Integrating various fields of mathematics in the process of developing multiple solutions to the same problems in geometry

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Introduction

The solution of problems and the provision of proofs have always played a crucial part in mathematics. In fact, they are the heart and soul of this discipline. Moreover, the use of different techniques and methods of proof in the same mathematical field, or by combining fields, for the same specific problem, can show the interrelations between the fields, as well as the richness, beauty and elegance of mathematics.

Mathematics educators agree that linking mathematical ideas by using more than one approach to solving the same problem (e.g., proving the same statement) is an essential element in the development of mathematical reasoning (Polya, 1973; Schoenfeld, 1985; NCTM, 2000). Problem-solving in different ways requires and develops mathematical knowledge (Polya,1973), and encourages flexibility and creativity in the individual's mathematical thinking (Krutetskii, 1976; Silver, 1997; Tall, 2007; Liekin & Lev, 2007).

In addition to the specific roles of proof in mathematics, we suggest that attempts to also prove a certain result (or solve a problem) using methods from several other different areas of mathematics (geometry, trigonometry, analytic geometry, vectors, complex numbers, etc.) are very important in developing deeper mathematical understanding, creativity, and appreciation of the value of argumentation and proof in learning different topics of mathematics. Our approach, that of presenting multiple proofs to the same problem, as a device for constructing mathematical connections is supported by (Polya,1973,1981; Schoenfeld, 1988; NCTM, 2000; Ersoz, 2009; Levav-Waynberg & Leikin, 2009).

Very similar to our notion of 'One problem, multiple solutions/proofs' is the idea of multiple solution tasks (MST) presented by (Liekin & Lev, 2007; Levav-Waynberg & Leikin, 2009; Liekin, 2009). MSTs contain an explicit requirement for proving a statement in multiple ways. The differences between the proofs are based on using:

1. different representations of a mathematical concept;
2. different properties (definitions or theorems) of mathematical concepts from a particular mathematical topic;
3. different mathematics tools and theorems from different branches of mathematics;
4. different tools and theorems from different subjects (not necessarily mathematics);
5. different strategies of problem solving.

In our case, we apply the third type of differences between the proofs; we shall present various solutions to the problems using the tools and theorems of Euclidean geometry, analytic geometry, trigonometry, and vectors. Adding the concept of multiple solutions/proofs for one problem into the curriculum of mathematics studies, as well as MSTs, allows the development of connected mathematical knowledge not only for students but for their teachers as well.

Based on many years of experience in teaching mathematics in high school, in academic courses and from the process of constructing and training pre-service teachers, we have established the major importance of possessing comprehensive command of different mathematical tools, which permit one to deal with complex problems and with challenges that require use of various solution strategies.

Reasoning by using a dynamic geometry software (DGS)

The introduction of dynamic geometry software (such as GeoGebra) into classrooms creates a challenge to the praxis of theorem acquisition and deductive proof in the study and teaching of Euclidean geometry. Students/learners can experiment through different dragging modalities on geometrical objects that they construct, and consequently infer properties, generalities, and conjectures about the geometrical artefact. The dragging operation on a geometrical object enables students to apprehend a whole class of objects in which the conjectured attribute is invariant, and hence, the students become convinced that their conjecture will always be true (De Villiers, 1998; Leung & Lopez-Real, 2002).

Student activity

In the framework of a course in a combination of subjects taught to pre-service teachers in mathematics, at the end of the school year we presented two geometric problems to a group of 20 students. Each student was asked to solve these problems using a large number of methods. It was suggested for them to solve it using: geometry of the plane, trigonometry, analytical

geometry, vectors and complex numbers. Each of them was required to solve it independently, and to try to find several methods. Indeed, many proofs were found, part of them were completely different and used different mathematical tools.

The students studied the beautiful geometrical property of the second problem, by using dynamic geometric software. The applet enabled dragging objects that caused the size of segments to be changed while conservation the property (Ben-Chaim, Katz & Stupel, 2016).

After presenting the problems we shall give some of the solutions found by the students.

Problem 1

Presentation of the problem

Given is the square ABCD, whose side length is a. Through vertexes A and B we draw straight lines, which form an angle of 15° with the side AB. The lines intersect at point O (see Figure 1).

