Connecting new knowledge to old:

Uncovering hidden premises in mathematical explanations

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This article describes the benefits of students making explicit connections between new concepts and their prior knowledge. In this way students are able to recognise the many connections in mathematics to increase their understanding.

Understanding is a major goal of mathematics. Indeed, it is one of the four proficiencies in the *Australian Curriculum: Mathematics* (AC:M). But what does it truly mean to "understand" mathematics? Peter Sullivan, past President of the Australian Association of Mathematics Teachers, states that:

> ...understanding really is about forming connections between what is known, what needs to be known, and how the various aspects of a particular concept are connected to other aspects of learning. (Sullivan, 2016).

This explanation emphasises that mathematical understanding means a coherent, or well-connected, view of mathematical ideas. Research has shown that those who understand a topic well have connected and organised their knowledge around key concepts (National Research Council [NRC], 2000, 2001). This helps them understand new concepts more quickly, retain their current knowledge more effectively, and transfer their knowledge to new situations more easily. In one large-scale study involving over 2000 students and 90 teachers, students whose teachers emphasised mathematical connections made greater gains on numeracy tests than students whose teachers did not emphasise connections (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997).

Despite the numerous benefits that stem from a coherent mathematical understanding, it is uncommon for teachers to encourage students to form a developed web of mathematical connections (Mousley, 1999). Additionally, when students explain their thinking, they frequently do not state how their reasoning connects to prior knowledge. This obscures important mathematical connections between new knowledge and prior knowledge, making it more likely that the class will view new mathematical knowledge as isolated pieces of information. When students lack mathematical coherence, they are more likely to forget new information and be unsuccessful in transferring their knowledge to new situations (Fennema & Romberg, 1999).

Even if teachers want to emphasise mathematical coherence in their teaching, they may not understand how to do so (e.g. Cohen, 1990). Further complicating the matter is the fact that lessons that do and do not emphasise connections can appear similar, distiguished only by subtle differences (Kazemi & Stipek, 2001). In this article, I present a framework to help teachers prompt their students to explicitly connect new ideas back to prior knowledge. Making these sorts of explicit connections as a regular practice will help students to achieve the understanding proficiency in the *Australian Curriculum: Mathematics* of "mak[ing] connections between related concepts and progressively apply[ing] the familiar to develop new ideas" (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2017, p.5).

Making connections more explicit — a framework

Whenever we reason or explain our thinking, we always assume certain things to be true. This gives us a starting point for our reasoning. These starting assumptions are called premises. Premises form the crucial link between our current reasoning and our body of prior knowledge. Ideally, when explaining our thinking to others, we would explicitly state all of our premises so that the connections we are relying on are evident. In reality however, our premises are often left unstated.

Informed listeners may be able to fill in the blanks in their own minds and recognise the unstated connections, but less-informed listeners may not be able to do so.

Table 1 shows three different versions of the same explanation. The first row is the 'Less Connected Explanation' and explains why an even number times an even number always results in an even number. This explanation leaves a premise (the sum of any two even numbers is always even) unstated. In the 'Connected' version of the explanation, this hidden premise has been explicitly stated (underlined for emphasis), drawing attention to how the explanation relies on prior knowledge. In the 'Most Connected Explanation', this premise has also been justified. By going a step further and explaining why the premise is true, the Most Connected Explanation reveals the rich and varied ways that the current reasoning connects with prior knowledge.

Table 1. Three versions of the same explanation of why the product of two even numbers is always even.

Less Connected Explanation: (premise unstated)	Multiplication is repeated addition, so an even times an even is actually a bunch of even numbers added together. The sum of all these numbers will be even.
Connected Explanation: (premise explicitly stated and underlined)	Multiplication is repeated addition, so an even times an even is actually a bunch of even numbers added together. The sum of any two even numbers is always even. The sum of all these numbers will be even.
Most connected explanation: (premise explicitly stated and underlined, premise justified in *italics*)	Multiplication is repeated addition, so an even times an even is actually a bunch of even numbers added together. *Even numbers consist entirely of pairs. When we add a bunch of pairs together, the result will also consist entirely of pairs,* so the sum of any two even numbers is always even. The sum of all these numbers will be even.

