

Fostering Middle School Students' Number Sense Through Contextualized Tasks

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Abstract

Number sense has been considered as one of the most important mathematical notions to be addressed in school mathematics in the 21st century. In this paper, we identify how students of a public middle school, located in a rural area in Mexico, showed several aspects of number sense by performing tasks involving arithmetic operations in a shopping context. Students invented mental and written calculation strategies that provide evidence of understanding regarding numbers and operations. Students used fantasy money as a representation system to decompose quantities in non-standard ways, or referred to strategies employed by unschooled people. Compensation strategies that involve modifying the quantities before applying the written algorithm for subtraction, in order to perform a subtraction without regrouping, also appeared. That is, it was identified a transfer from mental calculation strategies to a paper and pencil scenario for facilitating calculations during problem solving, which is an indicator of student's creativity and number sense. Additionally, the proposed tasks provided evidence that contextualized tasks could promote the understanding of decimal numbers.

Keywords: Number sense; Arithmetic operations; Contextualized tasks; Calculation strategies.

Introduction

One of the main objectives of mathematics education is to provide theoretical and methodological principles to guide the design and implementation of tasks that foster the development of students' mathematical understanding. Tasks design is important, since characteristics and inherent pedagogies of tasks are key aspects that determine the nature and qualities of students' learning (Sarama & Clements, 2009; Sullivan, Clarke, & Clarke, 2013) and, on the other hand, tasks constitute the main instrument that teachers have to help students understand mathematical ideas (Anthony & Walshaw, 2009). On this line of ideas, mathematics education research has obtained evidences that the greatest gains on mathematical understanding are related to tasks that require high levels of mathematical thinking and reasoning, and that engage students in "doing mathematics or using procedures with connection to meaning" (Stein & Lane, 1996, p. 50).

Understanding is not something that a person has or does not have; it is something that is always changing. Understanding is an important idea; since when a person understands something, this can be adapted and used to solve problems (Hiebert, et al., 1997). We consider that understand an idea, means realizing how it is connected or related to the mathematical resources that one can bring to bear to approach a problem (Schoenfeld, 1985).

[...] a student understands how to add 35 and 47 if she can relate this problem to other things she knows about addition and about the meaning of numerals 35 and 47. Knowing that 35 is 3 tens and 5 ones and that 47 is 4 tens and 7 ones helps her understand several ways of combining the numbers [...] evidence for understanding is often provided in the form of explanations for why things are like they are [...] (Hiebert, et al., 1997, p. 4).

From the previous conceptualization of understanding in terms of connections, it follows that there may be several levels of this, depending on the quantity and robustness of relationships that are established between a new idea and a person's prior knowledge. Understanding levels have been characterized for specific domains or mathematical subjects (Chimhande, Naidoo, & Stols, 2017; Jones, et al., 1996; Malloy, 1999; Munisamy & Doraisamy, 1998; Pitta-Pantazi, Christou, & Zachariades, 2007; Sarama & Clements, 2009; Sowder, 1992) and several models have been proposed to analyze the growth of mathematical understanding. One of such models was suggested by Pirie and Kieren (1994); it starts by considering students' previous knowledge, which is the base for the formation and internalization of mental images, formed through the actions that are executed on physical or symbolic objects. The next levels of understanding include observation of regularities, formulation of conjectures, justification and structuring results, and posing new questions. We will incorporate some elements of this model (referred as the P&K Model) to structure our conceptual framework, which will allow us to determine how students' number sense can be enhanced through a contextualized task that favor the invention of mental and written calculation strategies.

Number sense and contextualized tasks

To develop mathematical understanding, students should play an active role as part of a learning community, in which meanings taken-as-shared are constructed through reflection and communication activities (Cobb, et al., 1991; Hiebert, et al., 1997; National Council of Teachers of Mathematics, 2000). Communication allows students to defend their opinions, and question others' ideas, and during this process, "they are likely to recognize incongruities, and elaborate, clarify, and reorganize their own thinking" (Hiebert & Wearne, 1993, p. 396). Particularly, in the basic education (K-9 grades) it is important that students develop *number*

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sense through dealing with problematic situations that allow them to relate quantities with real life situations and develop skills oriented to decompose, group, as well as to use the commutative, associative and distributive properties of addition and multiplication to facilitate or simplify arithmetic calculations (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Number sense is a basic skill for the 21st century citizens (Devlin, 2017), since despite the existence of digital technologies, with which a person can perform calculations and carry out a wide variety of complex mathematical procedures in a few seconds, computational devices cannot interpret mathematical results for making informed decisions (Fuson, 1992).

On the other hand, "employers are demanding higher levels of mathematics skills from their employees in their workplaces than in the past" (Ali, 2014, p. 2) and for this reason, students with limited mathematics proficiency might have limited career or job opportunities. Hence, the increasing importance of mathematical knowledge in professions and everyday activities is a compelling reason for including contextualized tasks in mathematics curriculum (Niss, 1987). "Students will begin to acquire number sense if they are engaged in purposeful activities requiring them to think about numbers and numerical relations and to make connections with quantitative information seen in everyday life" (Reys B. J., 1994, p. 115).

It is important to note that, generally, in school mathematics, procedural and algorithmic aspects of arithmetic have been emphasized. A consequence of this approach is that, nowadays, a high proportion of the worldwide adult population shows difficulties for solving problems that involve carrying out basic arithmetic operations, including the calculation of percentages or probabilities (Paulos, 1988). Hence, some mathematicians, mathematics educators and curricular developers have proposed to encourage the development of number sense in mathematics classrooms (National Council of Teachers of Mathematics, 2000; Yang & Wu, 2010). Additionally, few instructional interventions have implemented tasks contextualized in real situations, or similar to those of students' natural or social environments, to foster number sense. We argue that tasks framed in real or hypothetical scenarios provide a robust environment, which can promote the development of students' creativity. The context of a task has a twofold purpose:

- 1) shows how mathematics is used to solve problems of the everyday life and
- 2) motivates students to solve mathematical tasks (Clarke & Roche, 2018). Contextualized tasks can help students to bring their informal knowledge into the classroom and to connect it with the institutionalized one, so that students can see how mathematics might help them to make sense of their surrounding world (Clarke & Roche, 2018).

