Teaching place-value: Concept development, big ideas and activities

Judy Hartnett
Making Maths Reasonable
<judyhartnett@bigpond.com>

This paper was originally presented at the 25th Biennial Conference of AAMT, 'Capital Maths', and published in the Conference Proceedings. The author discusses the development and beginning implementation of a detailed sequence of concepts, with the intention of supporting teachers to support students to understand place value.

Introduction

Working as a mathematics education consultant in schools, both as a system-based advisor and private consultant, I am regularly asked to support teachers to work with students who do not understand place value. As reported by other researchers including Rogers (2013), I, too, am regularly dismayed at the superficial understanding so many primary students exhibit in relation to place value. I have also observed repeated amazement by teachers when I demonstrate lessons, or discuss the 'big ideas' relating to place value in comments such as "I have never thought about it that way".

Primary school teachers teach a number of subjects, and while they continue to learn and access professional development, it is not surprising that they do not have an indepth understanding of the many concepts that underpin mathematical topics such as place value. This paper outlines the development of a list of concepts I hope will support teachers to develop their knowledge of this topic to help them plan and present lessons to assist their students to understand the concepts.

Teacher knowledges

Research has recognised that teachers require a range of types of knowledge. Shulman (1986) described seven types of teacher knowledge: knowledge of content, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of students, knowledge of educational contexts and knowledge of educational ends, purposes and values. Based on the work of Shulman and colleagues (e.g., Shulman & Grossman, 1988), Borko and Putnam (1995) proposed a model of teacher knowledge organised around three domains of knowledge: general pedagogical knowledge, subject matter knowledge and pedagogical content knowledge. Table 1 outlines these domains and their components.

Table 1. *Domains and components of the knowledge base of teaching* (Borko & Putnam, 1995).

Domains	Components
General pedagogical knowledge	• Learning environments and instructional strategies • Classroom management • Knowledge of learners and learning
Subject matter knowledge	• Knowledge of content and substantive structures • Syntactic structures
Pedagogical content knowledge	• Overarching conception of teaching a subject • Knowledge of instructional strategies and representations • Knowledge of students' understandings and potential misunderstandings • Knowledge of curriculum and curricular materials

Teachers I had been supporting appeared to lack mathematical subject matter knowledge, and also welcomed suggestions about pedagogical content knowledge. Teachers have access to Australian Curriculum documents and often commercial maths programs and resources. These documents provide descriptions, glossaries and activities but lack the mathematical depth to help the teachers develop their subject matter knowledge. A sequential list of big ideas and concepts that under-pin mathematics could be a valuable support for teachers. Table 2 provides an example of how several concepts can underpin one Australian Curriculum description. I started to draft such lists for mathematics topics in a number of areas about which I had received requests for assistance, including place value.

Big ideas

Place value is not a single concept. Schmittau and Vagliardo (2006) used concept mapping to describe place value as a complex system. Price (1998) described how the development of connected memory structures or schema can assist students to understand complex numeration concepts like place value. He described how assisting students to develop powerful schemas to understand the complexities of place value was attractive to mathematics educators but commented that for many students such schemas had not developed. My anecdotal observations led me to suspect that many of the teachers I was supporting may not have developed a rich schema for a deep understanding of our base-ten number system themselves. This lack of subject knowledge would likely inhibit their ability to design learning activities and identify common misconceptions in their students.

Many researchers have identified concepts that need to be understood for students to become place value experts (Rogers, 2012; Ross 1989). Rogers (2012) conducted a comprehensive search of literature on the topic and focussed on the work of Rubin and Russell (1992) and Ross (2002) to identify seven components of place value. Rogers noted that there is no developmental order implied in the list:

