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Designing a research-based detection test for eliciting students' prior understanding on proportional reasoning

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Abstract

In the Swedish Prison Education Program only two out of ten reach a passing grade in their mathematics courses. Large variation in prior knowledge makes it difficult to meet the students at their level. This paper reports on a project aiming to enhance students' possibilities to access mathematics through individualization. Research findings on the development of the pervasive mathematical idea of proportional reasoning are used to construct a test on proportional reasoning, designed to work specifically with students with large variation in prior knowledge. The test presented here, combined with a follow-up clinical interview, can be used in adult education in general as a basis for individualizing instruction.

Key words: proportional reasoning; adult education; prison; individualized instruction

Introduction

Already in 1968, Freudenthal drew attention to the inequity of an unbalanced mathematics education system that provides mathematics education to only a small group of citizens who "fit" within the system. He proposed that instead of worrying that the highest level may be too low for the gifted students, we should strive for educational justice for all people in society. Educational injustice is particularly prominent in mathematics (Stinson, 2004). The illusion that mathematics is only for people with special gifts has lived on for at least 2300 years, as illustrated by Plato's claim that "We shall persuade those who are to perform high functions in the city to undertake calculation, but not as amateurs." (Plato, trans. 1996, p. 219). In our modern society, mathematics acts as a key for passing through the gates to economic access, full citizenship, and higher education (Stinson, 2004). This gatekeeping function makes equitable access to mathematics education a question of democracy. To enhance democracy, mathematics needs to be an inclusive instrument for empowerment rather than an exclusive instrument for stratification. Instead, in most school systems, some students become disqualified from mathematics while others qualify (Jurdak et al., 2016; Popkewitz, 2002). When these disqualified students return to mathematics as adults we cannot expect success by repeating educational practices that did not work in their past. Therefore, an equitable access to mathematics education for such a disadvantaged group needs to acknowledge not only prior knowledge, but also life conditions alongside personal study narratives.

The Swedish prison education program is an educational environment where students with without an upper secondary diploma cluster. Close to 50 % of the prisoners lacks an upper secondary diploma (Riksrevsionen, 2015). In the prison education program a second chance to obtain this diploma is provided. Nevertheless, only two out of ten students pass their mathematics courses. This is disturbing in itself, but particularly in light of the resources available. The teachers are university trained upper secondary school mathematics teachers and the students sign up voluntarily. These conditions should ensure a high potential for effectiveness. However, the student group shows significant variation in age, ethnicity, socioeconomic and school backgrounds and life experience in general. These factors may account for such low pass rates, and justifies, a plausible reason to focus on such individualized differences.

It is an unchallenged claim that teaching accounts for an individual's prior knowledge, his [1] life conditions and his rationale for studying mathematics have better prospects to succeed than a “one size fits all” teaching practice. When teaching adults returning to mathematics, such considerations are of particular importance. Given that a teaching approach aiming at individualized instruction [2] requires information on the individual's prior knowledge, his life conditions and his rationale for studying mathematics, teachers need strategies to collect the necessary data. This paper focuses on a project aiming to elicit one piece of information: the students’ prior knowledge of mathematics. The setting is the special context of adults studying mathematics in prison. While students in lower- and upper secondary school can be screened on the content taught last year, as a means to provide a picture of the current state, adults with many years away from school possess far more unpredictable knowledge. Therefore, adults returning to mathematics after several years away from formal schooling present a particular challenge. A prerequisite for individualization of courses is that teachers have the opportunity to find out students' prior mathematical experience and their mathematical competencies and adapt their teaching accordingly. Realizing this opportunity hinges on the teachers’ competencies, e.g., they need to action their teaching competency more effectively (Niss & Højgaard, 2011). Further, it is reasonable to believe that teachers’ practical ability to investigate students’ prior knowledge increase if they are provided with support. They may need a test to investigate the students’ prior knowledge, to support teachers’ individualization process.

The test needs to fulfill certain properties. It needs to give information on students’ prior knowledge both vertically, in relation to progress throughout school years, and horizontally throughout taught topics in compulsory school. A full investigation would be far too time-consuming, but since proportionality may be the most important, pervasive and powerful idea in elementary school mathematics (Behr et al., 1992; Hilton, Hilton, Dole, & Goos, 2013; Karplus, Pulos, & Stage, 1983; Lamon, 2007; Sowder et al., 1998), proportional reasoning was chosen as the basis for a test combined with a follow-up clinical interview (Ginsburg, 1981). Furthermore, the test has to give information that discriminates among students. Discovering that all students suffer from the same lack of reasoning skills will not help teachers to individualize instruction. We want to discover students’ difficulties as well as be provided with information on how students tackle known thresholds for the development of proportional reasoning. By implementing results from the extensive research on the topic, we can relate students’ results to the learning goals and core content in the Swedish national curricula. Therefore, students’ ability to reason proportionally can provide an access point for individualized instruction. The aim of this study is to investigate if it is possible to design a test that together with a clinical interview gives information on students’ prior knowledge on proportional reasoning so as:

- discriminate among students and,
- provide teachers with an access point for designing individualized instruction.