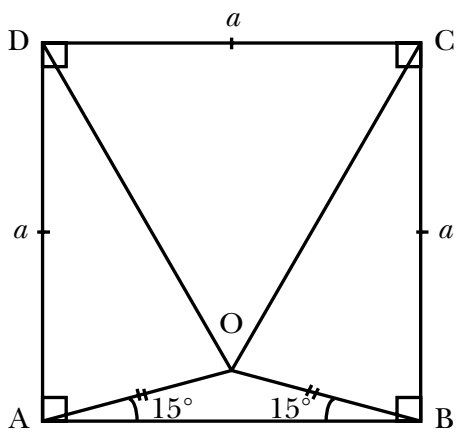

Figure 1

Prove that the triangle ΔDOC is equilateral.

Method A

The triangles ΔDOA and ΔCOB are congruent according to the Side–Angle–Side theorem. From the congruence it follows that OD = OC. On the side AB we build the equilateral triangle ΔABE directed outwards (see Figure 2). The quadrilateral AOBE is a kite, therefore its diagonals, which intersect at point F, are perpendicular. From angle calculations, ∠EAO = ∠AOE = 75°, therefore EA = EO = AD = a.

DA||OE (due to their perpendicularity to AB). Therefore the quadrilateral DOEA is a parallelogram (due to two equal and opposite sides), but since in

this parallelogram there are two equal adjacent sides (DA = EA = a), it is a rhombus. Hence also DO = a, and the triangle is equilateral.

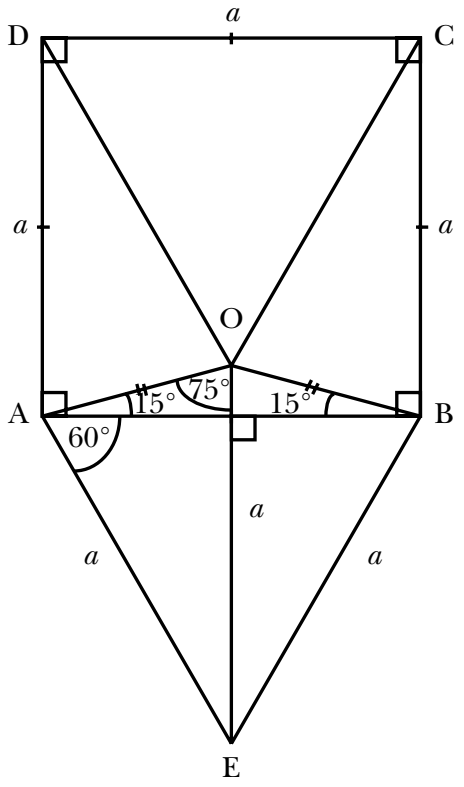

Figure 2

Note: Above we signed that OD = OC and so the triangle ΔDOC is isosceles. Therefore it is enough to prove that one of its angles is 60°. This fact is used later in methods B, D, and F.

Method B

Through the vertices B and C we draw straight lines that form an angle of 15° with the side BC. The lines intersect at point O'. See Figure 3.

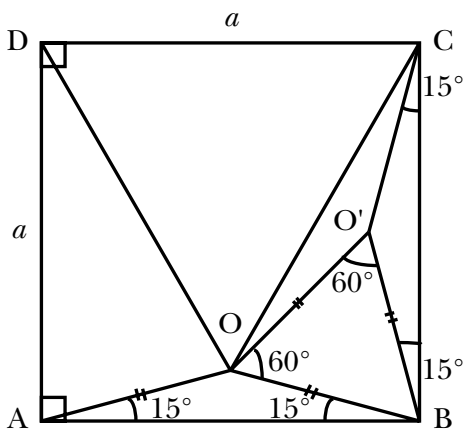

Figure 3

Then the isosceles triangles ΔAOB and ΔBO'C are congruent according to the Angle–Side–Angle theorem, therefore BO = BO'. ∠OBO' = 90° – 15° – 15° = 60°. So the triangle ΔBOO' is equilateral and OO' = O'B = O'C, ∠OO'B = 60°. By calculating the angles, we obtain: ∠BO'C = 150°, ∠OO'C = 360° – 60° – 150° = 150°, ∠OCO' = 15°. It follows that ∠OCB = 30°, and therefore, ∠DCO = 60° and the triangle ΔDOC is equilateral.

