Using magic squares to teach linear algebra

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Introduction

This article summarises activities that happened during the first three weeks of a fictitious high-school-level linear algebra section that used magic squares as a teaching tool to inspire students to further investigate the topics. The author has been working with students from high school and college levels for years, and although this situation can be considered very realistic, it was not based on a single classroom, it is a product of imagination based on actual experience.

This study focuses in the development of activities related to linear algebra with use of technology. An active learning method was used as the main strategy to motivate discovery. It aligns with the Australian Curriculum not only for the encouragement of using computers and calculators but also for helping the student understand the concepts and techniques in matrices and apply reasoning skills to solve problems with matrices, which are learning outcomes from Unit 2 Specialist Mathematics.

Active learning

Educators have embraced innovation to better serve a constantly changing student population. Methods that once seemed to be the only rule, now are being challenged. With interactive technology available at virtually any daily task, students seem to not respond well to the outdated passive learning method anymore. Robert Zemsky [16] states that the current undergraduate structure is outdated and inadequately services the present and changing student demographic.

Prince (2004) defines active learning as “any instructional method that engages students in the learning process.” This definition, in theory, includes many traditional classroom activities, such as instructor-centred lectures. However, in general, modern active learning processes commonly involve peer-assisted and problem-based learning approaches. Michael (Michael, 2006) highlights the success for active learning, especially the flipped-classroom method. This approach requires students to gain knowledge before going to class, where with peer and faculty support a higher level of cognitive work can be performed.

Bonwell and Eison (1991) conclude that active learning leads to better students' attitudes and improvements in students thinking and writing. More anecdotal stories seem to support the claim that students learn best when they engage with course material and actively participate in their learning.

Our method of instruction follows the standard practices in most recent articles. In order to develop competence in a topic, research suggests that students need to:

- have factual knowledge;
- understand facts and ideas in the context of a conceptual framework;
- organise knowledge in ways that facilitate retrieval and application. (Bransford, 2004)

Students in our linear algebra class will achieve this by the following elements:

- gaining first exposure prior to class: with YouTube videos, pre-lesson assignments, or reflection upon a topic;
- receiving incentive to come well-prepared to class: part of students' semester grade is dependent on how well they are prepared for the time in classroom;
- performing in-class activities that focus in higher-level cognitive learning: once students are well-prepared, class time is used to promote deeper learning and to increase their skills at using new material.

About the course

This special and optional linear algebra section was aimed at senior students. Class met for three hours a week, divided into two meetings of 75 minutes each time. It mocked a college-level class structure and lasted 15 weeks during a spring semester. There were 26 students registered, all of them willing to have a college degree: 8 saying they will pursue math major, 16 computer science major, and 2 preferring a Biology program.

Demographic data was not available, but students were in majority white and Hispanic.

Terenzini et al. (2001) shows that students taught in a way that incorporates small-group learning procedures generally achieve better knowledge retention than students working by themselves. Hence, during the very first day of classes, students were assigned into six different groups, ranging from four to five members.

For privacy, names shall be omitted. In our report, when referring to the different student groups, we use letters: A, B, C, D, E, and F. And to the students as numbers within the group: A1, A2, and so on.

Although the class requires a textbook, during the first weeks of classes, students were encouraged not to use it. The magic square theme was used to assist the delivery of the first two chapters only: Systems of Linear Equations and Matrix Algebra. Later chapters in the course were: Vector Spaces, Inner Products, Linear Transformations, and Eigenvalues and Eigenvectors.

About the 'magic module'

The goal of starting the course with the 'magic module' is a way of introducing several linear algebra concepts through an effective approach so students would develop solid ideas and intuitions regarding the material. By no means the module may substitute the formal and precise approach to linear algebra required by a course at this level.

After the completion of the three-week introductory module, students would dedicate more time to proofs, and more mathematical precision in manipulating matrices, and in how to compute rank, determinants, and more.

Mathematical concepts

For the first weeks, magic squares were used as a theme to motivate deeper inquiries regarding some linear algebra concepts, such as solutions of systems of linear equations, and some applications of key characteristics of a matrix (specifically, rank and inverse).

The following material was only presented to the students after our active learning activities.

Magic squares

Magic squares have fascinated mathematicians for centuries. The earliest known magic square was a 3 by 3 one recorded around 2800 BC in China. Fuh-Hi described the "Loh-Shu", or "scroll of the river Loh". Since then, many people in many nations have enjoyed, studied, and recorded magic squares.

Recently, several research papers have been published in questions involving magic squares. For instance, Benjamin and Yasuda (1999) had studied interesting properties for squaring numbers formed on the magic squares. Benjamin also published an article on a magazine for magicians where he teaches how to create four by four magic squares based on a volunteer's birthday (Benjamin, 2006). Other studies relate magic squares with: dominoes (Springfield & Goddard, 2009), vector spaces (Ward, 1980), weights and centre of mass (Behforooz, 2012), and even the US election (Behforooz, 2009).