Theoretical underpinnings for the development of a test

Mathematical reasoning is essential for doing mathematics. It is one of eight competencies for identifying and analyzing students’ mathematical knowledge, described in the Danish KOM-project (Niss & Højgaard, 2011). “The mathematical reasoning competency consists, first, of the ability to follow and assess mathematical reasoning, i.e., a chain of arguments put forward – orally or in writing – in support of a claim.” (Jankvist & Niss, 2015, p. 264). The kind of mathematical reasoning called proportional reasoning is a prerequisite for successful further studies in mathematics and science, since multiplicative relations underpin almost all number-related concepts studied in elementary school (Behr et al., 1992; Lamon, 2007). A proportion is defined as a statement of equality of two ratios $a/b = c/d$. Proportion can also be defined as a function with the isomorphic properties $f(x+y) = f(x) + f(y)$ and $f(ax) = af(x)$ (Vergnaud, 2009). A function, $A(x,y)$, can also be linear with respect to several variables, (n-linear) functions. For example the area functions for a rectangle with sides x and y is bilinear (2-linear) since $A(x,y) = xy$ and it is easy to check that this function is linear with respect to each of its variables when the other is considered constant.

Proportionality is a key concept in mathematics education from elementary school to university (Lamon, 2007). Despite the pervasive nature of proportional reasoning throughout the school years it is well known that children around the world have considerable difficulty in developing the mathematical competency to reason about fractions, percentages, ratio, proportion, scaling, rates, similarity, trigonometry, and rates of change (Behr, Harel, Post, & Lesh, 1992; Lamon, 2007). Typically, proportional reasoning problems come in the shape of a missing value problems or comparison problems (Lamon, 2007). In the former, a multiplicative relation is present where three elements are provided and the fourth is to be found. The latter asks the student to compare which ratio is the bigger or smaller.

Key points for the development of proportional reasoning and the building of multiplicative structures can be identified from the accumulated research (c.f. Behr, Harel, Post, & Lesh, 1992; Fernández et al., 2012; Lamon, 2007; Shield & Dole, 2013; Van Dooren, De Bock, Vleugels, & Verschaffel, 2010; Vergnaud, 1983). To illustrate the conceptual structure of the proportionality concept, four central key points are presented below.

Key point 1. Students need to be able to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present. The ability to distinguish additive from multiplicative comparisons constitute a major stumbling block for students (Van Dooren et al., 2005). Students need to be able to recognize that a proportional situation exists when the comparison is multiplicative (Shield & Dole, 2013). In Sweden, students get acquainted with additive strategies for reasoning about quantities in grades 4 to 6. For example, an increase in price by 10% can be calculated in two steps. First, calculate how much 10% is and then add this to the original price. A transition from an additive to multiplicative thinking approach is introduced in grades 7 to 9. The new price can now be approached in one multiplicative step: the original price multiplied by the factor 1.1, to find the new price. The error of using additive strategies on proportional situations increases during primary school and decreases during secondary school (Fernández et al., 2012). A desirable development in students' reasoning would be that they, after being introduced to multiplicative reasoning, still hold on to their ability to use additive strategies when appropriate. However, research findings show that once students have been introduced to multiplicative strategies they tend to overuse this approach on everything that resembles a proportional situation (Van Dooren et al., 2005). Further, non-integer ratios cause more errors than integer ratios (Fernandez et al., 2012; Gläser & Riegler, 2015), while the non-integer situations can be considered to require a more developed knowledge of rational numbers.

Key point 2. Students need to be able to draw connections to the algebraic rules for fractions when working with part/part ratios, part/whole fractions and proportions, $a:b = c:d$. Many situations require that students can relate to part/part ratios and part/whole fractions (Vergnaud, 1983). For example, if a company employs 11 women and 31 men, the part/whole fractions 11/42 and 31/42 represent the relation of women and men related to the whole. If asked to determine the company's gender distribution, it is instead the part/part ratio 11:31 between women and men that is relevant. When a ratio connects two parts of the same whole, students may not adequately recognize the difference between part/part and part/whole relationships (Clark, Berenson, & Cavey, 2003). It is not easy for students to approach situations that require shifting from part/part to part/whole situations. Moreover, students need to connect mathematical ideas. Since ratios can be written in fraction form, they obey the same mathematical laws as fractions (Shield & Dole, 2002).

Key point 3. Students need to recognize and use a range of concrete representations for proportions, e.g., tables, graphs, formulas and drawing pictures. Students tend to apply linear proportional reasoning on scaling in two- and three dimensions, without considering the nature of the item. Van Dooren et al. (2010) found that students tend to use linear proportional reasoning even when it is inappropriate e.g., in word problems where a real word context is required to solve the problem. For example: Farmer Gus needs 8 hours to fertilize a square pasture with sides of 200 meters. Approximately how much time will he need to fertilize a square pasture with sides of 600 meters? Recognizing this as a missing value problem i.e., three values given and one unknown, this problem will trigger a cross-multiplication type solution which gives the wrong answer of 24 hours. Since scale is one of the major themes that span mathematics, chemistry, physics, earth/space science and biology it is crucial for students to gain knowledge of the concept of scale. Scale in one, two, and three dimensions is a central unifying concept that crosses the science domains, crucial for understanding science phenomena (Taylor & Jones, 2009).