Using this method the proof can be presented in the style of a proof without words (PWW; see Figure 4).

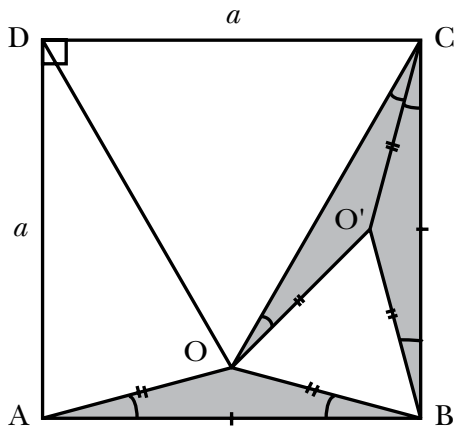

Figure 4

Method C

We present an indirect proof. The triangle ΔDCO' is equilateral. Hence, the angle ∠O'CB = 30° is a vertex angle in the isosceles triangle ΔO'CE (see Figure 5).

Therefore, ∠CBO' = 75°. Since the angle at the vertex B is of 90°, this requires that the vertices O and O' coincide—in contradiction to the datum, i.e., ΔDOC is equilateral.

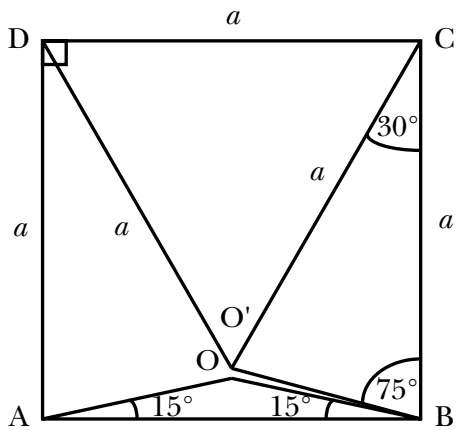

Figure 5

Method D

We draw the diagonals of the square, and denote by E their point of intersection. The continuation of AO intersects the diagonal at the point F and the side BC at the point G (see Figure 6). By calculating the angle, we obtain ∠CAO = 30°. We connect point F with the vertex C. FD is a mid-perpendicular to the diagonal AC.

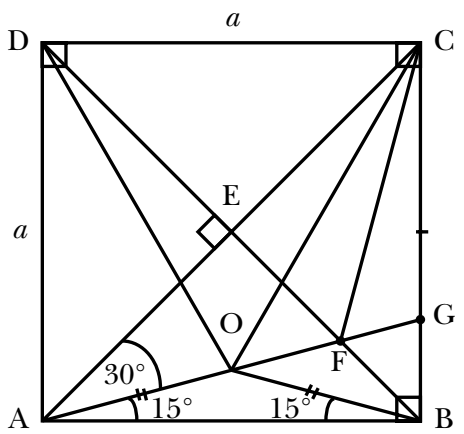

Figure 6

From the properties of a mid-perpendicular, we obtain:

$$\angle EAO = \angle ECF = 30°;\ \angle EFA = \angle EFC = 60°.$$

From a calculation of the angles we obtain:

$$\angle CFO = \angle CFB = 120°,\ \angle FCB = 15°,\ \angle FOB = \angle FBO = 30° \Rightarrow FO = FB$$

The triangles: ΔCFO and ΔCFB are congruent according to the Side–Angle–Side theorem. Hence ∠OCF = ∠FCB = 15° and ∠OCB = 30° ⇒ ∠OCD = 60°, therefore the triangle ΔODC is equilateral.

Method E: Proof by trigonometry

We denote by FE the mid-perpendicular to the sides AB and DC (see Figure 7).