It is important to emphasise that all three versions of the explanation are mathematically sound. The only difference is how explicit the connections to prior knowledge are. The Most Connected Explanation clearly lays out how it builds upon established knowledge of even and odd. The Less Connected Explanation assumes that its listeners can identify these connections on their own. If the Less Connected Explanation took place in a classroom that had just discussed the additive properties of even and odd, then the listeners would probably be able to recognise the implicit connections on their own. However, if the class had not recently discussed the additive properties of even and odd, then it is doubtful whether all listeners could mentally make these connections. In other words, the amount of detail that must be provided to help students make connections will always depend on the particular group of students and their current level of mathematical knowledge. A premise that is well-established and clearly recognised among one group of students may not be so for a different group.

Suppose a Less Connected Explanation is given and the teacher is unsure if students are connecting it to their prior knowledge. What can be done in this situation? Challenging the explanation at the right point will usually cause a premise to be explicitly stated. For example, if the Less Connected Explanation from Table 1 was given, the teacher could ask, "How do you know that the sum of these even numbers will be even?" The student will probably reply that the sum of even numbers is always even. At this point, the explanation has moved to the 'Connected' level because the premise has now been explicitly stated. The teacher could then draw out a justification for the premise by asking, "Why is the sum of two even numbers always even?" Once this justification has been given, the explanation has moved to the Most Connected level because the premise has been stated and justified. The previously implicit connections to prior knowledge are now visible for the entire class. Listeners are no longer required to make these connections for themselves as they try to follow the reasoning.

When trying to draw out the implicit connections in an explanation, teachers may find that the students are unable to recognise their own premises or justify them. In the rest of this article, I provide a few examples of this happening. These examples come from a U.S. classroom of fifth-grade students (ages 10–11) and all occurred within the first few weeks of the school year.

Example 1

In this situation, the class was reviewing the various properties of numbers using the number 15 as an example. When Emma mentioned that 15 was odd, the teacher asked for an explanation of why.

Teacher: Why is 15 odd?
Emma: 5 is odd.
Teacher: Keep going! This isn't the number 5. This is the number 15.
Emma: 5 is odd, so 15 is odd because it ends in the number 5.
Teacher: Why? I don't understand.
Emma: So 5 is odd, and 15 is odd. So 15 is odd, because it has 5 as the last number, and 5 isn't divisible by 2.

Teacher: Okay, but this is the number 15.
Emma: I know. The 5 is not divisible by 2. The ten doesn't matter.
Teacher: Why not? Why doesn't the ten matter? It seems kind of important. Without it, we don't have the number 15.
Emma: But it doesn't matter when you're deciding even and odd.
Teacher: Why doesn't it matter? Why can I just focus on the ones place? (long pause)

Table 2 shows Emma's explanation using the framework. The initial explanation was very brief and did not state the premise, making it a Less Connected Explanation. By challenging and questioning the explanation, the teacher drew out an explicit statement of the premise. By stating that "the ten doesn't matter" and that "15 is odd because it ends in the number 5," Emma's premise became evident: the ones place determines whether a number is even or odd. The explanation was now a Connected Explanation. Trying to move to the Most Connected level, the teacher asked Emma to justify this premise. In response, Emma stated, "[The tens place] doesn't matter when you're deciding even and odd." This was not an explanation of why the premise was true, but was simply a restatement of the premise. The teacher once again asked for a justification, this time to the entire class. After a long pause, it became evident that the class did not understand why the ones place determined whether a number was even or odd. Because of this disconnect in the class's mathematical knowledge, the explanation could not reach the Most Connected level. In response, the teacher decided to review place value with her students and work towards making an explicit connection between it and even/odd. After some time, students recognised that the tens digit always represents an even quantity. Combined with the additive properties of even and odd, students eventually understood the Most Connected Explanation shown in Table 2 and recognised a connection between two areas of maths that they had previously assumed were unrelated.

Example 2

In this example, Noah explained how he multiplied 6 × 20 to get 120.

Noah: I did 6 times 2 and then I added a zero.
Teacher: He said, "I multiplied 6 times 2, which is 12, and then I added zero." (writes 12 + 0 = 12). So is this what he just did? Is 12 plus 0 equal to 12? So he's saying the answer is 12. Do you agree ?