- **Count:** Counting forwards and backwards in place value parts (e.g., 45, 55, 65 is counting using the unit ten). Bridging forwards and backwards over place value segments (e.g., 995 and one more ten requires bridging forwards over hundreds to thousands).
- **Make/represent:** Make, represent or identify the value of a number using a range of materials or models—these may be proportional, non-proportional, canonical and non-canonical.
- **Name/record:** Read and write a number in words and figures (e.g., 75 is written as 'seventy-five'). Identify the value of digits in a number (e.g., the value of 3 in 345 is three hundreds). Rounding numbers to the nearest place value part (e.g., round 2456 to the nearest thousand).
- **Rename:** Recognise and complete partitions and regrouping of numbers. (e.g.,1260 has 126 tens).
- **Compare/order:** Compare numbers to determine which is larger or smaller and place them in descending or ascending order.
- **Calculate:** Apply knowledge of place value when completing calculations (e.g., 45 by 10 is 45 tens)
- **Estimate:** Use knowledge of magnitude of numbers when estimating (e.g., estimate how many oranges fill a classroom: 10? 100? 100 000?)

Other authors have identified concepts relating to place value which theoretically fit within the components above. Examples include the 'Odometer Principle' (YuMi Deadly Centre, 2014) which states that in any place-value position, numbers count the same as in the ones place, counting forwards from 0 to 9 and then back to 0 with the digit to the left increased by 1; the 'recursive HTO pattern' (Siemon, Beswick, Brady, Clark, Farragher & Warren, 2011); 'composite units/super-unitising and sub-unitising' (Siemon et al., 2011, Baturo, 1998); and the 'recursive multiplicative relationship' between the places where "10 of these is one of those" (Siemon et al., p. 302). Place value is a complex mathematical topic with a multitude of big ideas and connected concepts to be understood. A list of these big ideas of place value would need to include all these big ideas presented in a way that made sense to teachers.

Conceptual development

When learning mathematics, students progress through topics and associated concepts which get increasingly more sophisticated. This educative progression has been identified as back as far as Piaget (1952). Carpenter and Fennema (1991) noted that teachers need an understanding of the stages students move through when developing concepts and procedures. The Queensland University of Technology (QUT) YuMi Deadly Centre (2014) stated that it is "essential for teachers to know what mathematics precedes, relates to and follows what they are teaching" (p. 2). Recent studies in Australia have used this idea to identify learning progressions; e.g., Growth Points (Clarke et al., 2002) and Learning Assessment Frameworks (Siemon, Izard, Breed & Virgona, 2006).

The idea of a list of the progression of concepts for place value as a teaching tool appealed to me as a way of supporting teachers who lacked subject matter knowledge of this topic. I began to draft a conceptual development list for place value based on ideas and concepts gathered over many years working with teachers, researching in mathematics education and working with other researchers. My intention was to write the list in teacher-friendly words rather than 'academic speak' and for the list to progress from early concepts through to more advanced concepts. The statements are intended to be what we would like students to understand about place value rather than what we would like them to do. My intention was for this list to be a reference document for teachers. Teachers could then plan learning activities to help the students achieve these understandings. Figure 1 shows an early section of the progression which currently is over three pages long and still being drafted.

The concept development list extends through whole number place value concepts to concepts for decimal place value, while aiming to include all big ideas including the structure of our number system; reading, writing and ordering numbers; the relationship between the places; the role of the decimal point and use of zero etc. Figure 2 provides part of the conceptual development sequence as it extends into decimals.