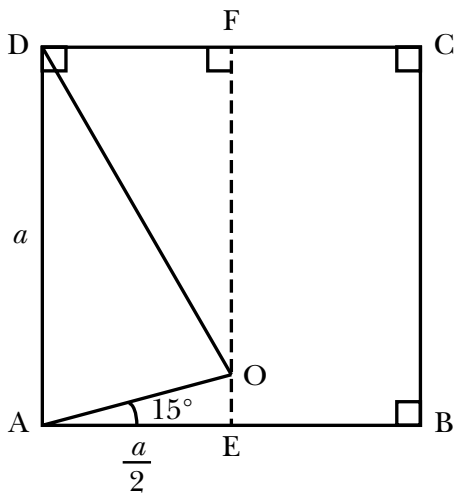

Figure 7

Since $$\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$$

we obtain $$\tan 15^\circ=\frac{1-\cos 30^\circ}{\sin 30^\circ}=\frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}=2-\sqrt{3}$$

From triangle ΔAOE $$\tan 15^\circ=\frac{OE}{AE}$$

$$\Rightarrow OE=\frac{a}{2}\cdot\tan 15^\circ=a\cdot\frac{2-\sqrt{3}}{2}$$

$$\Rightarrow FO=a-OE=\frac{a\sqrt{3}}{2}$$

Then $$\begin{aligned}DO^2&=DF^2+FO^2\\&=\left(\frac{a}{2}\right)^2+\left(\frac{a\sqrt{3}}{2}\right)^2\\&=a^2\\\Rightarrow DO&=a\end{aligned}$$

So the triangle ΔDOC is equilateral.

Method F: Proof using analytic geometry

We consider the square ABCD in a Cartesian coordinate system (see Figure 8).

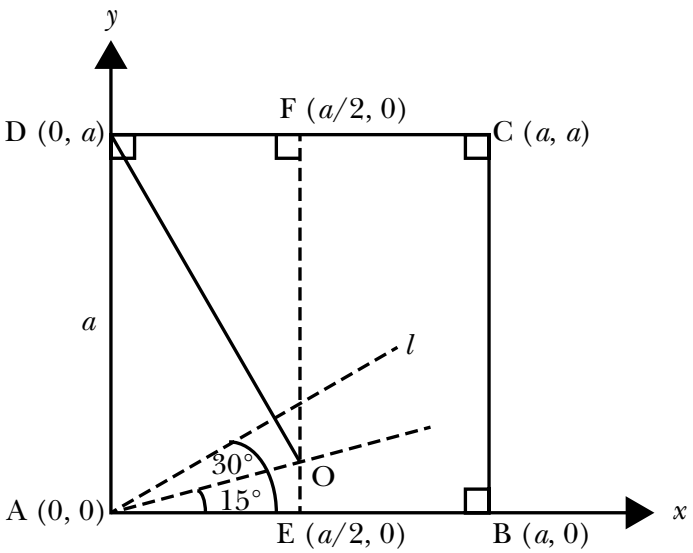

Figure 8

FE is the mid-perpendicular to the sides AB and DC. Through vertex A we draw the straight line l which forms an angle of 30° with the side BC. Then the equation of l is

$$y = \frac{\sqrt{3}}{3}x \text{ or } \sqrt{3}x - 3y = 0$$

The equation of the straight line AB is $y = 0$. Then according to the formula of the bisectors of the angles between two straight lines, one can easily obtain the equation of straight line AO: $y = \left(2-\sqrt{3}\right)x$. So the coordinates of point O are:

$$\left(\frac{a}{2}, \frac{2-\sqrt{3}}{2}a\right)$$

Therefore

$$\mathrm{OF} = a - \frac{2-\sqrt{3}}{2}a = \frac{a\sqrt{3}}{2}$$

$$\Rightarrow \tan\angle\mathrm{FDO} = \frac{a\sqrt{3}}{2} : \frac{a}{2} = \sqrt{3}$$

$$\Rightarrow \angle\mathrm{FDO} = 60^{\circ}$$

Problem 2

Presentation of the problem

Given is a square ABCD whose side length is a. A straight line is drawn through the vertex A, which intersects the side DC at the point E. The bisector of the angle $\angle$EAB is drawn through the vertex A, which intersects the side BC at the point F (see Figure 9).

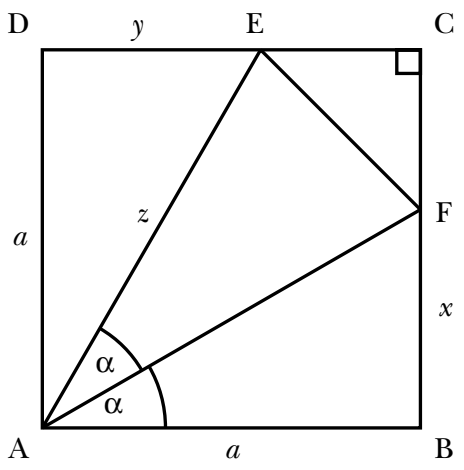

Figure 9

Prove that BF + DE = AE ($x + y = z$).