Table 2. Three versions of Emma's explanation that the number 15 is odd.

Less Connected Explanation: (as stated by student)	15 is odd because 5 is odd.
Connected Explanation: (premise explicitly stated and underlined)	The ones place determines whether a number is even or odd. 15 is odd because 5 is odd.
Most Connected Explanation: (premise explicitly stated and underlined, premise justified in *italics*)	*A digit in the tens place always represents an even quantity since it is being multiplied by ten and can therefore be evenly divided by two. Since even + even = even, and even + odd = odd,* this means the ones place determines whether a number is even or odd. 15 is odd because 5 is odd.

Table 3. Three versions of Noah's explanation that 6 x 20 = 120.

Less Connected Explanation: (as stated by student)	I did 6 times 2 and then I added a zero and got 120.
Connected Explanation: (premise explicitly stated and underlined)	I first decomposed 20 into 2 × 10. I did 6 times 2. Adding a zero is equivalent to multiplying by ten. I added a zero and got 120.
Most Connected Explanation: (premise explicitly stated and underlined, premise justified in *italics*)	I first decomposed 20 into 2 x10. I did 6 times 2. Adding a zero is equivalent to multiplying by ten, *since multiplying by ten moves each digit up to the next-larger place value.* I added a zero and got 120.

Noah: You don't add a zero, you like, put a zero behind the twelve.
Teacher: So I put a zero behind the twelve. Like that? (writes 12 0)
Class: No.
Teacher: Why am I being so silly here?
Natalie: You're adding a zero to the ones place.
Teacher: (writes 12 + 0 = 12) But here I'm adding a zero to the ones place, right? You know what you're doing. I want you to be explicit in your thinking.

In Noah's original explanation, the premise (adding a zero is equivalent to multiplying by ten) was not explicitly stated, so the explanation was at the Less Connected level. The teacher began challenging Noah's explanation

to draw out a statement of the premise. However, unlike the previous example, the class struggled to articulate the premise. Although Noah and others initially appeared to know that adding a zero to the end of a number was a shortcut for multiplying by ten, they were unable to clearly state this. Despite being able to use this computational shortcut, the students were unable to incorporate it as part of an explanation, suggesting that their understanding of the shortcut was superficial and isolated from their other mathematical knowledge. Even after some additional 'pair-share' time, students remained hesitant about what was happening mathematically when a zero was added to the end of a number. The lack of coherence in students' mathematical knowledge had caused the explanation to become stuck at the Less Connected level. Recognising this disconnect, the teacher reviewed place value in a way that helped them to connect it to multiplication.

Benefits of explicit connections

When students can use various strategies to perform calculations and solve problems, it can be easy to assume that they understand the underlying concepts and have connected these concepts to their prior knowledge. However, as the examples illustrate, this may not be the case. Even if students can use a strategy correctly, on a conceptual level that strategy may be an isolated piece of knowledge in their minds. As both examples also illustrate, it is not uncommon for students to leave their premises unstated and unjustified, obscuring the connections to prior knowledge. When these connections are obscured, other students miss out on valuable opportunities to form a more coherent understanding of mathematics.

Drawing out implicit connections can also be a diagnostic tool that allows the teacher to identify areas where students have fragmented knowledge. This is particularly useful at the beginning of the academic year for determining if the class needs to step back and review fundamental concepts and connections. Although both examples in this article occurred in a fifth-grade classroom, much of the content the class discussed came from earlier years (e.g. place value, multiplying by ten). The press for explicit connections, done at the beginning of the school year, familiarised the teacher with her students' mathematical foundations. By addressing disconnects in their knowledge, the teacher enabled her students to more fully participate in future mathematics.

As a final benefit, making connections explicit helps prepare students for the increasingly complex and formal reasoning, known as proofs, that they must employ in their future maths classes. Researchers believe that emphasising explicit connections early can ease this transition to proofs by familiarising students with the type of thinking that is required (Knuth & Elliot, 1998). At all levels, as students recognise the many connections that permeate mathematics, they will begin to achieve the Understanding Proficiency in the AC:M and "build a robust knowledge of adaptable and transferable mathematical concepts … progressively apply[ing] the familiar to develop new ideas" (ACARA, 2017, p.5).

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