- A base-ten number system uses only 10 symbols to represent any number large or small.
- The Hindu-Arabic number system uses ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
- Then counting forwards, the digits in a place are named in order, from 0 to 9 and then back to 0 in that place. When the digit in a place returns to 0 the place to the left increases by 1 (Odometer Principle) (e.g., 17, 18, 19, 20, 21). Numbers beyond ten can be considered as one ten and some extras…
- Teen numbers consist of a full ten and extra ones, but not enough extra ones to make another ten.
- Multiples of ten consist of several groups of ten and are written to show the number of tens with zero ones e.g., 4 tens = 40.
- Two-digit numbers consist of a number of tens and a number of ones with the tens recorded in the place to the left of the ones (e.g., 45 is 4 tens and 5 ones.)
- Two-digit numbers are ordered according to the number of tens and then the number of ones.
- Adding a multiple of ten to a number will increase the number of tens by 1 but not change the number of ones (e.g., 45 + 30 = 75).
- There are 10 tens in one hundred.
- The place of a digit in a number indicates its value (e.g., 4 in the tens place is worth 4 tens).
- The value of a digit is determined by multiplying its face value by the value assigned to its place in the number.
- Zeros are used to show when a number has none of a particular place value (e.g., 30 is 3 tens and 0 ones; 405 has 4 hundreds, 0 tens and 5 ones).
- Zeros are used a place holders to maintain the place value structure of a number (e.g. 304 is a three-digit number that consists of 3 hundreds, 0 tens and 4 ones).

Figure 1. An early section of the Conceptual Development for Place Value list.

- The structure of the Hindu-Arabic number system extends to the right to allow us to show parts of whole numbers using place value.
- The structure of the place-value system remains constant—moving numbers to different positions in the place value chart will change the value of the digits. The place-value chart structure including the decimal point does not move.
- Multiplying and dividing numbers by powers of ten will move the numbers in relation to the place-value chart and will change the value of digits.
- The groups of three (HTO) structure continues to the right of the ones place although it is rarely used to describe decimals (e.g., tenths, hundredths, one thousandths, ten thousandths etc.).
- The groups of three (HTO) structure is reversed in the decimals because the first place is 1÷10 ($\frac{1}{10}$) which is tenths.
- Mixed numbers are fractions that have a whole component and a part component. With fractions the number of parts (denominator) can be any number (e.g., $2\frac{1}{10}$).
- The decimal point is used to mark the ones place so numbers can be read and interpreted using place value (e.g., 4.56 is 4 wholes, 5 tenths and 6 hundredths, or 4 wholes and 56 hundredths).
- The decimal point marks the separation of the whole component of a number and the part component (e.g., 4.5 is 4 ones [whole] and 5 tenths [part]).
- Zeros are used as place holders in decimals numbers to show there are no digits of a particular value in a number (e.g., 4.06 is 4 ones, 0 tenths and 6 hundredths).
- Zeros placed to the right of decimals do not change the value to the number as they do not add any further places of value (e.g., 5.6 is the same as 5.60).

Whole numbers					Parts of wholes			
Thousands		Ones			Parts		Thousandths	
T	O	H	T	O	T	H	O	T
					$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10\,000}$
				1	1÷10	1÷10÷10	1÷10÷10÷10	etc

Figure 2. A latter section of the place value concept list including decimal concepts.

Activities

The teaching of number concepts requires the student to abstract the learning from examples and activities provided by the teacher. Number concepts cannot be perceived directly with the physical senses. They can be represented or symbolised but the meaning must be abstracted by the learner (Price, 1998). The activities that teachers prepare need to provide examples and stimulus for students to abstract the concepts. The choice of resources can support student understanding. There has been discussion in the literature about the benefits of particular resources for teaching number and in particular teaching place value.

Resources for teaching place value

The use of hands-on materials is widely recognised as beneficial to the development of students' conceptual understanding in mathematics. Price (1998) reported that 96% of teachers he surveyed believed that materials benefitted children's learning and that curriculum documents recommended the use of materials. However, he commented that the use of materials to support place-value learning needs careful planning and that teachers cannot assume that students are making sense of the representations the same way as the teacher.

Place-value blocks, also known as Multi-Base Arithmetic Blocks (MAB), are the most common classroom hands-on material used to support the learning of place-value concepts (Price, 1998). The structure of these blocks models the base-ten number system as the size of each block is proportional to the value it represents. However, the effectiveness of these blocks for teaching place value concepts has been questioned by several authors including Booker, Bond, Sparrow & Swan (2010); Fuson, (1990); Siemon et al., (2011); Miura and Okamato (2003); Price (1998); and Rogers (2009). Using MAB to model place-value concepts requires the students to see the relationship between the different blocks; that is, they require an understanding of area to see that the hundred is equivalent to 10 tens, and an understanding of volume to see the thousand block as 10 hundreds. I have observed many students counting to check how many segments there are in a ten block, not trusting or knowing that it is being used to represent 1 ten. I have also observed students who believe the thousand block to be equivalent to 6 hundred as they see the faces as 100 but fail to recognise there would be more cubes 'inside' the block.