Number sense has been conceptualized in different ways. Some authors consider that it refers to "general understanding of number and operations, along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful and efficient strategies for managing numerical situations" (Reys R. , Reys, McIntosh, Emanuelsson, Johansson, & Yang, 1999, p. 61). Other authors conceive number sense as "a good intuition about numbers and their relationships" (Nickerson & Whitacre, 2010), or as a well-organized conceptual network that allows people to relate the properties of numbers and operations, and it can be recognized when students operate numbers in creative

and flexible ways. Another view from authors is that number sense represents a way of thinking, rather than a body of knowledge that can be transmitted to others (Sowder, 1992). Despite the different conceptualization of number sense, an important component of it is the flexibility to use methods to calculate, depending on characteristics of the problem being solved, or depending on the particular numbers involved in a problem (Blöte, Klein, & Beishuizen, 2000) and includes making qualitative and quantitative judgments, or recognizing unreasonable results before making calculations, as well as using non-standard procedures (Sowder, 1992). At the heart of flexibility, there is a solid understanding of the different representations of numbers. For instance, the strategy of adding tens and units separately, and then adding the partial results, is more appropriate to calculate $55 + 24$ than for adding 33 and 29. On the other hand, rounding a number to tens before making an addition is a more efficient strategy for calculating $33 + 29$ (Threlfall, 2002).

The relevance of number sense can be noticed by the amount of research literature regarding this subject. Several researchers have focused their attention on behaviors that characterize those who have number sense. For example, Reys et al. (1999), consider that a student with number sense is able to compare quantities, estimate or approximate results, and to propose procedures to carry out arithmetic operations in alternative ways to the standard written algorithms, as well as to judge the relevance of the obtained results. On this line, Sowder (1992, pp. 4-6), recognizes that number sense is developed through several levels of understanding, and it is dependent on the number systems, since it is possible to have good number sense for whole numbers, but not for other kind of numbers such as fractions or decimal numbers. On the other hand, Greeno (1991) argues that reflection regarding number sense requires a theoretical analysis rather than a definition, and proposes a set of abilities for characterizing those who possess number sense.

According to Reys et al. (1999, p. 62), number sense is structured around six components each of which describes essential elements that students need to develop and use in the process of approaching tasks successfully, even if they use digital technologies. These categories will be described in detail in the research framework section and employed to evaluate the students' development of number sense. Taking the above into account, the research question that guides this work is: *What strands of number sense show students of a middle school, located in a rural community of the state of Hidalgo, Mexico, when they approach tasks that involve performing arithmetic operations framed in a context of shopping and a ludic classroom environment?* This question is relevant, since number sense is the most important mathematical concept in the 21st Century for K-12 education (Devlin, 2017) and is a valuable way of thinking in daily life (Reys R. E., Reys, Nohda, & Emori, 1995); hence, it "should permeate all aspects of mathematics teaching and learning" (Reys B. J., 1994, p. 114). On the other hand, deficiencies in students' knowledge are largely due to a lack of connection between school knowledge and the informal or intuitive knowledge they possess (Fuson, 1992; Sarama & Clements, 2009); based on this, several researchers and curricular designers propose implementing contextualized tasks as a means to favor students' mathematics understanding (Sullivan, Clarke, & Clarke, 2013).

Theoretical framework

The research framework that guides this work is structured around three dimensions: (i) ontological, (ii) epistemological and (iii) didactic. Regarding the ontological dimension, we consider that mathematics is the science of patterns

(Steen, 1988) and that learning mathematics consists in developing a disposition to see the world through the lens of a mathematician (Clements & Sarama, 2013; Schoenfeld, 1992; Sarama & Clements, 2009). That is, learning mathematics is more than just memorizing facts and acquiring fluency to carry out algorithms or routine procedures; it involves a students' disposition to experiment, explore mathematical relationships, formulate conjectures, justify results, communicate ideas, solve problems by different routes (Polya, 1945) and systematically formulate questions, provide answers and pose new problems (Berger, 2014). In summary, learning mathematics consists, before all, in putting into practice the *fundamental elements of mathematical thinking*.

Concerning the epistemological dimension, we adopt a socio-constructivist perspective (Simon, 1994). We suppose that each person actively constructs her/his own knowledge when approaching problems that disequilibrate their cognitive structures, regardless of the context or the presence and nature of the teaching process; and that learning is a continuous process that takes place into a *community of practice* (Wenger, McDermott, & Snyder, 2002), where students construct meanings *taken-as-shared* (Cobb, et al., 1991). In such a community, each student interprets her/his experience in terms of particular cognitive structures, which are different from that of her/his peers. That is, learning is a social process, in which cultural environment and its productions determine the characteristics of the constructed knowledge (Wertsch, 1993). Thus, opportunities to learn, or situations that disequilibrate students' cognitive structures, arise from the systematic attempts that students make to reconcile personal and their peers points of view, as well as communication processes carried out during problem solving activities (Jones, et al., 1996).

The didactic dimension refers to characteristics of knowledge that we consider as desirable and the conditions that favor the configuration of such characteristics. In this regard, we are interested in students' development of *learning with understanding* (Hiebert, et al., 1997), which implies the construction of a robust structure (connections) between a new knowledge and students' resources (facts, procedures, and strategies). Additionally, we argue that to understand an idea or concept requires, necessarily, *using this knowledge to develop new ideas or to solve problems*.

The construction of mathematical understanding requires that students carry out successive cycles of *action, observation, formulation of conjectures and justification of results* (Zull, 2002). This basic cycle was originally proposed by David Kolb in his book *Experiential Learning* (Kolb, 1984), and revisited by Zull (2002) who, at the beginning of his encounter with the cycle, was skeptical about its usefulness to comprehend learning and teaching processes, since he thought that the cycle was too simple and arbitrary. However after reviewing it carefully, he found that the cycle has biological grounds. Particularly the structure of the cycle is analogous to the way that some regions in the brain are activated and connected during the learning processes. The cycle that we propose (Figure 1) is based on Zull's cycle and complemented with ideas from the P&K Model.