At the first stage the students were given an applet which allowed the point E is dragged along the side BC, including the possibility of dragging the point outside the square to the right of the vertex C. During each stage

of dragging the lengths the segments and the sum of the relevant segment lengths appear on screen. The experience with the applet shows that at each stage the properties are conserved. It is possible to reach the applet using the following link: https://www.geogebra.org/m/VnpDPzg4

We denote: $\angle EAB = 2a$, $BF = x$, $DE = y$, $AE = z$.

Method A

From point F we drop a perpendicular FG to the segment AE (see Figure 10). From the property of a point on an angle bisector, we have $BF = FG = x$.

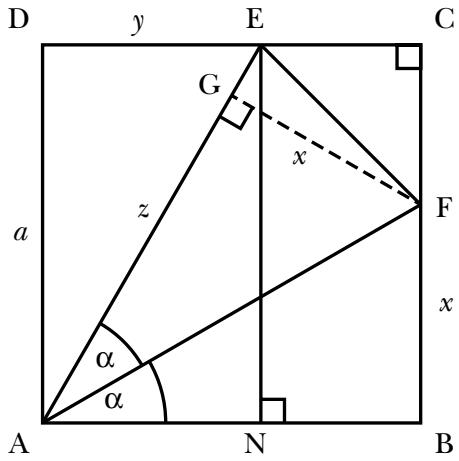

Figure 10

We calculate the area of the square ABCD from the areas of the four triangles from which it is composed:

$$S_{ABCD} = S_{\Delta ABF} + S_{\Delta FCE} + S_{\Delta EFA} + S_{\Delta ADE}$$

$$a^2 = \frac{a \cdot x}{2} + \frac{(a-x)\,(a-y)}{2} + \frac{z \cdot x}{2} + \frac{a \cdot y}{2}$$

After simplification we obtain $a^2 = x(y + z)$.

By using the Pythagorean theorem in the triangle ADE, we have $a^2 = z^2 - y^2$.

By substituting this relation we obtain $z^2 - y^2 = x(y + z)$,

i.e., $(z - y)(z + y) = x(y + z)$.

After cancelling $(y + z)$ from both sides, we obtain $x + y = z$.

Method B

From the point E we drop a perpendicular EN to the side AB. We denote by M the point of intersection of this perpendicular with the angle bisector AF (see Figure 10). By using the angle bisector theorem in the triangle AEN, we obtain:

$$\frac{z}{y} = \frac{EM}{MN} \tag{1}$$

The triangles AMN and AFB are similar, therefore we can write the proportion:

$$\frac{y}{a} = \frac{\text{MN}}{x}$$

$$\Rightarrow \text{MN} = \frac{x \cdot y}{a} \tag{2}$$

$$\text{EM} = a - \text{MN} = \frac{a^2 - xy}{a} \tag{3}$$

We substitute the relations (2) and (3) in (1), to obtain $a^2 = x(y + z)$.
Using the Pythagorean theorem in the triangle ADE, we obtain $x + y = z$.

Method C

Proof based on the properties of the angle bisector and the Pythagorean Theorem (see Figure 11).

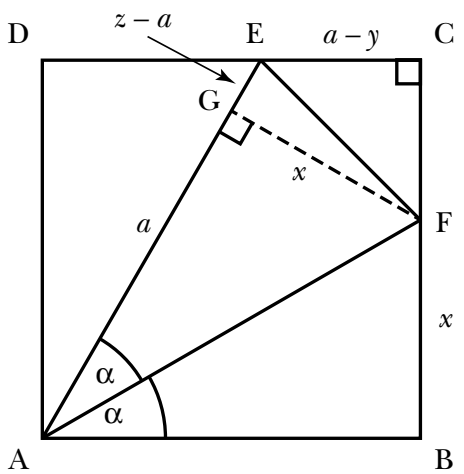

Figure 11

From the properties of the angle bisector:

$$\text{AB} = \text{AG} = a, \text{FB} = \text{FG} = x$$

From the Pythagorean Theorem in the triangle ECF:

$$\text{EF}^2 = (a - y)^2 + (a - x)^2$$

From the Pythagorean Theorem in the triangle EGF:

$$\text{EF}^2 = (z - a)^2 + x^2$$

From equating the values of EF^2, we obtain:

$$(z - a)^2 + x^2 = (a - y)^2 + (a - x)^2$$

From the Pythagorean theorem in the triangle EDA:

$$a^2 = z^2 - y^2$$

and by cancelling terms, we obtain $z = x + y$.

Method D

We copy ΔABF in such a manner that the side AB coincides with the side AD (see Figure 12).

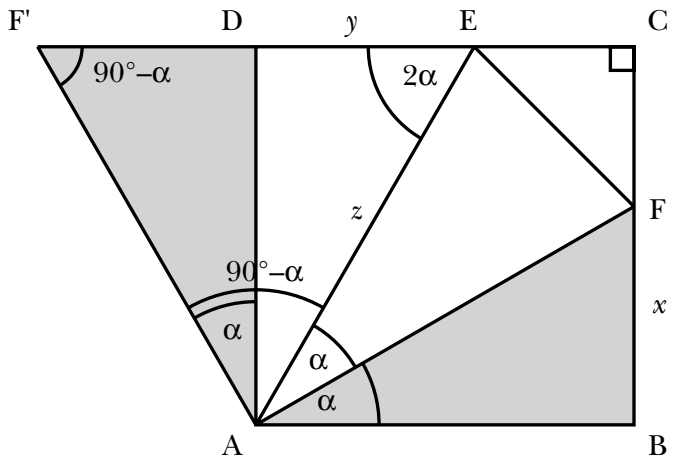

Figure 12

The triangle ΔF'EA is an isosceles triangle since it has two equal angles:

$$\angle EF'A = \angle EAF' = 90° - \alpha$$

Conclusion: $EF' = EA \Rightarrow x + y = z$

Method E: Proof by trigonometry

From triangle ΔAFB: $x = a \cdot \tan a$

From triangle ΔADE: $y = a \cdot \cot 2a$

$$z = \frac{a}{\sin 2\alpha}$$

It remains to prove that:

$$\frac{a}{\sin 2\alpha} = a \cdot \cot 2\alpha + a \cdot \tan\alpha$$

We multiply both sides by sin 2α and obtain:

$$\begin{aligned} a &= a \cdot \cos 2\alpha + \frac{a \cdot \sin\alpha \cdot 2 \cdot \sin\alpha \cdot \cos\alpha}{\cos\alpha} \\ &= a \cdot \left(1 - 2\sin^2\alpha\right) + \frac{a \cdot \sin\alpha \cdot 2 \cdot \sin\alpha \cdot \cos\alpha}{\cos\alpha} \\ &= a \end{aligned}$$

Method F

We use the rules of analytic geometry. We locate the square in a system of coordinates in such a manner that the vertex A coincides with the origin and the sides AB and AD lie on the axes x and y, respectively (see Figure 13).

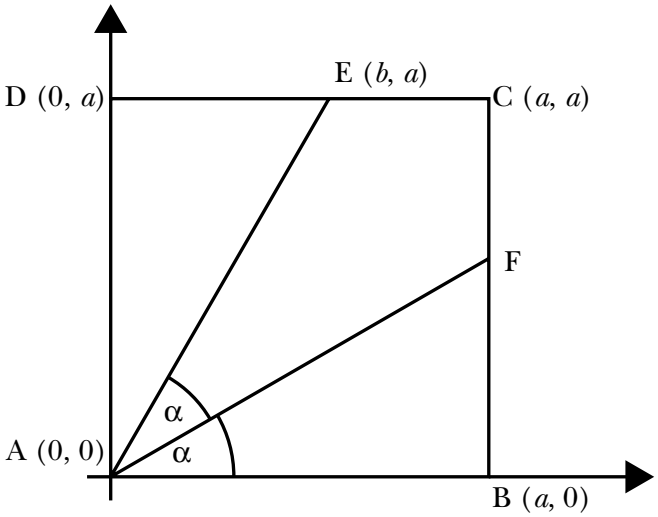

Figure 13

The coordinates of the vertices are A (0, 0), B (a, 0), C (a, a), D (0, a).
We select a point E on the side DC, whose coordinates are E (b, a), such that $b < a$.
The equation of the side AB is $y = 0$.
The equation of the line AE is $y = \frac{a}{b}x$.
Two straight lines are given by their general equations:

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

The equation of the angle bisectors are:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

In accordance with the general equations of the lines AF and AB:

$$\begin{cases} ax - by = 0 \\ y = 0 \end{cases}$$

we obtain that the equations of their angle bisectors are:

$$\frac{ax - by}{\sqrt{a^2 + b^2}} = \pm y$$

The explicit equation of the angle bisector ∠EAB (the one with the positive slope that corresponds to the straight line AF) is:

$$y = \frac{a \cdot x}{b + \sqrt{a^2 + b^2}}$$

Therefore the coordinates of the point F are:

$$\left(a,\ \frac{a^2}{b+\sqrt{a^2+b^2}}\right)$$

From the coordinates of the vertices A, B, C, D, E, F we obtain the lengths of the segments:

$$BF = \frac{a^2}{b+\sqrt{a^2+b^2}}$$
$$AE = \sqrt{a^2+b^2}$$
$$DE = b$$

Then

$$\begin{aligned} BF + DE &= b + \frac{a^2}{b+\sqrt{a^2+b^2}} \\ &= b + \frac{a^2(b-\sqrt{a^2+b^2})}{(b+\sqrt{a^2+b^2})(b-\sqrt{a^2+b^2})} \\ &= b - (b+\sqrt{a^2+b^2}) \\ &= \sqrt{a^2+b^2} = AE \end{aligned}$$

Method G: Proof using vectors

We denote: $\overrightarrow{AD} = \underline{u}$, $\overrightarrow{AB} = \underline{v}$, $|\underline{u}| = |\underline{v}| = a$ (see Figure 14).

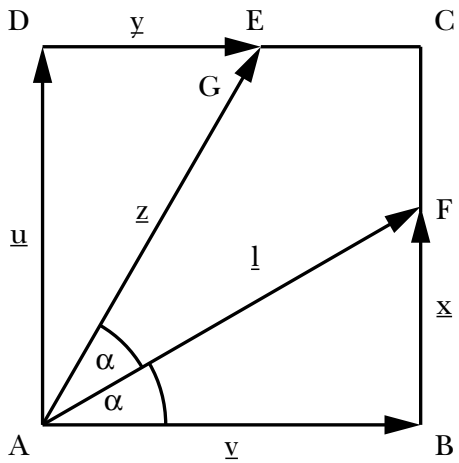

Figure 14

$\underline{z} = \underline{u} + \underline{y}$, $\underline{l} = \underline{v} + \underline{x}$, $\underline{z} \cdot \underline{l} = \underline{u} \cdot \underline{v} + \underline{y} \cdot \underline{v} + \underline{u} \cdot \underline{x} + \underline{y} \cdot \underline{x}$

$\underline{u} \cdot \underline{v} = \underline{y} \cdot \underline{x} = 0$ (since the vectors are perpendicular).

$\underline{y} \cdot \underline{v} = |\underline{y}| \cdot |\underline{v}| = y \cdot a$ (since the vectors are parallel).

$\underline{u} \cdot \underline{x} = |\underline{u}| \cdot |\underline{x}| = x \cdot a$ (since the vectors are parallel).

Hence: $\underline{z} \cdot \underline{l} = |\underline{z}| \cdot |\underline{l}| \cdot \cos\alpha = z \cdot l \cdot \cos\alpha = z \cdot a$

And hence: $z \cdot a = x \cdot a + y \cdot a \Rightarrow z = x + y$

Method H: Proof using complex numbers

We place the square in a system of coordinates, where the horizontal axis is real and the vertex A is at the origin (see Figure 15).