Another resource that can be used to represent the multiplicative relationship inherent in place value and to represent the idea that 'ten of these is one of those" described by Siemon et al. (2011) is ten frames. I have used ten frames successfully to teach early place-value concepts and my experience demonstrates that the multiplicative nature of place-value is clearer for students to comprehend with this resource than with MAB. Ten-frames are a frame that has space for ten objects. As the frame is always visible, the relationship to ten can be reinforced whether the frame is full or not. When the frame is full there is no need to count to find the quantity of objects. Siemon et al. (2011) describe how ten-frames can help students gain a sense of ten. Double ten-frames provide a clear representation of teen numbers as 1 ten and extras (see Figure 1).

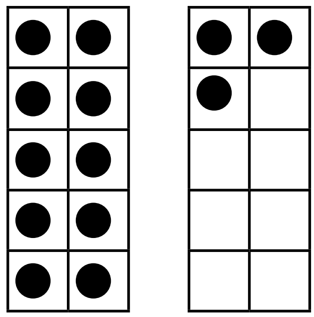

Figure 3. Double ten-frames representing the teen number 13 as 1 ten and 3 ones.

Multiple ten-frames can be used to represent two-digit numbers where the total quantity is visible as well as the multiples of ten place-value structure. Figure 2 shows ten-frames used to represent the number 45 in three different ways (canonical and noncanonical partitioning) and how a digit-only focus showing a 4 and a 5 is clearly not representing the quantity. This material does get cumbersome when representing three-digit numbers although the multiplicative relationship of 1 hundred as being the same as 10 tens can be represented clearly. Additionally, students can benefit from this representation as a support of their understanding of the multiplicative structure of our number system from which further abstract concepts can be built (see Figure 3).

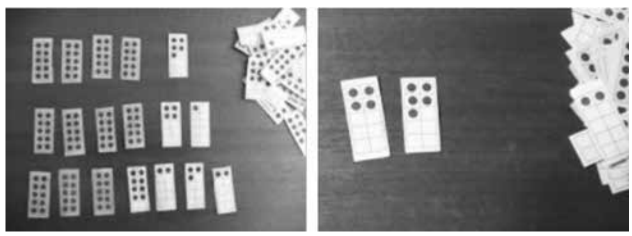

Figure 4. Using ten-frames to represent 45 in many ways and how a digit focus is clearly not 45 dots.

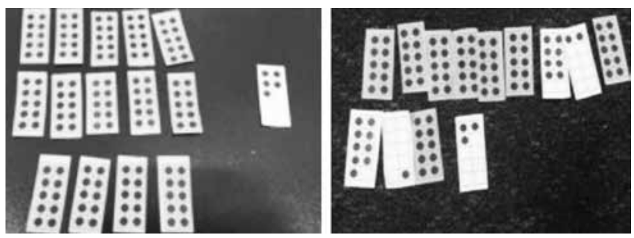

Figure 5. Representing numbers beyond 100 using ten-frames.