In the action phase, students interact with the data of the task, identifying and representing information that is helpful, or ignoring unnecessary information. Students also represent and interpret data, based on her/his previous knowledge, quantify attributes, or add auxiliary elements to expand the available information to approach the task. This initial phase, which permeates the whole cycle, is very important, since the development of knowledge is grounded in physical experiences (seeing, touching, listening, etc.) and in the mental images that students form from these experiences. Without reference to events or physical and symbolic objects (includ-

ing semiotic representations) mathematical ideas have no sense. That is, "physical experiences and images are required in order to understand anything at all" (Zull, 2002, p. 6). In the first phase of the cycle, the actions that students carry out on physical or symbolic objects lead them to the construction of mental images, process named as *Image Making* level in the P&K Model. A next stage is the internalization through a reflection process of the acquired images in the previous stage (*Image Having* level). In a third stage (*Property Noticing* level) within the action phase includes operating with internalized images adapting them to specific contexts. We recognize that images that can be formed in this phase and their meaning depend on each individual mental schemata.

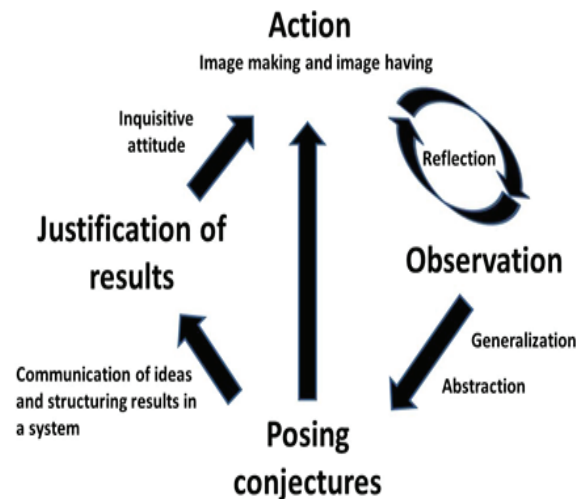


Figure 1. Basic cycle for developing mathematical understanding.

In the observation phase, relationships between data and unknowns are identified, as well as patterns and regularities (*Formalizing* level in the P&K Model). The action and observation phases are intertwined and this relation is represented by arrows in two directions (Figure 1), because the identification of relationships and patterns may require a continued interaction with physical or symbolic objects in which those relations are embedded.

In the phase of formulation of conjectures, students, by a process of identifying patterns, generalize the results observed in the previous phase and formalize their observations in mathematical terms. This second phase of the proposed cycle corresponds to the *Observing* level in the P&K Model. Finally, in the phase of justification, students communicate their conjectures to other members of the community of practice expressing different arguments to support those conjectures: visual, empirical, transformational and formal (Harel & Sowder, 1998). Additionally, students propose ways of structuring the results they discovered into a system which might become a theory (*Structuring* level) and extend results or formulate new questions and problems that will lead to a new cycle of action for developing mathematical understanding (*Inventing* level).

In an empirical research, a person's understandings are always part of productions that are accessible to the researchers as a set of empirical data, represented in a documentary form (text, symbols, figures, pictures, video or audio recordings, etc.), and not the actual understandings that took place in the person's mind (Lithner, 2003). Since we do not have direct access to students' thinking, we consider as evidence of their understanding, the justifications provided by them in the form of oral or written productions. As part of those productions, we identify components or strands that charac-

terize number sense (Table 1), as a tool that helped us to determine the strands of number sense that students, as a group, can show by approaching "The Store" task.

Table 1. Description of number sense strands.

Components	Description
1. Understanding the meaning and size of numbers.	Ability to identify and represent known quantities to compare with others.
2. Understanding and use of equivalent representations of numbers.	Identifying equivalence of numbers in different representation registers (representation of numbers in a number line, as the number of elements in a collection of objects, as numerals in different bases).
3. Understanding the meaning and effect of operations	Recognize diverse type of changes of numbers when applying operations.
4. Understanding and use of equivalent expressions.	Representing numbers in different ways using arithmetic operations, such as sum and product, or identifying rational numbers with their decimal representation.
5. Flexible computing and counting strategies for mental and written computation, including flexible use of electronic devices.	To decompose numbers in different ways to make calculations more efficient.
6. Measurement benchmarks.	Identifying useful benchmarks to perform arithmetic operation and numerical estimations.

Source: Prepared by the authors based on Reys et al. (1999).

Methodology

Participants

The proposed instructional task is called "The store". This task was implemented during the 2016-2017 school year, by developing three weekly special sessions, beyond the regular school schedule, with a group of 24 students (9 boys and 15 girls), from grade 7 to grade 9, of a middle school located in a rural area in the state of Hidalgo, Mexico. This middle school is part of a sub-system called Telesecundaria, in which a single instructor teaches all subjects of a school grade, based on series of pre-recorded lessons transmitted through closed television channels; and whose objective is to offer middle education to students in rural communities with less than 2500 inhabitants.

Most of the students who participated in this research attended primary education (grades 1-6) in single-teacher or two-teacher schools (United Nations Educational, Scientific and Cultural Organization, 1989). Most of those students, in the opinion of their regular teachers, showed learning deficiencies regarding: (i) reading and comprehension in Spanish, and (ii) fluency to perform algorithms to operate with whole and rational numbers. Prior to the participation of the students in the research, their parents were officially notified and asked for their approval to videotape students' work. Parents were informed that the videos would only be observed by the research team and that the participants will be referred in the research reports by a pseudonym.