Key point 4. Students need to acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity. Proportionalities can be represented in different ways, e.g., with words, pictures, algebraically, with graphs or tables. Shield and Dole (2013) enhance the use of a range of representations to promote students' learning. If students are given the opportunity to work with graphs, tables and other diagrams that illustrate the proportional situation present in the mathematical task, their conceptual knowledge is promoted (Vergnaud, 2009). Further, their ability to see connections between problems that are based on the same mathematical idea is enhanced, e.g. to see that missing value problems on similarity, proportional functions and speed problems can be illustrated with different representations but approached with the same mathematical idea.

Several concepts are in play when students reason with proportional quantities. The set of intertwined concepts required for the development of proportional reasoning constitutes a *conceptual field* (Vergnaud, 2009). A conceptual field is a set of situations and concepts tied together. As the theory of conceptual fields show, together with other well-known theoretical frameworks for conceptual knowledge, the meaning of a single concept does not come from one situation only (Sfard, 1991; Tall & Vinner, 1981; Vergnaud, 2009) but from a variety of situations demanding mathematical reasoning related to the concept in question. The conceptual field of intertwined concepts in play in proportional reasoning cover at the least "linear and *n*-linear functions, vector spaces, dimensional analysis, fraction, ratio, rate, rational number, and multiplication and division" (Vergnaud, 1983, p. 141). It is the complexity of the concepts in play together with the pervasive nature of proportional reasoning from elementary school to university that makes proportional reasoning suitable for the design of a test.

Methodology

The test was developed and tried out in the context of the Swedish prison education program, where approximately 650 adult students take mathematics courses each year. The vast majority, 80 %, study basic courses where proportional reasoning spans across all core content in the national mathematics curricula topics: *Understanding and use of numbers, Algebra, Geometry, Probability and statistics, Relationship and changes* [3]. The content given in the national curricula for school years 3, 6 and 9 indicate a clear progression of proportional reasoning. And correspondingly, an analysis of the national assessment given in grade 9 in the year 2013 showed that 69% of the items could be solved with applying proportional reasoning. The basic mathematics course for adults, corresponding to the 9 years of Swedish elementary school, is divided into four parts. By the end of part four the students are supposed to master: methods for calculating with rational numbers, fractions and ratios; the concept of a variable; formulas; methods for equation solving; scale in one, two and three dimensions; similarity; and uniform probability. When students pass the fourth part they get a grade for the entire course. Thus opportunities for individualization open up. Depending on students' prior knowledge they can start on different levels in the elementary course. Hence, this opportunity supports the need for a test that discriminates among students' and provides teachers with an access point for designing individualized instruction.

An important design choice for the test was to use a multiple-choice design. Even though open response tests are a powerful method to elicit students' knowledge, the advantages of multiple-choice tests were in this case considered to be the best option. An open response test can be a negative experience for students with low prior knowledge, since they may be unable to supply any answers. If possible, we want to avoid negative experiences in the beginning of a mathematics course. A multiple-choice test is easy to take for the students. Even when they do not have the mathematical competencies to reason and solve an item, they can still provide an answer by intuition or chance. Since multiple-choice tests "may be designed so that wrong answers clearly indicate well-known misconceptions or specific incorrect strategies of solution" (Ginsburg, 2009, p. 113), common errors, known from accumulated prior research and experience, are incorporated among the answer alternatives.

The items in the test were chosen from published research papers, with the intention to draw on knowledge from the research field on proportional reasoning. The rationale for my choices is as follows: a) the items have already been proved to work well for giving information on students' knowledge, and b) extensive background information of the nature of the mathematical reasoning in play are provided as well as analyzes of students results. Referring to the key points presented in the theory section, the potential reasoning related to each item involves several concepts and competencies, yet the items can still be categorized as referring mainly to one of the four presented key points:

- *Key point 1*. Students' ability to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present, is always required for carrying out proportional reasoning, however is mainly tested by items 1, 5, 6, 7 and 11 (see Appendix).
- *Key point 2*. Students' ability to draw connections to the algebraic rules for fractions when working on part/part ratios, part/whole fractions and proportions, a:b = c:d, is mainly tested by items 2, 4, 12 and 16.
- *Key point 3*. Students' ability to recognize and use a range of representations for proportions, e.g., tables, graphs, formulas, and pictures is mainly tested by items 3, 9 and 15.
- *Key point 4*. Students' ability to acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity is mainly tested by items 8, 10, 13 and 14.

Several errors on items referring to the same key point indicate a lack of knowledge that should be investigated further in the follow-up clinical interview. The test items are also adapted to mirror the progression throughout the basic course. Items 1 and 4 refer to content taught in part two of the elementary course. Items 2, 3, 6 and 10 deal with content from part three, and part four is reflected in items 7, 8, 9 and 11-16 (see table 1).