Table 2. *Example of a possible lesson heading using the concept list.*

Lesson focus	The place of a digit determines its value
Australian Curriculum	Year 2 ACMNA027 Recognise, model, represent and order numbers to at least 1000
Concept	• The most important place in our number system is the Ones place. As long as we know which digit is in the ones place we can read any number, large or small. • The value of a digit is determined by its place. Without a place-value chart the value of a digit is not known unless the ones place can be identified. If there is only one digit the digit is assumed to be the ones place (e.g. a 4 is assumed to be 4 ones). • Recording a digit in a particular place does not represent the number with the same value until all the places to the ones place have been filled (e.g. 4 in the hundreds place is not the number 4 hundred. It needs zeros in the empty tens and ones places to make the number 400).
Activities	Give each student a three-digit number place value chart with headings H, T and O. Ask: "What do you think the H, T and 0 are short for? (Hundreds, Tens and Ones) "What do you think we are going to put into the columns?" (Many will say numbers. Lead them to realise that they will use digits.) "What is a digit?" (Accept a range of answers and ask student for more information… discuss this important structural aspect of our number system. Digits are used to make numbers. They are the building blocks of numbers…compare to letters for words…letters make words, digits make numbers. We can have one-letter words, and we can have one-digit numbers) "How many digits are there in our number system?" (Ten 0–9) etc. …

Other materials used to represent place value concepts include unifix cubes and bundling sticks which allow the building and unbuilding of tens. With both of these materials the ten is not immediately visible, requiring counting to check if there are 10 cubes or 10 sticks in a ten. Use of a range of representations and materials in a teacher's program is to be encouraged, in conjunction with knowledge of the concept the activity or lesson intends to teach. If the concept is for students to understand that 10 ones is 1 ten then bundling sticks would be a valuable resource. If the concept is to move from counting all to identifying ten without counting then ten-frames would be valuable.

Implementation

The conceptual development sequence described in this paper along with draft sequences for other mathematics topics have been shared with four schools in Queensland. Teachers are using the document to support their planning of mathematics activities. Some teachers are using the conceptual statements as 'learning intentions' as part of a 'visible learning focus' (Hattie, 2012). Others are using the statements as checklist items to guide observational formative assessment. The use of ten-frames to develop early place-value concepts, in particular the multiplicative 10× relationship between adjacent places, appears to be helping students to understand how place value works ahead of discussions about the structure of the whole number system and beyond. Table 2 shows how one school has been using the concept statements to plan and record lessons connected with the Australian Curriculum.

Other teachers have begun using the conceptual statements to differentiate learning experiences by helping them to identify prior or later concepts or concepts that individuals or small groups of students might be missing.

Looking ahead

The development and implementation of these concept lists is a work in progress. The schools that are trialling the lists are using them in a variety of ways as discussed above. Anecdotal observations indicate a positive initial response and that teachers are extending their mathematics subject matter knowledge and pedagogical content knowledge. In this way the lists of concepts are providing personalised professional development and empowering the teachers to plan lessons to focus on what they want students to understand, rather than what they want them to do.

I have recently extended the teacher support for using the place value concept list to include a set of inquiry style lessons like those in Table 2, that focus on concepts from the list. I have also developed a set of place value diagnostic assessments that are currently being trialled. The assessments have items linked to particular concepts on my lists and then link to lessons. The assessments can be used as pre/post-tests to provide formative assessment information and then support for classroom teaching using the lessons and activities. There is potential for more formal data collection and research in the use of these resources based on the conceptual development statements which I hope to gather and report on in the future.

References

Australian Curriculum, Assessment and Reporting Authority (2016). *Foundation to year 10 curriculum: Mathematics*. Retrieved from http://www.australiancurriculum.edu.au/mathematics/curriculum/f-10?layout=1

Baturo, A. (1998). The implication of multiplicative structure for students' understanding of decimal number numeration. In F. Biddulph & K. Carr (Eds.), *Proceedings of the 20th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 90-97). Rotorua, NZ: MERGA

Booker, G., Bond, D., Sparrow, L., & Swan, P. (2010). *Teaching Primary Mathematics* (4th ed.) Frenchs Forest: Pearson Australia.

Borko, H., & Putnam, R. T. (1995). Expanding a teacher's knowledge base: A cognitive psychological perspective on professional development. In T.R. Guskey & M. Huberman (Eds.), *Professional development in education: New paradigms and practices* (pp. 35–65). New York: Teachers College Press.

