara, 2004). Therefore, all units were transcribed. All statements in each support unit were continuously numbered in order to be able to recognize whether a scene was more likely to occur at either the beginning or end of the support unit. Subsequently, the scenes in each support unit in which the at-risk child in particular is required to individually work or to actively take part in the discourse with the normal-achieving child were filtered out. Each scene was then analyzed by a team of three researchers in order to ascertain whether the intended learning steps (Steps 1 to 4) had actually been achieved. The following scenes were selected for the in depth-analysis as they allow in only few statements and actions a cautious diagnosis of the individual learning steps of the children.

Results

Ilay's individual learning trajectory

The at-risk child Ilay (I) worked together with his normal-achieving classmate Nelli (N). Ilay clearly showed that he was a very communicative child who liked to contribute to discussions and really enjoyed explaining how he arrived at his solution.

Step 1: Informal thinking starting from students' resources

During the first support unit, Ilay knew many multiplication problems from the times tables by heart, but still solved them by means of reciting the particular row. While doing so he usually started to recite the rows from one, something that was very time consuming. This meant, for example, that he needed a total of 45 seconds to provide the correct solution to 6 x 7. While reciting, he raised individual fingers one after another, which indicated additive calculation. He explained how he arrived at the solution as follows.

56	I	Because when you now, when you now … aahmmm … then that's 7 [*points to his thumb*] plus 7 [*points to his index finger*] are then 14. And then you work out these 2 [*points to his middle finger and his ring finger*] are also 14. That is then 28 in total. And then you only need to add this 14 [*holds up the next two fingers*]. That is then around 42.

57 T And what comes out at the end of this problem? [*points to the problem 7 x 7*]
58 I Around . . . 48 [*he starts to count his fingers forward once again, whispers*] 14, 28, 42. So . . . [*after 25 seconds*] 49.

Ilay worked out the result by using an additive strategy (Turns 56 and 58) because he added his intermediate results. He was not able to derive the solution for the new problem, 7 x 7, from the previous problem, 6 x 7. He had to "start from the beginning" (Turn 58) and did not derive. He also used the same method to solve the subsequent problem, 8 x 7, and all other problems given to him during the first support unit. In doing this, Ilay displayed typical additive and no derived facts strategies.

Step 2: Focusing first on meaning-related vocabulary for understanding multiplication

During the second support unit, the meaning-related vocabulary for the multiplicative groups was introduced. In the following scene, approximately 10 different rectangular arrays were laid out in front of the children, which also included the correct array for the task, 3 x 5. It is important to remember that Ilay and Nelli got to know this problem as part of the five times tables.

07 T I will now give you the problem. Do you know the game, I spy with my little eye? We will now play using the cards: I spy with my little eye lots of 3 groups of 5 [*the teacher places a note in front of the children with the phrase "3 groups of 5" written on it*].
08 N 3 groups of 5?
09 T Do you have any idea which dot image I mean?
10 I [*points uncertainly at the card showing the dot image relating to 3 x 5*]
11 T Why could that match?
12 I Yes, because here are 5 [*points to the uppermost row*]
13 N . . . because there are 5 in each row.
14 I . . . because there are 5 in each row, so . . .
15 N . . . that's 15.
16 I So when you calculate [*takes the rectangular array 3 x 5*], 1, 2, 3 [t*aps each dot in the first column*], there are 3 below and here, calculate 1, 2, 3, 4, 5 [*taps each dot in the first line*], then that is 15 and here are 3 groups of 5 [*points to the note with the same sentence*].
17 T Okay, so now here are the 3 groups of 5?
18 I 3?
19 T You have already said that that is 3 and 5. Show me the groups of 5 once again.
20 N Here at the top [*moves along the top line*].
21 I Exactly, and we then have a group of 5 here [*moves along the top row*], two groups of 5 together, 3 groups of 5 together [*moves along both the other two rows*].

It can be seen that the children initially reacted in an uncertain manner to this unknown vocabulary (Turns 8 and 10). Even when Ilay directly found the dot image he was looking for (Turn 10), he was still not able to immediately explain why the description "three groups of five" matched this picture. He pointed to the first column and the first row (Turns 12 and 16). This focus is seen in many at-risk children and indicates a limited conceptual understanding (Battista et al., 1998). Nelli, on the other hand, seemed to understand the content immediately, which is often seen in the case of normal-achieving children (Götze, 2018, 2019). She emphasized that in each row there must be five dots (Turn 13), which Ilay then repeated word for word (Turn 14). That Ilay had not yet understood the content of this formulation was indicated when the teacher posed a follow-up question (Turn 17). The teacher recognized that although Ilay pointed to the numbers three and five of the term in the rectangular array in his customary manner, he did not argue in cardinal groups of always five dots. But when Nelli pointed out the group of five in the dot image to him, Ilay seemed to understand this new perspective (Turn 21).