Number Sense in Telesecundaria Curriculum

In the Mexican National Curriculum for Mathematics one of the standards for mathematics is *number sense and algebraic thinking* and it states that "As a result of the study

of Mathematics, it is expected that students: Use mental calculation, estimation of results or operations written with whole numbers, fractional or decimal numbers, to solve additive and multiplicative problems." (Secretaría de Educación Pública, 2011, p. 14). However, the tasks proposed in the textbooks are basically algorithmic in nature; and additionally, the use of rules to operate numbers are emphasized, as can be observed in the following vignettes taken from the first year mathematics textbook (Secretaría de Educación Pública, 2016, pp. 107, 117).

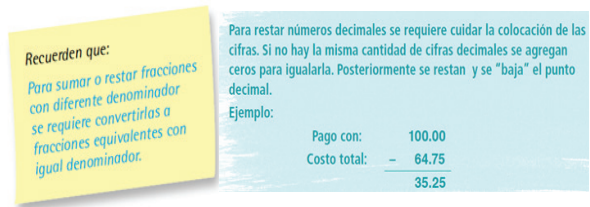


Figure 2. Rules and procedures highlighted in the textbooks.

However, even some tasks proposed in textbooks are framed in shopping contexts, the implementation of the tasks do not emphasize mental calculations in order to develop number sense, which differs from some international proposals (Ulu & Özdemir, 2018). Hence, the research participants' mathematics instruction was essentially based on algorithmic procedures to approach tasks.

Research Design

During the design of tasks we followed some suggestions of Clarke and Roche (2018), who argue that contextualized tasks should connect students' interests and experiences, to involve important mathematics, and to promote students' disposition for appreciating usefulness of mathematics to make sense of the world around them. Hence, "The Store" task included the organization of a classroom instructional setting that simulated a situation that students can encounter in their daily lives (shopping experiences). In the first stage of task implementation (two hours), the students bought and sold diverse products using fantasy money (action phase). This was done to promote the need of performing mental or written calculations to obtain amounts to pay and remaining money (observation phase). Students were asked to bring to school goods that were not used at home (clothes, shoes, music or movie CD's, USB flash drives, soccer balls, cell phones, etcetera), and they had to search for their commercial prices, assuming that products were new. The products were labeled with their prices and some with discounts identified through colored tags (Figure 2). We decided to use real objects and prices, following a suggestion of van den Heuvel-Panhuizen (2005), who proposes to avoid excessively simplified situations (a store in which the products are drawings) or situations unfamiliar for students; instead, the context was designed to be as realistic as possible.

The students were organized in two teams. Team A consisted of the store assistants and team B of the customers. Four students were appointed to be the cashiers of the store and four more were designated as warehouse managers. Each customer was given the amount of 2 000 "Mexican pesos" and each cashier was given 2 500 "Mexican pesos". The customers had to spend most of the money and all the participants had to record the transactions in control sheets. After 30 minutes, the students exchanged roles, from store assistants to customers and vice versa. The cashiers and the warehouse managers exchanged roles; new control sheets were given to each participant, as well as new amounts of money.



Figure 3. Students interacting in "The Store".

During task implementation, the main activity of the researcher, who oriented the students' activity (*instructor*), and who is also a teacher of the middle school, was to supervise, with the support from two colleagues; that the participants follow the indications. The instructor also promoted an environment that offered opportunities to reflect on, understand and communicate mental or written calculation strategies. The formulation and communication of the strategies, as well as the justification of their utility, were central in the process of developing mathematical understanding, according to the research framework. In a next stage of the task, the instructor formed four teams, combining students of each grade. The instructor asked students to indicate two products that they had purchased, and he wrote the prices on the blackboard (e.g. a pair of pants and a soccer ball, \$425 and \$215 pesos, respectively). Then, the instructor posed questions as the following: how much money are you left with? When you add or subtract numbers, should you always start by adding units and continue with tens and so on? What are the positional values of 5 in the number 575.50? If you have 42 pesos, how much money do you need to get 100 pesos? The instructor highlighted the use of mental calculation strategies since those strategies can enhance the discovering of relationships among quantities, including groupings, compensations or decompositions that are useful to transform initial quantities in a way that facilitate calculations (Reys R., Reys, McIntosh, Emanuelsson, Johansson, & Yang, 1999).

After the discussion inside the teams, a representative of each team went to the blackboard to explain a strategy of mental calculation for addition (phase of formulation of conjectures), the challenge was that the next player should present a strategy different from those already discussed. Each participant explained his own calculation strategy, commenting on its advantages and proposing some examples (phase of justification). The other teams' members expressed their doubts or identified some disadvantages of the presented strategy. The same dynamic was carried out for the other strategies.

In a third stage (two hours), it was organized a competition, with the aim to evaluate the permanence of the strategies. In each round, a player solved a contextualized problem and, for each correct result and explanation, the player received two points which were added to her/his team score. The competition stage was important to support mathematical understanding, since it generated the need to use personal strategies to perform arithmetic operations in flexible ways, with the aim of winning the competition. We decided to include a competition activity, since we agree with Anthony and Walshaw (2009) regarding that games could be a means for developing fluency, and that games provide appropriate

feedback and challenge for participants. On this line, Angileri (2006, p. 14) considers that "one of the most effective ways to encourage mental work and classroom discussion is through the use of games and puzzles, inside and outside the classroom".

The justification phase, based mainly on the communication of results, was carried out when the students discussed within the groups and later communicated the reason for using the strategies they proposed. In this phase, a central element of a socio-constructivist approach was implemented, since independently of the activity, time was allocated to analyze the different routes to obtain the answers, point out the range of possible strategies and highlight the most efficient or appropriate strategies (Qualifications and Curriculum Authority, 1999, p. 19).

Data Collection and Analysis

The data collection tools were video recordings of the session, which were subsequently transcribed, and field notes taken by one of the researchers. Based on the transcriptions, we identified the calculation strategies employed by the students and then analyzed these strategies with the aim to identify the strands of number sense immersed in them.

Results

In this section we describe the calculation strategies invented by the students and evaluate the permanence of the strategies through the competition stage. Also, we identify the strands of number sense that students showed, as means to determine the level of understanding regarding number sense developed by the students as a group. These data are the source to answer the research question: What strands of number sense are showed by students of a middle school, located in a rural community of the state of Hidalgo, Mexico, when they approach tasks that involve performing arithmetic operations framed in a context of shopping and a ludic classroom environment?