Carpenter, T., & Fennema, E. (1991). Research and cognitively guided instruction. In E. Fennema, T. Carpenter & S. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 1-16). Albany, GA: State University of New York Press.

Clarke, D., Cheeseman, J., Gervasoni, A., Gronn, D., Horne, M., McDonough, A., Montgomery, P., Roche, A., Sullivan, P., Clarke, B., & Rowley, G. (2002) *Early numeracy research project final report.* Melbourne: Mathematics Teaching and Learning Centre, Australian Catholic University.

Fuson, K. C. (1990). Issue in place-value and multi-digit addition and subtraction learning and teaching. *Journal for Research in Mathematics Education, 28*(2), 130–162.

Hattie, J. (2012). *Visible learning for teachers: Maximising impact on learning*. Oxon: Routledge.

Miura, I. T., & Okamoto, Y. (2003). Language supports for mathematics understanding and performance. In A. J. Baroody, & A. DOwker, (Eds.). *The development of arithmetic concepts and skills.* New Jersey: Lawrance Erlbaum Associates.

Piaget, J. (1952). *The child's conception of number.* New York: W.W Norton.

Price, P. (1998). *Year three students place value misconceptions: Another look at MAB.* Paper presented at the Teaching Mathematics in New times Annual Conference of the Mathematics Education Research Group of Australasia, Gold Coast, QLD. Retrieved from http://www.merga.net.au/publications/counter.php?pub=pub_conf&id= 1803

Rogers, A. (2012). Steps in developing a quality whole number place value assessment for years 3–6: Unmasking the "experts". In *35th Annual Conference of the Mathematics Education Research Group of Australasia (*pp. 1–8). MERGA, Inc. Also in J. Dindyal, L. P. Cheng & S.F. Ng, Mathematics education: Expanding horizons, Proceedings ... (pp. 1-8). Singapore: MERGA.

Rogers, A. (2013). Entering the 'new frontier' of mathematics assessment: Designing and trialling the PVAT-O (Online). In V. Steinle, L. Ball & C. Bardini (Eds.). *Mathematics Education: Yesterday, Today and Tomorrow. Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia* (pp. 586-593). Melbourne: MERGA.

Rogers, A. (2009). Place Value: Working with children at risk in Year 3. In C. Hurst, M. Kemp, B. Kissane, L. Sparrow & T. Spencer (Eds.). *Mathematics: It's mine. Proceedings of the 22nd AAMT Biennial Conference.* Fremantle: Australian Association of Mathematics Teachers.

Ross, S. (1989). Parts, wholes and place value: a developmental view. *The Arithmetic Teacher, 36*(6), 47–51.

Ross, S. (2002). Place value: problem solving and written assessment. *Teaching Children Mathematics, 8*(7), 419–423.

Rubin, A., & Russell, S. (1992). Children's developing concepts of landmarks in the number system. In W. Geeslin & K. Graham (Eds.), *Proceedings of the 16th Annual Conference of the International group for the Psychology of Mathematics Education (Vol 3,* pp. 136–140). Durham, NH: PME

Schmittau, J., & Vagliardo, J. J. (2006). Using concept mapping in the development of the concept of positional system. In *Concept Maps: Theory, Methodology, Technology. Proceedings of the Second International Conference on Concept Mapping*. (pp. 590–597) San José, Costa Rica: Universidad de Costa Rica.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Shulman, L., & Grossman, P. (1988). *Knowledge growth in teaching: A final report to the Spencer Foundation.* Stanford, CA: School of Education, Stanford University.

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R., & Warren, E. (2011). *Teaching mathematics: Foundations to middle years.* South Melbourne: Oxford.

Siemon, D., Izard, J., Breed, M., & Virgona, J. (2006). The derivation of a learning assessment framework for multiplicative thinking. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *30th Conference of the International Group for the Psychology of Mathematics Education* (pp. 113-120). Prague: PME

YuMi Deadly Centre (2014). *YuMi deadly maths: Number.* Prep to Year 9. Kelvin Grove: Queensland University of Technology.