Table 1. Items in relation to key points and progression from part 2 to 4 in the elementary course.

	Distinguish additive from multiplicative reasoning	Draw connections to the algebraic rules for fractions	Recognize and use a range of concrete representations	Acknowledge the properties of geometrical objects
Course part two	Item 1	Item 4		
Course part three	Item 5 Item 6	Item 2	Item 3	Item 10
Course part four	Item 7 Item 11	Item 12 Item 16	Item 9 Item 15	Item 8 Item 13 Item 14

The sources for the test items are: Gläser and Riegler (2015); Fernadéz et al. (2012); Niss and Jankvist (2013a; 2013b) and Hilton, Hilton, Dole, and Goos (2013). The items from Hilton et al. were already designed as two tier multiple-choice test items i.e. students should first decide whether an assertion is true or false then choose one statement out of four that supports the student's reasoning. The other items were adapted from their original design to a multiple-choice design, using incorrect answer alternatives either reported in the original studies or answer alternatives recalled from my experience from teaching.

The test design was tried out in two steps. First, a pilot version of the test consisting of 22 items was tried out in April 2016, by 12 voluntarily participating students with ongoing mathematics courses. This step was performed to get information about the clarity of the item’s description and how long the test takes to complete. Feedback from the participants provided the insights that the test was a bit too long and that some of the items were difficult to interpret. In the second step, a revised version consisting of 16 proportional reasoning items was distributed to colleagues of mine during a teacher gathering in May 2016. Colleagues and I offered our ongoing mathematics students’ to take the test voluntarily. A total of 32 students chose to participate. This aimed to check that the test discriminated among students (see results section). It was not followed up with student interviews. This step was important since a test that gives similar results among the vast majority of students would fail to give information on students’ individual prior knowledge. The test took between 20 to 40 minutes to complete, without any time pressure.

A test can only produce limited information of cognitive processes (Ginsburg, 2009). Therefore, It is of great importance to follow-up the test with a clinical interview, where the students' reasoning can be confirmed, following the proposed idea in the answer alternative chosen, or adjusted if there is a misunderstanding of the task (Ginsburg, 1981). This is an important step since many students do not have Swedish as their mother tongue, which of course may cloud their interpretation of the items. Many of the students also have concentration difficulties, so a written test alone may not give a satisfactory picture of students’ prior knowledge. In the interview the students get opportunity to orally communicate their reasoning to confirm or refute the found prior knowledge and potential lack of knowledge, which is a powerful method for learning about students’ underlying cognitive mathematical competence (Ginsburg, 2009). The interviews are semi-structured and adjusted to the situation. The starting point is that the student should account for all his or her answers, both correct and incorrect. However, if there are a lot of errors the interview is adjusted to a sample of the items e.g. if incorrect answers is given for all part 4 items and a lot of part 3 items, the interview can be focused around the correct answers and the part 3 incorrect answers. For such a case the incorrect part 4 answers can be postponed until later on in the course. The interviews are documented with notes during the interview and afterwards, the rewritten notes are documented in the student's folder. The test together with the follow-up clinical interview gives information that provides an access point for individualized instruction, as to be shown by three cases in the results section below.

Results

Test results from the 32 participants in step two in the test design, spread between 0 to 13 correct answers, showing that the 16 items discriminate well among students. None of the students came up with the exact answering profile as another student. Two students scored zero at the test, but with different incorrect reasoning on most items. However, information from teachers gave reason to believe that the zero results may be a consequence of not having Swedish as mother tongue rather than poor prior knowledge solely. Furthermore, the analysis showed that some items caused incorrect answers from most students, i.e. 9, 11 and 14, (see appendix). The line-item (9) was solved by only three students, the dice-item (14) by two students. Also the steep hill-item (11) caused a lot of errors, solved by only four students. I will comment on these items in the discussion.

Table 2. Test results from 32 students

	Incorrect answers	Correct answers
Running laps	50% (16)	50% (16)
Number line	62.5% (20)	37.5% (12)
Riding home with Jens	50% (16)	50% (16)
End-of-term activities	57% (18)	44% (14)
Washing days	69% (22)	31% (10)
Orchestra	50% (16)	50% (16)
Loading boxes	57% (18)	44% (14)
Drawing insects	50% (16)	50% (16)
Line	91% (29)	9% (3)
Circle	66% (21)	34% (11)
Olive oil	78% (25)	22% (7)
Steep hill	87.5% (28)	12.5% (4)
Pizza	62.5% (20)	37.5% (12)
Dice	94% (30)	6% (2)
Cylinders	69% (22)	31% (10)
Tractor	66% (21)	34% (11)

Addressing the students' prior knowledge on proportional reasoning, in relation to the given theoretical underpinnings, three cases will be displayed. These cases show how the test and the follow-up clinical interview with the students provided information that provided an assess point for individualized instruction.