Mental and Written Calculation Strategies

Students were encouraged to make payments, verifying the change that they got back after buying certain products, reviewing products in stock and taking advantage of the discounts offered. "The Store" task helped students to formulate mathematical and extra-mathematical questions such as, how much money can a low income farming family spend on the purchase of goods and services? How could we set up a bazaar to gain some money? During the development of the task some students who were responsible of a teller had problems to determine the amount of money that they should deliver to a customer. In these cases, other students provided support in the form of suggestions to solve the problem. One of such suggestion was to count forwards starting from the smaller number. Students commented that this strategy is commonly used by some of their grandparents (illiterate individuals), for convincing that they receive the correct exchange when they buy products in the market.

Customer: Please charge me 238 pesos [He gives 250 pesos to the cashier]

Cashier: Um... do I charge you 238? [He hesitates about how much money he should give back to the customer]

Student: [taking the money from the box, he explains how to give the change], plus one peso, there are 239, plus another peso, there are 240 and with this 10 pesos coin we have 250 [putting the coins on the table]

Cashier: [Reflecting a little bit he takes the coins and delivers them to the customer] it's ok; here you have your change [twelve pesos].

Students proposed four main strategies for addition and four for subtraction. Some of these strategies are already reported in the literature concerning mental calculation

(Anghileri, 2006), and others are novel (Table 2, 3 and 4). Most of the strategies were mental calculation strategies, while others made a skillfully use of fantasy bills as a representation system to decompose quantities in nonstandard ways. Another strategy consisted in decomposing the minuend in a subtraction, before applying the standard written algorithm, employing ideas of mental calculation. All the strategies were invented by the students, without the instructor having previously enunciated or exemplified them. We observed that students were dealing with decimal numbers in a holistic way when executed mental operations, and then decimal number and arithmetic operations became meaningful for the students, unlike when they employ standard written algorithms, since in the latter case, they operate on separate digits without a reflection regarding positional value of numbers.

Table 2. Strategies for addition.

Strategy	Example
1. Adding hundreds, tens and units separately, and then adding partial results (addition by parts).	Instructor: You bought a blouse that costs 315 pesos and one pair of pants that costs 285 pesos, how much did you pay for those items? Students: What I did was to start from the bigger quantities 300 plus 200 which give 500. Now I continue with the tens. Then, there are 10 [units] and 8 tens, or 80 units, I would get 90. And finally, the units, 5 units and 5 units would give 10 units, and I added all together... 90 plus 10 gives 100; plus 500, gives 600.
2. Addition by splitting one addend.	Instructor: You bought a pair of pants that costs 318 pesos and one blouse that costs 224, how much did you pay for both products? Student: I added 318 and 200, and obtained 518. Now to those 518, I added 24. First I added 20 and then added 4... it would be ... 538, it would be... 542.
3. Rounding the addends up to tens.	Instructor: We bought one pair of pants that costs 318 pesos and one blouse that costs 224 [pesos], how much did you pay for both products? Student: First of all, I completed to tens... [The student wrote $318+2$ and $224+6$]. Then, 320 plus 230 [he writes the new addends and perform the addition following the standard written algorithm] the result is 550. From this result I subtracted eight [the students pointed the +2 and +6 that he summed to the initial addends] and obtained 542.
4. Rounding the addends down to tens.	Instructor: A Jacket is going to be bought whose cost is 443 and a pair of ladies shoes whose cost is 512, how much is it going to be paid? Student: here I subtracted 3 and then [I subtracted] 2...this... to get 440 left and 510. Here I put 440 from that and from here 510 from where... I added and I got 950, after I added 3 and 2 [he pointed out -3 and -2] and I get 5 and from here I got 955.

Source: Prepared by the authors based on video transcriptions.

Three students (Memo, Julio and Angy) showed some difficulties to understand the idea of positional value of numbers. They tried to combine representations that are used to execute the written addition algorithm, with ideas from mental calculations. For example, to add the numbers 318 and 236, they added the hundreds correctly, but when they added the tens, only considered the absolute value $1 + 3$ (instead of $10 + 30$). In addition, it was identified that when placed the "new" addends, they did not take into account the place value of the numbers (Figure 4). By contrast, some other students were able to modify the quantities, using mental calculation strategies, before applying the standard algorithm with the aim to simplify calculations, as in the case of the fourth strategy for

subtraction (Table 3, Figure 5). We noted a trend of some students to prefer standard written algorithms since they consider that these procedures provide reliable results regarding mental calculation strategies.

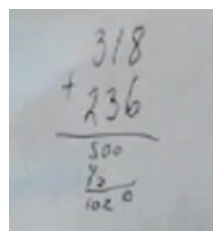


Figure 4. Error committed in a written strategy for addition

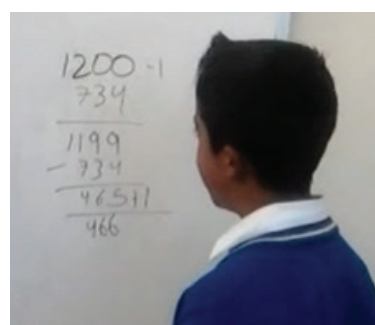


Figure 5. Modifying the written standard algorithm.

Table 3. Strategies for subtraction.