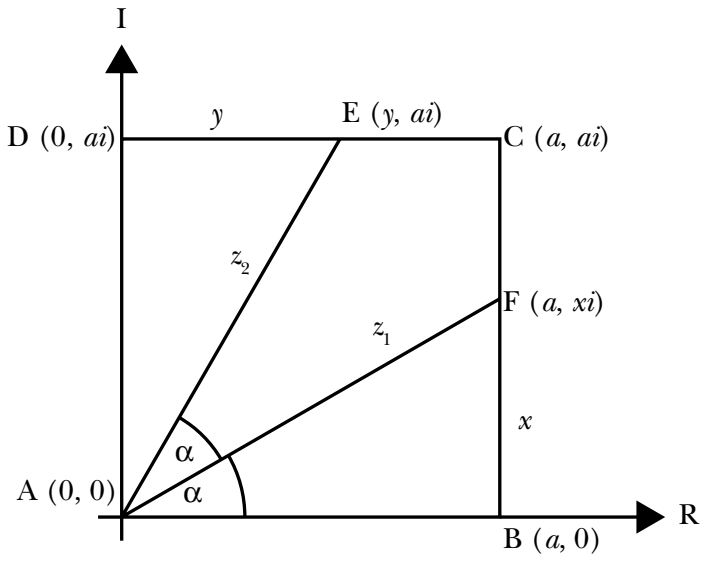

Figure 15

$z_1 = a + xi$, $z_2 = y + ai$, $z_1^2 = a^2 - x^2 + 2axi$

Since z_2 and z_1^2 both have an angle of 2α, one can write down: $z_1^2 = k \cdot z_2$, where k is real.

We equate the expressions: $a^2 - x^2 + 2axi = k(y + ai)$

and obtain two equations: $a^2 - x^2 = ky$ (1)

$2ax = ka \Rightarrow k = 2x$ (2)

We substitute $k = 2x$ in the first equation and obtain $a^2 - x^2 = 2xy$.

Since $a^2 = z^2 - y^2$, then $z^2 - y^2 - x^2 = 2xy$ and $z = x + y$.

Methodical discussion with the students

Towards the end of the activity the students presented the different solutions to the entire group. A methodical discussion was held on each solution from different aspects, such as: the degree of difficulty of the solution method, the elegance and simplicity of the solution, the creativeness of the solution method, integration between different fields in mathematics and their tools, the extent to which the solutions match the knowledge of the students and the program of studies. Subsequently we presented to the students the concept of 'conservation', based on the conserved property of the sum of the lengths of two segments along a third segment by shifting the point E on the side DC.

From this point on, a discussion was held concerning the importance of finding several methods of solution of the same task, the fact that in each mathematical theorem, and especially in Euclidean geometry, there is a

single property for several properties that are conserved. A comprehensive discussion was held on the geometry conservation properties. By applying GeoGebra applets, other examples of conservation properties were presented and has been emphasized their contribution to solving problems in geometry (Ben-Chaim, Katz & Stupel, 2016).

In addition, a discussion was held on the importance of using the computerised technological tool for testing the correctness of different hypotheses.

The distribution of the different solutions of the students

The distribution of the number of proofs found by the students for problem 2 (for which were suggested more various methods of solution) appears in Table 1.

Table 1

Number of different proofs*	1	2	3	4	5	6
Number of students	2	8	5	3	2	1

* The overall number of different proofs in the same mathematical field.

The distribution of the types of proofs by the mathematical field found by the students for problem 2 appears in Table 2.

Table 2

Mathematical field of the proof	Geometry	Trigonometry	Analytical geometry	Vectors	Complex numbers
Number of students	14	9	3	2	1

Conclusions from the data in the tables

1. Only six students managed to solve the task using four methods or more.
2. Most of the students (70%) solved the task using the geometrical method.
3. 45% of the students solved the task using a trigonometric method.
4. A small number of the students solved the task using analytical geometry, vectors and complex numbers.

Feedback from the students

- The different proofs found for the task reinforce the perception of mathematics as a field built from different intertwined branches.
- At a first glance this seems to be a task in geometry, and without the requirement of solving it using traditional methods we wouldn't have tried to solve it using analytical geometry, vectors and complex numbers.

- The proof that the students liked the most, of those shown at the end of the activity, is method D (by rotating a triangle), followed by method A (by calculating areas).
- There is much similarity between the proofs using vectors and the proof using complex numbers.

Summary

The authors believe that this activity suggests that the students need to absorb the importance of solving tasks using several different methods.

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