Case 1: Bill

It had past 16 years since Bill studied mathematics the last time. Therefore, there was reason to believe that Bill had forgotten a lot of the school mathematics as well as gained some “life experience mathematics”. The test showed that Bill reasoned correctly on 10 items and incorrectly on 6. No errors were caused by test items for key point (1) Students’ ability to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present. Bill’s errors clustered in key point (4) Students’ ability to acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity, where he gave incorrect answers on all four items testing the key point. The other two errors were given for items 9 and 12. In the follow-up clinical interview we discussed the items where incorrect answers were given, to elicit the students reasoning. It became clear that Bill, in line with the given answers in the test, had difficulties with scaling in two- and three dimensions. Also, the algebraic representation of proportional functions, with focus on the limitations for the mathematical notation $y = ax$, as well as average speed, was conceptually weak in Bill’s conceptual field for proportional reasoning. For the line-item Bill claimed that all lines can be written on the form $y = ax$. When I asked him “What about a line that follows the y-axis?”, he could not interpret that a line following the y-axis is not possible to write on the form $y = ax$, if a is a constant. He could not say how the constant a affected the line but he recognized the representation. For the steep-hill item (item 12 in appendix) students shall decide on the average speed for a whole walk with different average speeds. One way with 3 km/h and back with 6 km/h. Bill explained to me that he added 3 + 6 and divided by 2. So, he calculated an average of the averages, which indicates that he may have a weak conceptualization of the rate, average speed. Speed is proportional to the distance, as time is kept constant. However, item steep-hill (12) involves two average speeds, over the same distance. Therefore, to just manipulate the $s = vt$ formula, fails as a method for calculating the answer.

The result on Bill’s prior knowledge indicated that the assess point could be a teaching sequences to catch up on geometrical properties and scaling in two- and three dimensions to be followed up with a teaching sentence with rates, such as speed, density and price per unit. Then Bill hopefully may be prepared to start the first upper secondary course, with extra attention on the properties of linear functions.

Case 2: Caesar

While Bill’s results clustered in key point four, Caesar’s incorrect answers spread almost equally among the four key points. Like Bill, it has been several years since Caesar studied mathematics in upper secondary school, without reaching a grade. He reasoned correctly on 7 items and incorrectly on 9 items. Compared to Bill, Caesar’s results showed a different profile on his prior knowledge. His incorrect answers clustered on items testing mathematical content from part four in the elementary course, indicating that the course design should start with catching up part four before moving on to the first upper secondary course. The follow-up clinical interview confirmed that Caesar lacked conceptual knowledge to tackle the items testing the mathematical reasoning competency required for part four in the elementary course. This became visible when Caesar despite scaffolding with questions could not reach the correct reasoning. His conceptual field for proportional reasoning seemed to lack pivotal pieces of conceptual knowledge. For the steep-hill item, also Caesar told me that he calculated an average on the two different average speeds, incorrectly concluding that the average speed for the whole walk was 4,5 km/h. When we discussed the line-item Caesar claimed to never had seen that representation before. He had no idea on how to approach the item. During the interview Caesar told me that he had not followed his class during the mathematics lessons in lower secondary school. Instead he was instructed in a small group, that to his understanding did not encounter the same challenges as his peers in the regular mathematics classroom. Of course, also this information was found most valuable for the individual design of Caesar’s mathematics course. A reasonable access point for Caesar would be to start on part four, and then move on to the first upper secondary course.

Case 3: David

David was young and had recently left upper secondary school to start serving his prison sentence. His results on the test were weak, showing possible lack of knowledge in all four key points for the development of proportional reasoning. He gave correct claims on 5 of the items; where two items concerned mathematics for part two, and two items for part three. Surprisingly he also gave the correct claim on one the items that had shown to cause incorrect answers from the vast majority of students taking the test, the pizza-item (13). In the follow-up clinical interview it turned out that he had picked the answer by chance and had no idea how to approach the item mathematically. There is a probability of 0.125 to pick the correct answer by chance i.e. chose the correct true or false claim and thereafter the right reasoning alternative. In the follow-up clinical interview, David told me that he had been absent most of the time in both lower- and upper secondary school and that he perceived mathematics to be the most difficult school subject. Still, he was motivated to take up mathematics, though he believed that mathematical competencies are of great importance for life in society. Although David was a native Swedish speaker he had trouble to interpret the items. He told me that he in general is a fairly good reader, but whenever a mathematical word appeared in the text he got confused. Also David used the incorrect average on an average reasoning on the steep hill item. When discussing the line-item, David told me that he felt very uncomfortable with characters in mathematical tasks. He just picked an answer without trying to interpret the item. The test combined with the follow-up clinical interview showed that his proportional reasoning competencies partly covered part three. This raised an intricate dilemma. Shall David start on part four with some extra attention for lost mathematical content in part three or shall he start with part three, although he already seem to manage some of the mathematical content taught in part three. In dialogue we agreed to start with part three, because David preferred a smooth start on his studies.

Discussion

The aim of this study was to investigate if it is possible to design a research-based test that combined with a follow-up clinical interview gives information on students' prior knowledge on proportional reasoning so that it discriminates among students, and provides teachers with an access point for designing individualized instruction.