Strategy	Example
1. Subtraction by decomposing the subtrahend.	Instructor: When Antonia arrived to the Store she had 900 pesos. She paid 587 for some products she bought, how much did she have left? Student: Well... to the hundreds that are 900 I will subtract 500 and I have 400 left, then I take away 80 which are the tens and I have 320 left, finally I take the units away which are 7 and I have 313 left.
2. Counting forwards from the smaller number (reported in Anghileri, 2006).	Instructor: I arrived to the store with 600 pesos and I have to pay 302 pesos, how much would I have left? Student: I add 8 [to 302] and I get 310, to 310 I add 90 and I get 400. [Finally] I add 200 [to 400] and I get 600, then I add this [points out the complements +8,+90 and +200], and I got 298
3. Fantasy money used as a representation system	Instructor: If you have 1200 and you pay 715 pesos in the store, how much do you have left? Student: For paying 715 I would use one 500 bill, one 200 bill and one more of 100... and it would add up to 800 and I would keep 400 out of 1200...and first I subtracted 700 from 800 and I have 100 pesos left, and from those 100 pesos I subtracted 15 and I would have... 85 left. Then from 800 I subtract 715 and I would have 85 left and I add 400 to that I have from the 1200 and I would be given 485 of change.
4. Change of subtraction with regrouping to subtraction without regrouping adjusting, units, tens and hundreds	Instructor: I arrived to the store with 1200 pesos and I have to pay 734 pesos, How much do I have left? Student: Ehh... Well, I subtracted 1 to 1200 obtaining 1199, then I subtracted 734 to 1199 and I obtained 465, after that I added 1 which I had subtracted above and I obtained 466.

Table 4. Strategies for multiplication.

Strategy	Example
1. Multiplication as a repeated addition	Student 1: Finding the price of two shirts is equal to multiplying by 2 the price of a single shirt; that is, multiplying by 2 is equivalent to finding twice the quantity: two hundred and fifteen times two equals to two hundred and fifteen plus two hundred and fifteen, equals four hundred and thirty Student 2: Finding the amount of four shirts, is equal to multiplying by 4 the price of an individual shirt; that is, it is equivalent to finding twice the double of the individual price: $215 \times 4 = (215 + 215) \times 2 = 430 + 430 = 860$.
2. Multiplication by 10 adding a zero or moving a zero or moving the decimal point one place to the right, one place to the left	Student: To multiply a number by 10, I add a zero or I move the decimal point one place to the right.
3. Multiplication by 5, halving the product 215×10 .	Student: Finding the amount to pay by five shirts is equal to multiplying by 5 the price of one shirt; that is, it consists of multiplying by 10 and then calculating the half, example: $215 \times 5 = 2150 \div 2 = 1075$.

Source: Prepared by the authors based on video transcriptions.

In the competition stage (Table 5), it was observed that the more common strategies for multiplication were to multiply by ten, as well as doubling and halving quantities. It is particularly important to notice that students made sense of the distributive property as well as the decimal numbers such as 0.5 when they related this quantity with the process of halving.

Instructor: In the following multiplication 128×7.5 , tell me, what did you do?

Student: First, I multiplied seven by one hundred, that gave 700, then [seven] by 20, are 140; and then [seven] by 8, are 56, which results 896. Then a half of 128 is 64, and added 896 and 64, and obtained 960.

Table 5. Strategies employed in the competition stage.

Problem statement	Example
A kilogram of sugar costs 12 pesos, how much would we pay for twelve and a half kilograms?	Student: Well...I multiplied 12 times 10, and gives 120. Then I multiplied 12 by 2 and gives me 24 and finally, point five we know is a half, then a half of 12 is 6. Instructor: Half of a kilogram? Student: Yes ... And I added everything and the sum is 150. Instructor: Sure? How did you add? Did you add 4 plus 6 and you carried out one [ten]? Student: No ... 120 plus 24 gives me 144, plus 6 ... equals 150.
Imagine that you have 200 pesos, and you buy 5 kilograms of apples and 4 kilograms of bananas. The kilogram of apples costs 17 pesos and the kilogram of bananas costs 16.50. How much money do you have left?	Student: Here ... We put a zero [he points the number 17] ...because it is 10 kilos and you have to take out the 5, then here, half of 170 are 185. [She writes that amount on the blackboard]. Then, here [the bananas] its price is 16.50, twice of this is 33, plus 33 is 66. Then, here I add what I paid for the 5 kilos of apples and the 4 kilos of bananas. 80 plus 60 are... 110, 120, 130...no! [Correcting] 140! Plus 5 [divides the 6 into 5 + 1] are 150, plus 1 ... 151, then here I had 200 pesos to buy this, so here I have left ... 49.
A person goes to the market and takes 200 pesos, she buys 4 kilos of bananas at 13.50 pesos each and 5 kilos of apples, at 19 pesos each, how much money will she have left?	Student: well ... we divide the price [points to 13.50] and we only have 10, then 4 times 10 is 40, and 4 times 3 is 12 and 40 plus 12 is 52, plus half of 4; there are 54 left. Then, we also divide this other price [notes 19], we remove the 9 and I have 10, then 5 by 10 are 50 and 9 by 5 are 45, then 45 plus 50 are 95. Then we just add these two prices, which gives us ... 90 plus 50 gives 140 and 5 plus 4 are 9, then I have 149. Then I pay with one of 200 and I have 149. [Write +1 in front of 149] plus 1 is 150, we have 150 for 200 I am missing 50, then it's 51.

Source: Prepared by the authors based on video transcriptions.

The mental calculation strategy for addition, most employed, was the addition by splitting one addend (second strategy, Table 2). For example, when they added $48 + 64$, they first added 48 plus 60, which gives 108, then added 4, obtaining 112. In the case of subtraction, the most used strategy was counting forwards from the smaller number (Table 3). For example, the subtraction $50-17$ was carried out by first adding 3 to 17 to complete 20, and then adding 30 to obtain 50, finally, the solution was $3 + 30$ equal to 33. This also could be interpreted as the use of the commutative and associative properties of the sum, by translating to: $50-17+3-3=50-(17+3-3)=50-30+3=33$.