The overall evaluation of the data from this support unit demonstrated that this scene appeared to be a key scene in the learning process for Ilay. He seemed to be given a new way of looking at the rectangular arrays by means of the meaning-related language being offered. But substantial repetition was needed so that Ilay could make this new way of looking at the rectangular arrays a part of his own conceptual understanding of multiplicative grouping.

92	I	[*The card with the rectangular array 5 x 10 is on the table.*] It is difficult [*quietly counts the dots in the upper row, then the dots downward*]. 10 times the 5, eh, 5 times 10.
93	T	Why 5 times 10?
94	I	Because 1, 2, 3, 4, 5 [*counts the dots downward, but also on the right-hand side*] and 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 [*counts the dots in the upper row*].
95	T	And how would you guys describe that with the groups? I see lots out of . . .
96	N	5 times the 10, 5, 10, 5 groups.
97	T	Exactly 5 groups and how large are the groups?
98	N	10.
99	T	Exactly, we have groups of 10.
100	N	5 groups and 10.
101	T	Here we have an entire group of 10 [*circles around the upper line*].
102	I	This is the entire group of 10 [*points to the upper line*]. And here another [*points to the upper line*] and here another, and another and another [*points to each line with a group of 10*].

Both children initially displayed uncertainties when using the meaning-related vocabulary. They repeatedly returned to using the usual number-focused pointing to the two numbers 5 and 10 in the dot image as the first column and the uppermost line (Turn 94). The new vocabulary of the groups had to be illustrated in the rectangular array (Turns 99 and 101). Subsequently, Ilay was able to orally explain the grouping of equal groups and see this in the dot image.

The constant pointing to and describing of the group number and group sizes was possibly initially a process of practicing. After all, prior understanding overlapped with emerging understanding until the new understanding appeared to be increasingly stabilized.

Step 3: Independent use of meaning-related vocabulary

In the third support unit, Ilay seemed to start to integrate into his own argumentation processes the meaning-related vocabulary that he had previously been provided. For example, he explained how and why the term 6 x 6 matched the corresponding rectangular array without having to be asked to do so by the teacher.

123	I	So I took this one here because that is 6, so 6 is here [*moves along the left column with his finger*] and there are 6 groups [*moves his finger from left to right several times*] . . . And here 1, 2, 3, 4, 5, 6 [*taps on each of the dots in the first row*], there are 6 in a group, and when you count down [*moves his finger from top to bottom*], then 1, 2, 3, 4, 5, 6 [*circles each of the six rows*]; that's actually really easy.

First, he addressed the multiplier by pointing to the six groups in the right corner ("there are 6 groups"). Subsequently, he described the meaning of the second six in the term ("there are 6 in a group").

Step 4: Using meaning-related vocabulary to initiate derived facts strategies

In order to obtain first indications about a possible development of strategies, in the fourth support unit, the children were supposed to try to make deductions between the tasks. In the following scene, Ilay was asked again to think about why 7 x 5 is always seven more than 7 x 6 without using manipulatives.

47	I	Because, that [*points to the multipliers*] stays the same and that [*points to the multiplicands*], that should, if you jump over one, then that should now also be 7s.
48	T	I don't quite understand that. What do you mean, if you jump over one?
49	I	If you count one to here [*points to the two multiplicands*], then that will, then it is, if you, a 7, if you [*points to both multiplicands*] count up to 7.
50	T	So if we have 7 times 5 and 7 times 6. Why is it 7 more?
51	I	Nelli, you can also give an answer.
52	T	That's right. But I am not convinced yet. 5s become 6s.
53	I	Because here, you have 6. And then one always comes in here, [*points to both multiplicands*], if we do that with the dots, it only comes, so, so one row . . . so one dot in each row becomes more.

At the beginning of the scene, Ilay initially tried to explain the correlations with his own linguistic means and using many pointing gestures (Turn 47). Indeed, he also occasionally used meaning-related vocabulary such as *sevens* but did not explain the content-related correlation of 7 x 5 and 7 x 6. After the direct reference of the teacher in Turn 52 ("fives become sixes") he changed his argumentation. So, he explained the multiplicative (rather than additive!) correlations very independently using imagined rectangular arrays (Turn 53: "one dot in each row becomes more"). Astoundingly, in contrast to the first support unit, he did not use his fingers to count the rows. Ilay showed this conceptual understanding of multiplicative grouping independently several times in the fourth support unit.