The test showed that it discriminates well among students both in relation to the development of proportional reasoning throughout school years, and in relation to the four key points for developing proportional reasoning, i.e. be able to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present; be able to draw connections to the algebraic rules for fractions when working with part/part ratios, part/whole fractions and proportions, $a:b = c:d$; recognize and use a range of concrete representations for proportions, e.g., tables, graphs, formulas and pictures, and acknowledge the properties of geometrical objects in two and three dimensions for calculation of scaling and similarity. The results from the second step in the design of the test distributed the students between 0 and 13 correct answers out of 16. The student cases presented above showed that the test, together with the follow-up clinical interview, provided valuable information on the students' prior knowledge, so that it could serve as an access point for designing one to one individual instruction.

The conceptual knowledge of the algebraic representation of a proportionality, average speed and scaling in three dimensions appeared to be generally weak among the participants. The line-item (8) calls for interpreting the algebraic representation of a proportionality and reason about the boundaries for the constant a, in $y = ax$. Only three students solved it correctly. This may indicate that use of different representations needs extra attention in general. Also the steep hill-item (12) caused major trouble, solved only by four students. This may indicate that the concept of average speed also needs some extra attention. The ability to calculate an average speed, given the distance and time does not require a reification of the concept average speed. You just apply the procedure of using a formula. But, for solving the steep hill-item students need to grasp the essence of the concept average speed, the relationship between distance and time. The reasoning on the three displayed cases were based on the same incorrect assumption that average speed for the whole walk can be calculated by adding (3 km/h + 9 km/h) / 2 = 4,5 km/h. This concept image may indicate that these students have a weak conceptualization of Average speed. Conceptual knowledge does not come easy for students. They need to be challenged with different situations to adjust their concept images (Tall & Vinner, 1981) and be given opportunities to develop their conceptual field of proportional reasoning (Vergnaud, 2009). Finally, the dice-item (14) caused severe trouble, solved by two students. However, if you have the information that there exist hurdles with scaling it is easy to help students realize the state of affairs with scaling from one- to three dimensions. Building blocks help students to model the situation. The block representation transforms the elusive abstraction of scaling in three dimensions to a concrete visible hands-on situation.

The three cases provided three different pictures of student's prior knowledge on proportional reasoning. Being the teacher of all three students I can tell that due to the results three different individualized course designs were set up. Both Bill and Caesar entered the course by a teaching sequence with model eliciting activities (Lesh & Doerr, 2003) giving them opportunities to investigate the properties of geometrical objects and the effects of scaling in one- two- and three dimensions. Then we paid some extra attention to the line item and other situations requiring a conceptual knowledge of average speed. Thereafter, Bill continued with number theory, while Caesar was given instruction on the difference between additive and multiplicative reasoning and hand-in assignments with tasks requiring him to distinguish additive from multiplicative reasoning and to recognize when a multiplicative relation is present. David on the other hand was introduced to real world problems where his solutions should be presented to me in oral communication with mathematical words to enhance David's mathematical language. Previously, despite having the ambition to individualize the teaching, it was difficult for me to know where to start. Students often got to start out with chapter one in a commercially produced textbook. The information provided by the test presented here gave opportunities to design individualized instruction and adapt the teaching accordingly to the students. I was given the opportunity to put my teaching competency into play (Niss & Højgaard, 2011).

Not all aspects of mathematical competencies are possible to elicit in a multiple-choice test, where students do not produce any written mathematical arguments. The views on mathematical competencies presented by Niss & Højgaard (2011) include that each of the eight competencies has a passive and an active side. The test solely requires the students to follow and assess mathematical reasoning, the passive side of the reasoning competency. It says nothing about the students' competencies to devise and carry out mathematical reasoning on the active side. However, although the test focuses on the mathematical reasoning competency it also informs on students' mathematical thinking competency, problem-handling competency and modeling competency, since these competencies are intertwined and overlapping with the reasoning competency. Together these four competencies make up one out of two overall competences associated with mathematics: The ability to ask and answer questions in and with mathematics (Niss & Højgaard, 2011). The other overall competence: The ability to deal with mathematical language and tools, covers the intertwined competencies representing competency, symbol and formalism competency, communication competency and aids and tools competency. When using the test, it is important to keep in mind that the scope of it does not cover the ability to deal with mathematical language and tools, or the ability to ask and answer questions in and with mathematics. These competences were taken up elsewhere within the course design.

While this study took place in the prison education program, this approach to test and interview students so that they can be provided with individual instruction that departures from their prior knowledge may be useful in other educational contexts for adult mathematics education.

Notes

[1] This paper concerns imprisoned students in Sweden, of which a majority are men (95%); this is why I will use the personal pronouns he and his throughout the paper.

[2] By individualized instruction I refer to an individual course design in line with both the students' prerequisites as well as the national curriculum. Teaching and learning still take place in a social setting where mathematics students communicate to justify their reasoning on modeling problems.