A magic square is a display of numbers in a square form having nice mathematical properties. Normally, the sum of all numbers in any row or in any column or in any diagonal is always the same number. One other 'nice' property that some magic squares may show is having all numbers different, in fact, some will even display consecutive integers starting at 1. Some magic squares may have other collections of numbers whose sum adds up to the same specific number. Or, in a variation of the problem, a volunteer may specify a number that the rows/columns/diagonals/special areas of the magic square will be equal to (in this case, we may have to repeat numbers within the square).

Let us think of an $n \times n$ square and answer some key questions.

Problem 1

How many numbers will it contain?

Answer 1

Since there are n rows and n columns, there are n^2 numbers in this magic square.

Problem 2

If we display the consecutive numbers 1, 2, … n^2, what is the expected sum of each row/column/diagonal?

Answer 2

The total sum of all numbers is $S = 1 + 2 + \ldots + n^2$. Let's use a small trick to determine a nice formula for it:

$$\begin{aligned} S &= 1 + 2 + \ldots + (n-1)^2 + n^2 \\ &= n^2 + (n^2 - 1) + \ldots + 2 + 1 \end{aligned}$$

Hence: $2S = (n^2 + 1) + (n^2 + 1) + \ldots + (n^2 + 1) + (n^2 + 1)$. So, $2S = (n^2 + 1)\,n^2$.

Thus $$S = \frac{(n^2+1)n^2}{2}$$

This expression gives the sum of all numbers.

If we expect each of the n rows to have the same sum, then the number must be

$$t = \frac{S}{n} = \frac{(n^2+1)n}{2}$$

Hence for a 3×3 magic square, the sum will be $t = \frac{(3^2+1)\times 3}{2} = 15$. For a 4×4 magic square $t = 34$. And for a 5×5, $t = 65$.

System of equations

The first topic the class covered was solving systems of linear equations. For example, how we would find values for x, y and z that solve the following three equations simultaneously:

$$\begin{cases} x+2y-z=-2 \\ 2x+y+4z=8 \\ 3x+5y-3z=-4 \end{cases} \qquad \text{(System 1)}$$

The process is completed by rewriting the problem with matrices.

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 3 & 5 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \\ -4 \end{pmatrix}$$

The first matrix A is called the matrix of coefficients. The second matrix X is called the variable vector (a vector is simply a matrix having only one column). Finally, the matrix on the right-hand side can be simply called the right-hand vector, and we name it vector B. So the representation of this problem is:

$$A \cdot X = B$$

One strategy to solve is to find the inverse matrix A^{-1} (if it exists) and perform a left-multiplication on both sides (matrix multiplication is not commutative, that is, order is important, the matrix product $E \cdot F$ may differ from the product $F \cdot E$): $A^{-1} \cdot A \cdot X = A^{-1} \cdot B$. So the result is $X = A^{-1} \cdot B$.

Rank of a matrix

In order to understand the concept of rank of a matrix, we need to set some language:

Definition 1

A set of vectors is linearly independent if no vector in the set is:

- a scalar multiple of another vector in the set; or
- a linear combination of other vectors in the set, that is, if one vector is equal to the sum of scalar multiples of other vectors.

The rank of a matrix is the maximum number of linearly independent rows. In practice, the rank of an $n \times m$ matrix will be n if and only if all rows are linearly independent. That is, if there is no row that can be written as a linear combination of the others.

In a system of equations, each equation can be seen as a piece of information. Specifically, the left-handed side of each equation will provide a different row in the coefficient matrix, A. If a row can be written as a linear combination of the others, then in practice it means that a piece of information is being somehow repeated.

Take, for instance, the previous system and add a fourth equation $6x + 8y = 2$:

$$\begin{cases} x+2y-z=-2 \\ 2x+y+4z=8 \\ 3x+5y-3z=-4 \\ 6x+8y=2 \end{cases} \qquad \text{(System 2)}$$

It looks like that the new equation adds a new piece of information. However, it does not, the fourth equation is simply the addition of all initial three equations:

Equation 4 = Equation 1 + Equation 2 + Equation 3

So technically, this fourth equation is redundant.

The following result is valid. It is mentioned in the introductory module, but its proof is only presented later in the course.

Theorem 1

Let A be an $n \times n$ matrix, X be a vector with n variables, and B be a vector with n numbers. Then the system of equations represented by $A \cdot X = B$ will have a unique solution if and only if $\text{rank}(A) = n$.

Inverse of a matrix

Our approach to solve a linear system of equations relies on finding the inverse of a matrix. Let's start with some definitions (remember that the main diagonal is the one where the row position is the same as the column position).

Definition 2

A $n \times n$ matrix is called the $n \times n$ *identity* if and only if all entries are zero, except on the main diagonal, which displays only ones.

This name is consistent with its main property: an identity matrix times any other matrix will be equal to the given matrix, provided the multiplication can be done.

Not all matrices have inverse, only the ones having non-zero determinant. We will avoid the formal definition of the determinant (that implies notions of permutations) for now and we will concentrate instead on its use.

Only square matrices (the same number of rows and columns) may have inverse. An equivalent way of determining the ones with inverse is stating that, from all $n \times n$ matrices, only the ones with 'rank' equal to n will have inverse.