The above analysis shows that, regarding the first part of the research question, it is possible to foster a conceptual understanding of multiplicative grouping in at-risk children. At the beginning of the support, Ilay needed some time for practice and networking with concrete representations (Step 2) before he could use this way of thinking independently.

Leon's individual learning trajectory

Leon's (L) performance in mathematics was below average at the time of the support. It was possible that he would have to repeat the third year. He was working together with his classmate Rico (R).

Step 1: Informal thinking starting from students' resources

During the first support unit, Leon showed that he knew a relatively large number of tasks of the times table by heart. He always began with *one times, two times, three times,* etc. until he reached the desired task (skip counting strategy). Therefore, he gave the impression of not having built up any conceptual understanding of multiplication. Thus, in the first support unit, he had to explain why the term 3 x 5 fitted the corresponding rectangular array.

12 L Because here there are 3 [*points to the first column*] and here 5 [*points to the first row*].
13 T Can you show me that again?
14 L So here there are 3 [*points to the first column*] and here 5 [*points to the first row*].
15 T Okay. Do you see it like that as well, Rico?
16 R Yes. . .
22 T If we look at the task 3 times 5 in more detail now. And I cover a few of the points. The one in the middle [*covers the dot image, the first row and the first column can be seen*]. Does this dot picture still match the task 3 times 5?
25 L Because the 3 is still here [*points from top to bottom*] and here is at least one 5 still [*points to the first row*].
26 T You say at least one 5?
27 L So there is, the main thing is that one 5 can still be seen.
28 R You must be able to see the 5, otherwise it's not right.
29 L There doesn't have to be another 5 here [*points to the second row*] and here [*points to the third row*], it can be just one 5. That's enough.

Similar to Ilay, Leon did not explain the multiplicative situation in this illustrative model, which is typical for at-risk children (Battista et al., 1998).

Step 2: Focusing first on meaning-related vocabulary for understanding multiplication

To introduce the meaning-related vocabulary, the terms 3 x 5 and 3 x 4 and the accompanying rectangular arrays were laid in front of the children. In addition, there was a note on the table: One group of five, always five in a group. One group of four, always four in a group.

37 I Where can I see these statements in the dot image now [*pushes the note towards the children*]? Read it again.
38 L [*reads the note*] A group of 5, always 5 in a group. A group of 4, always 4 in a group.
39 T Aha, where does it fit?
40 R This fits here [*points to the first set and 3 x 5*] and that fits here [*points to the*

second set and 3 x 4].
41 T Why does it fit? Can you explain it to me again?
42 L I do this [*points to the first set and 3 x 5, starts to circle the upper fives in the rectangular array with a pen*].
43 T What is that?
44 L A group of 5. The second group of 5 is here [*circles the second five*]. The third group of 5 [*circles the third five*].

Leon interpreted the newly introduced vocabulary correctly. He could explain straight away that the rectangular array of 3 x 5 consisted of three groups of five (Turn 44). The second unit seemed to show an immediate effect with Leon. Leon's quick linguistic perception looked promising for reaching the third step.

Step 3: Independent usage of the meaning-related vocabulary

The third support unit, however, went differently from Ilay's. The teacher missed the fact that Leon could reach this step, because she fell into a type of *vocabulary training* without even making the meaning-related vocabulary somehow accessible for Leon. In the following scene, Leon correctly placed the rectangular array for 4 x 2 and Rico for 4 x 3. The teacher then placed the term cards 5 x 2 and 5 x 3 in the middle.

153 T Leon, can you explain that with our maths words? The number of groups is . . . [*points to the first column in the rectangular array*], the size of the group is . . . [*points to the first row in the rectangular array*].
154 L The number of groups is . . . 4 and the size of group . . . 2 and here the number of the group is . . . 4 and the size of the group . . . 3.
155 T Good. And what happens to the new tasks now? [*taps on 5 x 2 and 5 x 3*].
156 L The number of groups always increases by 1.

The teacher tried to focus on the conceptual understanding of multiplicative grouping. However, she did not address the basic meaning-related vocabulary of the groups of two and groups of three. Rather, technical terms such as *group size* and *number of groups* were tested. Indeed, Leon was continually asked to show the number of groups and the size of the group (Turn 153); nevertheless, only numbers without meaning were tested with this. The differentiation of multiplier and multiplicand was not clear. However, up to that point, he did not have enough opportunities to acquire a conceptual understanding of multiplicative grouping. After all, he would have first had to learn to understand the content of statements such as "the group increases by 1" or "the number of groups increases by 1". At this point in the support, Leon did not argue in equal groups independently, so to him a formulation such as "the size of the group is two" (Turn 154) seemed to remain without content-related understanding. Therefore, it was not possible for Leon to reach step 4.