[3] Problem solving is also core content, as well as a goal, but the national curricula do not specify what mathematical ideas to deal with in problem solving. It is left to the teacher to decide what problems from ordinary-, society-, school- and work-life to encounter.

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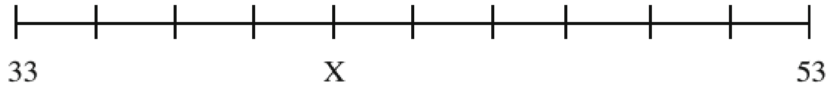
Appendix: Test items

1. **Running laps**: Sara and Johan were running equally fast around a track. Johan started first. When Johan had run 4 laps, Sara had run 2 laps. When Sara had completed 6 laps, Johan had run 12 laps.

True or **False** because (choose the best reason):

a) The further they run, the further Johan will get ahead Sara.
b) Johan is always 2 laps ahead of Sara.
c) Johan completes double the laps of Sara.
d) Sara has run 3 lots of 2 laps to make a total of 6 laps, so Johan must have run 3 lots of 4 laps to make a total of 12 laps.

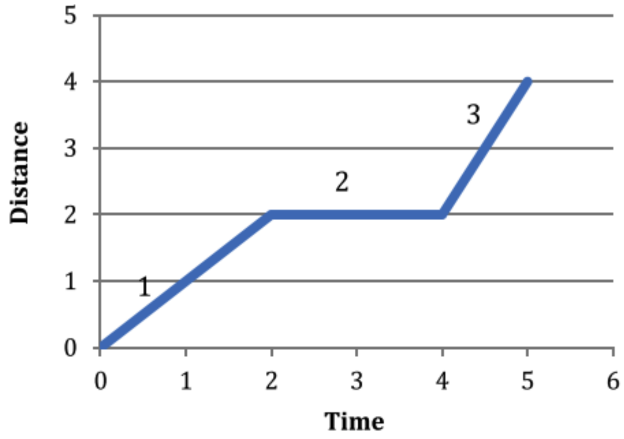
2. **Number line**: ***On this number line, X represents 37***.



True or **False** because (choose the best reason):

a) X is 4 cm along the line and 33+4=37.
b) X needs to be closer to 33 than 53.
c) X is nearly halfway so it looks OK.
d) X is 8 more than 33.

3. **Riding home with Jens**: Jens rode his bicycle home. He rode at a steady speed for a short time and then he had to rest. After his rest, he rode at double his original speed. He drew a graph to represent his journey. ***Jens' graph below is correct***.



True or **False** because (choose the best reason):

a) The distance covered in Part 3 is greater than the distance in Part 1.
b) Part 3 is twice as steep as Part 1.
c) Part 3 needs to be twice as long.
d) The times for parts 1 and 3 are the same.

4. **End-of-term activities**: This table shows the end-of-term activities voted by Year 5 and Year 6 students. ***Going to the beach is a relatively more popular choice with the Year 6 students than the Year 5 students***.

Year level	Students who chose the beach	Students who chose the movie	Total students
Year 5	8	14	22
Year 6	7	6	13

True or **False** because (choose the best reason):

a) More students in year 5 chose the beach.
b) Only 6 students in Year 6 chose not to go to the beach.
c) Fewer students in Year 6 chose the beach but there are fewer students in the class.
d) More than half of the students in Year 6 students chose the beach and less than half of the Year 5 students chose the beach.

5. **Washing days**: ***Washing powder A is the best value***.

True or **False** because (choose the best reason):

a) Washing powder A cost the least.
b) Washing powder B costs a little bit more but you get 10 more loads of washing.
c) The cost per load of washing is less.
d) Both washing powders are the same value.

6. **Orchestra**: An orchestra of 15 musicians takes 2 hours to play a concert. ***An orchestra of 30 musicians takes 4 hours to play the same concert***.

True or **False** because (choose the best reason):

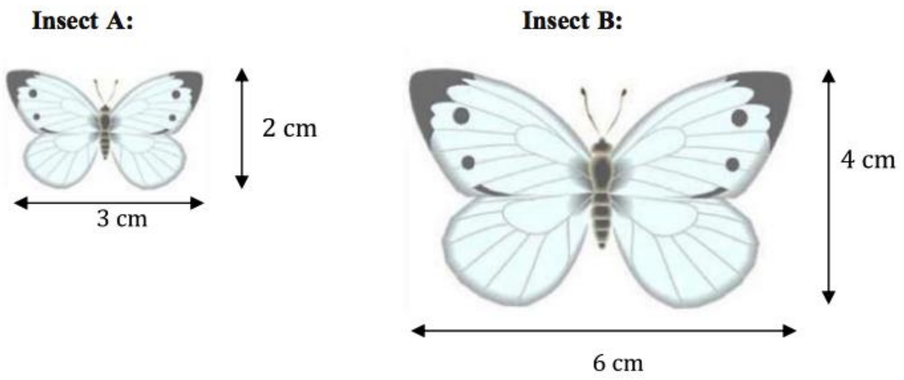
a) Doubling the number of musicians would double the time to play the concert.
b) Double the musicians should halve the time to play the concert.
c) The number of musicians does not affect the time to play the concert.
d) Adding more musicians increases the time to play the concert.