In the case of multiplication, the most used strategies were, multiplication by 4 (double the double of a quantity), multiplication by 10 (add a zero to the right of the quantity) and multiplication by 1.5 (sum the quantity plus its half). To multiply a number by 5, students multiplied the number by 10 and then halved the previous result. For example, to calculate 25×5 , first multiplied 25 by 10, equal to 250 and then calculating a half of 250 (half of 200 plus half of 50) $100 + 25 = 125$. It is interesting to note that students that showed better achievements in the activities identified flexible ways to transfer strategies for multiplying by 25, and then to transfer it to multiplication by 2.5. For example, $32 \times 25 = 800$ and $35 \times 2.5 = 80$ or $30 \times 25 = 750$ and $30 \times 2.5 = 75$. The multiplications by 12 and by 25, were those that required more time to perform, since in the case of multiplication by 12, students had to do two multiplications (first by 10 and then by two) and finally to compute the sum of the partial results. It is important to note that students were using the distributive property of the product respect to the sum. In the multiplication by 25, they multiplied by 10, then found the double (multiplying by 20), then calculated a half of the first product (multiplication by 5), and added both results (use of the associative and distributive property). An important aspect that stand out in this competition stage is that the students applied a great diversity of strategies of mental calculations, however, some of them (Alex, Fany, Luz and Fer) preferred to use written algorithms, since this gave them confidence about the obtained results.

Strands of Number Sense Shown by the Students

We identified that the tasks allowed students to show several strands of number sense. Students invented their own strategies to perform arithmetical calculations, some of which were mental calculation strategies; others were strategies that used the fantasy money as a representation system to decompose quantities in novel ways. During the use of these strategies, the six strands of number sense appeared. Most of the students exhibited number sense abilities during mental calculations; however, some others had difficulties to perform written calculations when they tried to combine representations used in the written algorithms, with ideas derived from mental calculations (Figure 4).

The "Store" task allowed students to show number sense strands through the strategies that they invented, for example, the meaning of decimal numbers in the strategies for multiplication. Particularly, in the subtraction strategies, students showed flexibility to operate with whole numbers and developed creativity to manage quantities, for instance, some students were able to modify standard written algorithms, as it is shown in the fourth subtraction strategy (Table 3). It is important to point out that fantasy money played an important role as a representation system that helped students to decompose quantities in novel ways. Regarding multiplication strategies, students showed all strands of number sense.

Table 6. Components of number sense included in calculation strategies.

Strategy	Components of number sense
1. Adding hundreds, tens and units separately, and then adding partial results (addition by parts).	The two strategies reflect an understanding of the meaning and size of the numbers, the evidence of this appears at breaking down the addends. In both strategies it is necessary to identify an order between the digits of the addends (e.g. the hundreds are greater than the tens and the units). The strategy implementation requires the use of equivalent expressions for numbers, as well as the use of the associative and commutative properties of the sum, which is an indicator of fluency and understanding of the effect of operations.
2. Addition by splitting one addend.	
3. Rounding the addends up to tens.	The implementation of these strategies requires understanding of the meaning and size of the numbers, as well as the use of benchmarks, incorporated in the activity of rounding the addends. These strategies also require the use of the associative and commutative properties of the sum and, implicitly, the property of the additive identity, which is an indicator of an understanding of the effect of the operations, and of fluidity.
4. Rounding the addends down to tens.	
5. Subtraction by decomposing the subtrahend.	The use of the strategy reflects an understanding of the meaning and size of the numbers, the evidence of this appears in the activity of decompose the subtrahend, and also indicates awareness of order between numbers and understanding of equivalent representations.
6. Counting forwards from the smaller number (reported in Anghileri, 2006).	By applying this strategy, students show an understanding of the meaning and size of numbers, and the use of benchmarks when performing calculations. Also there is evidence of equivalence of operations.
7. Fantasy money used as a representation system	The previous strategy demonstrates that students understand the meaning and size of numbers, such evidence appears at decompose a quantity in novel ways. The foregoing strategy is also indicative of the use of reference points.
8. Change of subtraction with regrouping to subtraction without regrouping adjusting, units, tens and hundreds	Employing this strategy requires the use of the associative and commutative properties of the sum, and implicitly uses the property of the additive neutral, adding and then subtracting the same amount, which is an indicator of an understanding of the effect of the operations, as well as the presence of fluidity. There is also an understanding of the subtraction algorithm on the problems involved when doing a subtraction that requires regrouping. It also takes advantage of the properties of the operations to facilitate the execution of operations with paper and pencil.
9. Multiplication as a repeated addition.	By applying this strategy, students show an understanding of the meaning and properties of the operations.
10. Multiplying by ten and then doubling and halving.	Employing this strategy requires the use of the associative and distributive property, as well as the use of benchmarks when performing calculations. The use of this strategy also involved the construction of meaning for decimal numbers.

Source: Prepared by the authors.

Discussion and Conclusions

The findings of this study support the fact that contextualized tasks could help students to develop number sense. We recognize that the implementation of a one task is not enough for students' to develop number sense,

since "number sense develops gradually" (Yilmaz, 2017, p. 892). However, we were able to identify which aspects of number sense emerge when students approach this type of tasks. The strands of number sense observed were immersed in the calculation strategies that the students invented, which could be the starting point to guide the design of didactic sequences that could allow students to develop number sense. During the students' engagement into a contextualized task, they were immersed in a problem solving environment (Polya, 1945), in which they used their previous knowledge to approach and solve dilemmas. Particularly, this environment supported the use of calculation strategies that correspond to the way students naturally think about numbers (Anghileri, 2006). The fact that the task has favored students' invention of their own strategies also supported the teacher's activity, since it is not a problem for the teacher to decide exactly which strategies should be taught, and in what order (Beishuizen & Anghileri, 1998), since students decided to employ the strategies they considered useful. That is, contextual tasks are a resource to help students gain both confidence and competence in doing mathematics (Burns, 1994). The invention and use of calculation strategies is an indicator of understanding and the developing of number sense, particularly when students transfer the strategies "from one task to another, even when the external cues are different" (Boaler, 1993, p. 17). This transfer is an indicator of understanding, since it shows that students can connect "the underlying processes which link the problem requirements and their significance in relation to each other" (Boaler, 1993, p. 17).