Step 4: Using meaning-related vocabulary to initiate derived facts strategies

The fact that Leon could not build up a conceptual understanding of multiplicative grouping was also shown in the fourth support unit. In the following scene, the terms 4 x 6, 4 x 5 and 4 x 4 lay in front of him. Leon named the results of the tasks straight away, but it remained unclear whether he had calculated or known them. In any case, he named the results very quickly, indicating that he had retrieved the result from memory.

87	L	The result is always reduced by 4.
88	T	Why?
89	L	Because the second number is also always reduced by 1.
90	T	Okay, the second number is reduced by 1, but why is the result reduced by 4 if the second number is only reduced by 1?
91	L	Because there, because then the 24 is 20 there and 4 there. So that's 4 less.
92	T	But why is the result reduced by 4?
93	L	I don't know, they thought it up.

Leon immediately determined that the results were reduced by four (Turns 87 and 91). He also assumed that this was linked to the decreasing multiplicands (Turn 89). However, he could not explain these changes in terms of content (Turn 93). Instead, he argued that tasks do not change systematically, but rather are "thought up" and are therefore linked arbitrarily (Turn 93). He did not use the newly introduced meaning-related vocabulary, so he could not argue in groups of the same size. Leon's attempts at an explanation then remained on a purely formal and number-focused level. In fact, Leon did not move beyond and did not even reach the second step of the hypothetical learning trajectory. His case shows that fostering meaning-related vocabulary does not automatically lead to a conceptual understanding of multiplicative grouping. It can help at-risk children to reach a new way of multiplicative understanding, but it is not a guarantee.

Discussion

The aim of the qualitative in-depth research project was to determine whether language-sensitive support can help at-risk students gain a conceptual understanding of multiplicative grouping. A second aim was to determine under which condition this support might lead to such an understanding.

Main findings

With regard to the first aim, this study indicates that at-risk children can gain a conceptual understanding of multiplicative grouping in a language-sensitive research project. At the end of the support, Ilay showed several times that he was able to argue in equal groups. Furthermore, he was able to use this understanding to derive tasks from other tasks and functions as a basis for developing derived facts strategies. However, it seems to be important that the introduced meaning-related vocabulary must be actively and independently used by the at-risk children. In Leon's case, vocabulary training was not very supportive. Thus, these two different learning trajectories indicates important design principles for developing further support units. For developing a conceptual understanding of multiplicative grouping it seems to be important that the children can (a) show the relevant groups and number of groups in concrete rectangular arrays, (b) describe the appearance of rectangular arrays, either mental or concrete, by describing the group size and number of groups and (c) use meaning-related vocabulary to explain correlations of terms or rectangular arrays. Thus, children do not need to follow the CRA approach step by step. Rather, they link and network the concrete actions, representations, and abstract signs again and again, which other studies have already demonstrated to be very beneficial to learning (Götze, 2019; Prediger & Wessel, 2013; Pöhler & Prediger, 2015).

Limitations, future research and practical implications

This study is laid out as qualitative didactical design research (Prediger & Pöhler, 2015) in the sense of a single-case study. The study provides the first indications that such a design could also help at-risk children learn to understand multiplicative grouping. However, the duration of the support with just four support units was very short. In fact, it is not clear how stable this new understanding of multiplicative groups is. How long the support should actually last for at-risk children to be able to independently argue in a content-related way also remains unanswered.

Future quantitative research must therefore check the hypotheses made in this article. The results of the study give helpful hints as to how content-related understanding can be stimulated in at-risk children and which errors should be avoided. Thus, in the project *Mathe sicher können* (*doing math confidently*; mathe-sicher-koennen.dzlm.de), with the help of the findings in this article, teaching materials have been developed and are currently being used to support a total of around 150 at-risk third graders in a longitudinal study. A second project based on the results of this article with more than 150 second graders in primary school (mainstream school children) has given empirical evidence that children who learned multiplication as a grouping concept showed significantly higher results in conceptual multiplication tasks at the end of the second grade than children who learned the rows traditionally (Götze & Baiker, 2019).

Consequently, there should be more focus on meaning-related vocabulary in, for example, school books or other teaching resources when introducing multiplication in primary school. However, this vocabulary should also be referred back to in the development of derived facts strategies. In this respect, teachers should be trained to link concrete actions and iconic and abstract representations via basic meaning-related vocabulary. It might then be possible that the introduction of multiplication will no longer be a "cutoff point" (Cawley et al., 2001) in the mathematical learning process for at-risk children.

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