7. **Loading boxes**: Petra and Tina are loading boxes in a truck. They started together but Tina loads faster. When Petra has loaded 40 boxes, Tina has loaded 160 boxes. ***When Petra has loaded 80 boxes, Tina has loaded 200 boxes***.

True or **False** because (choose the best reason):

a) Tina will always be 120 boxes ahead of Petra.
b) Petra loads faster than Tina.
c) Tina loads 4 times faster than Petra.
d) Tina loads with double speed.

8. **Insects**: Bob has drawn two diagrams. ***The area of insect B is twice that of insect A***.



True or **False** because (choose the best reason):

a) The area of insect A is 4 times greater.
b) Insect A is half the width of insect B.
c) Insect B is twice as long as insect A.
d) Bob has only doubled one dimension.

9. **Line**: We know that an equation on the form $y=ax$ (a is a constant) gives a line through (0,0) in a coordinate system. **Every line through (0,0) has an equation on the form $y=ax$, (a is a constant)**.

True or **False** because (choose the best reason):

a) There exist a line through (0,0) that cannot be written on the form $y=ax$.
b) The reasoning goes both ways.
c) All lines can be written on the form $y=ax$.
d) You cannot know without investigating an infinite amount of lines.

10. **Circle**: **Simon says that if you draw a new circle with half the diameter of another circle, the new circle will have half the perimeter and half the area of the other circle**.

True or **False** because (choose the best reason):

a) If the diameter is halved, the perimeter and area is halved.
b) The area will be ¼ and the perimeter ½ of the original.
c) You cannot know without knowing the length of the diameter in the new circle.
d) You cannot know without knowing the length of the diameter in the original circle.

11. **Olive oil**: You'll make a salad dressing according to this recipe for 100 milliliters (ml) dressing. **You need 110 ml olive oil to 150ml dressing**.

Olive oil	60 ml
Vinegar	30 ml
Soya sauce	10 ml

True or **False** because (choose the best reason):

a) It is far too much oil.
b) You will need 1.5 times oil.
c) Since the vinegar and soya sauce is 40 ml, the rest will be oil.
d) You need 100 ml oil.

12. **Steep hill**: There is a path up a quite steep hill in Athens. Rickard, who is in good shape, is going up the hill in an average speed of 3 km per hour. He goes down in double speed. **Richard's average speed for the whole walk is 4 km per hour**.

True or **False** because (choose the best reason):

a) The speed is higher than 4 km per hour.
b) The speed is 4 km per hour.
c) The speed is 5 km per hour.
d) You cannot know if you don't know the distance.

13. **Pizza**: A pizzeria serves round pizzas of various sizes. The smaller pizza has a diameter of 30 cm and costs 30 SEK. The larger has a diameter of 40 cm and costs 40 SEK. **The large pizza gives more pizza for the money**.

True or **False** because (choose the best reason):

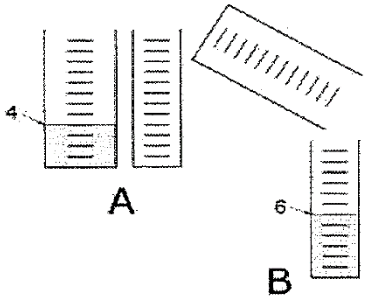
a) The small pizza gives more pizza for the money.
b) It is the same cost, 10 SEK per diameter pizza.
c) The small pizza is more expensive per area unit.
d) You cannot decide if you cannot see the pizzas.

14. **Dice**: A wooden dice where all edges is 2 cm weighs 4.8 g. **A wooden dice where all edges is 4 cm weight 19.2 g**.

True or **False** because (choose the best reason):

a) The weight increases 4 times if the edge doubles.
b) The weight increases 6 times if the edge doubles.
c) The weight increases 8 times if the edge doubles.
d) The weight doubles if the edge doubles.

15. **Cylinders**: Below are drawings of a wide and a narrow cylinder. The cylinders have equally spaced marks on them. Water is poured into the wide cylinder up to the 4th mark (see A). This water rises to the 6th mark when poured into the narrow cylinder (see B). Both cylinders are emptied (not shown) and water is poured into the wide cylinder up to the 6th mark. **The water would rise up to the 8th mark if it were poured into the empty narrow cylinder**.



True or **False** because (choose the best reason):

a) The answer cannot be determined with the information given.
b) It went up 2 more before so it will go up 2 more again.
c) It goes up 3 in the narrow for very 2 in the wide.
d) The second cylinder is narrower.

16. **Tractor**: When the big wheel of a tractor turns 5 times the small wheel turns 8 times. **When the big wheel turned 7 times the small wheel have turned 11 times**.

True or **False** because (choose the best reason):

a) The small wheel makes 3 more turns than the large one.
b) One has to find out the answer by observation.
c) The small wheel turns about one and a half time as often as the big one.
d) Both wheels will have turned two extra turns.