We observed a trend among some students' work to rely on algorithmic procedures. This behavior must be due, as was previously mentioned, to the fact that algorithms "provide reliable ways to compute and, therefore, to simplify potentially difficult calculations" (Burns, 1994, p. 473). The mentioned phenomenon contrasts with the evidence obtained in other studies, where students tended to use their own methods, although they have already been taught the standard algorithms (Anghileri, 2006). Standard algorithms have evolved over centuries to become efficient and accurate, but these algorithms are quite far from their conceptual underpinnings. Invented strategies, on the other hand, are generally derived directly from the underlying multidigit concepts. For example, some addition strategies were based on adding 30 and 20 or 3 tens and 2 tens rather than adding two numbers that appear in the same column (Carpenter, Franke, Jacobs, Fennema, & Empson, 1997, p. 5). The standard written algorithms are constructive, efficient, automatic, symbolic, analytic and generalizable; however, those methods are not easily grasped by students because of a lack of correspondence with informal methods. Mental methods, on the other hand, are fleeting, variable, flexible, active, holistic and constructive, and before all, those methods consider personal approaches for which students have developed ownership (Plunket, 1979).

The performance showed by the students when solving the arithmetic problems that appeared during the shopping activities is also an evidence of adaptive expertise, since they were able to invent new procedures, which is an evidence of a robust conceptual understanding (Marcovits & Sowder, 1994). However, it was also observed that some students committed mistakes when they tried to applied some strategies of mental calculations to written algorithms, since in this last scenario students tend to dissociate symbols from their referents when symbol manipulation is emphasized (Wearne, 1990). Hence, contextualized tasks provide opportunities to students to make sense of operations with numbers, as it is shown through the strategies we are reporting in this paper. Particularly,

students dealt with decimal numbers in a holistic way when executing mental operations, particularly when they operated with number whose decimal part is 0.5, it was interpreted by them as a halving procedure. This result contrasts with the fact that students treat a decimal number as if it were composed of two numbers separated by a period when they perform written algorithms (Steinle, 2004).

The decisions which teachers make when choosing tasks to support students' understanding are critical. Carefully chosen contextualized tasks can assist students to make connections between mathematics and its applications, and to see how mathematics can help to make sense of the world (Clarke & Roche, 2018). The organization of the activities with the participation of students of the three grades favored solidarity work and tutoring among peers. The context of the game and competition promoted that the students with more experience in performing calculations, provided support and advice to their peers who showed some difficulties, which favored the team performance in the competition phase. There were some students (Jorge, David, Alondra), who had been considered by the instructor as having poor academic achievements, however, their performance in the contextualized task allowed them to receive recognition from their classmates for their abilities to perform mental calculation. This was reflected in their election as team representatives in the competition stage. The recognition they received motivated them to take the tasks seriously and increased their participation in class. From this, we obtained evidence that a task, contextualized as a game, can be a useful strategy to encourage students' self-confidence regarding their ability to learn mathematics. It was also evidenced that the recognition of academic achievement may be the best stimulus to awaken the interest of those students considered "underperforming". Game activities allowed students to modify their conceptions towards mathematics. They generally perceive mathematics as a rigid and arid subject in the applications of rules and procedures. However, a contextualized task, with ludic components, was a platform to use the natural tendency of students to form teams, play and learn. This approach has allowed students to establish connections between the arithmetic of the classroom and the one that takes place in a context of daily life. In other words, they have grasped important elements to achieve learning with understanding.

The fantasy money employed during "The Store" task, turned out into a useful representation system (Duval, 2006) to perform decomposition of quantities in novel ways, supporting the generation of mental calculation strategies based on extracurricular knowledge, which generally do not appear in the classrooms. We argue that using representations with objects (fantasy money), along with didactic planning, can help students to establish connections between school mathematics and their experiences outside the classroom. The realization of mental calculations allowed students not only to develop comprehension regarding different aspects of number sense, but also to realize that a problem can be solved in different ways, and realize the importance to communicate their ideas to others and justify their solution routes. The standard curriculum ignores, or at least does not emphasize, the use of informal knowledge that students have developed (Mulligan & Mitchelmore, 1997), despite the fact that researchers such as Ausubel (2000) argue that prior knowledge is the base or anchor for the development of new meanings.

This work provides evidences that students' informal or extracurricular knowledge can be very useful to develop mathematical understanding, as well as being a starting point to generate novel mental calculation strategies or new procedures to perform calculations with written algorithms. The participants recognized that conventional algorithms are not the only way to perform calculations and linked school math-

ematics with those of everyday life. The mental calculation strategies allowed students to make sense of operations with decimal numbers, avoiding some confusion that arises with conventional algorithms.

We identified that the discussion of the strategies brought into practice by the students, allowed the instructor characterize the ways of thinking and reasoning that arose during the activities. The analysis of the ideas expressed by the students, leads to conclude that the teacher must possess skills to help students to verbally express their thoughts in a clear manner, so that the rest of their classmates understand their ideas. This characteristic and ability of the teacher has also been identified as relevant in other investigations (Swan & Sparrow, 2001). During the implementation of mental calculation strategies for addition and subtraction operations, students continuously showed preferences for the use of conventional algorithms. This could be due to the fact that a large part of their learning, related to basic operations, has focused on the development of algorithmic tasks, which has led to a nonsensical automatism in the handling of quantities, which is observed in a wide range of erroneous responses derived from the incorrect implementation of standard algorithms (Segura, 2015). The impeccable implementation of an algorithm does not guarantee that the executor has number sense, in contrast, the development of one own strategies to perform calculation is an evidence of this way of thinking (McIntosh, Reys, & Reys, 1992).

Finally, we want to point out that contextualized tasks favor the creativity of students, and help them to adapt the conventional algorithms to their necessities. In the development of the activity, it was possible to identify a transfer of knowledge and processes when mental calculations are performed, which uses paper and pencil, since the instructional context, generate a need for students to use different decompositions of quantities, unlike what happens when applying standard algorithms. The foregoing is an indicator of an understanding of quantities and arithmetic operations, as well as a context to connect inverse operations. We consider that mental calculation strategies make sense for students when those methods are the result of a need that emerges from a real world situation. Context problems have a high potential to develop fundamental elements of mathematical thinking and for that reason are extremely important to enhance mathematical learning with understanding.

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