The process of finding the inverse of a matrix can be elaborated. For our initial goals, we may use computational tools that will find them.

An example

Consider the system:

$$x + 2y - z = -2$$
$$2x + y + 4z = 8$$
$$x + 3y + 3z = 6$$

Although it looks like there are three pieces of information (each equation may be seen as a mathematical piece of information), a quick analysis reveals that the third equation is not a 'new' piece of information, since that equation can be the result of adding the first and second equations. This equation can be considered redundant, so it may be deleted. If we rewrite the system as $A \cdot X = B$, we conclude that the 'rank' of the matrix A is, then, not three. The remaining two equations are independent. Hence, the 'rank' is two. That is, there are only two pieces of information.

Since, we are trying to solve for three variables, according to theorem 1 having the rank of A equal to 2 would not be enough to get a unique solution. A way to overcome this issue would be giving a value for any of the variables as a third piece of information. That is, we would substitute an equation that is the result of manipulation of others by an equation that simply gives a value to a variable. For instance, if we say that $x = 4$, then instead of the third equation, it would be $x + 0y + 0z = 4$, and the third row of matrix A would be [1 0 0].

The rank of

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 1 & 0 & 0 \end{pmatrix}$$

is three, so this problem now has unique solution.

Theorem 2

Let A be a square matrix $n \times n$ whose rank is exactly equal to n, then the inverse matrix of A exists, represented by A^{-1}. Furthermore, if X is a vector of n variables, and B is a vector with n numbers, then the system of equations represented by $A \cdot X = B$ has a unique solution given by $X = A^{-1} \cdot B$.

Technology

There are several calculators, software, apps, websites, that would work with matrices. For instance, WolframAlpha (www.wolframalpha.com) may be useful.

Now going back to the first system of equations of this chapter, by typing "rank{{1, 2, −1}, {2, 1, 4}, {3, 5, −3}}" in WolframAlpha, for example, we certify that the rank is 3, so we may find unique solutions for the variables. We know the result is $X = A^{-1} \cdot B$, where A is the matrix of coefficients (this one we just computed the rank), and B is the right-hand side. The solution is $x = 3$; $y = -2$; $z = 1$.

The problem

3 × 3 magic square

From the above discussion, we want to display numbers 1, 2… 9 in such way that all rows, all columns and both diagonals have sum equal to 15:

a	b	c
d	e	f
g	h	i

The equations are:

$$\begin{aligned}
a + b + c &= 15 && \text{(first row)}\\
d + e + f &= 15 && \text{(second row)}\\
g + h + i &= 15 && \text{(third row)}\\
a + d + g &= 15 && \text{(first column)}\\
b + e + h &= 15 && \text{(second column)}\\
c + f + i &= 15 && \text{(third column)}\\
a + e + i &= 15 && \text{(main diagonal)}\\
c + e + g &= 15 && \text{(second diagonal)}
\end{aligned}$$

There are only eight equations for nine variables. We may want to create a ninth equation. We can accomplish this by giving value to a variable. For instance, let us say that $e = 5$, which is a reasonable number since it is the middle value and the middle cell. After adding this piece of information, we compute the rank (through an app, calculator or available websites), and we get rank = 7.

That is, from the 9 rows, we can eliminate two that are results of others. And we add two new values to variables. Without going on details of how to do, we point out that adding the first three rows gets the same result as adding rows 4, 5, and 6. So, we can eliminate one of these rows. Eliminate sixth row (basically, we noticed that row 1 + row 2 + row 3 – row 4 – row 5 = row 6).

Let us replace it by giving the value 1 to a cell. Do we want this low value on one of the corners, or in the middle? Let us try on the middle, for instance $b = 1$ (that is the sixth equation now).

Again, without many details at this point, we notice that adding rows 5, 7 and 8 we end up with: {1, 1, 1, 0, 3, 0, 1, 1, 1} which is the same as adding rows 1 and 3, and three times row 9. So:

$$\text{row } 5 + \text{row } 7 + \text{row } 8 = \text{row } 1 + \text{row } 3 + 3 \times \text{row } 9.$$

Hence, we can eliminate one of these rows and replace it with a row that represents giving value to a new variable (since we already gave value to b and

e, we can give value to any other variable other than h, since $b + e + h = 15$). Say we eliminate 8th row and let $a = 8$. The coefficient matrix (which we call A), the variable vector (X) and the right-handed vector (B) become:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, X = \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{pmatrix}, B = \begin{pmatrix} 15 \\ 15 \\ 15 \\ 15 \\ 15 \\ 1 \\ 15 \\ 8 \\ 5 \end{pmatrix}$$

Now, A has rank 9 (computed with a technology resource). Hence, we could use linear algebra and solve it. So, our magic square is the result of $A^{-1}B$ which is:

8	1	6
3	5	7
4	9	2

How did we guarantee that the numbers did not repeat? Well, that can be done by trial-and-error or little experience. Also notice that any 90 degree rotation or any vertical/horizontal reflection or combinations of these moves would also result in similar magic squares.

Notice the following nice properties (Benjamin & Yasuda, 1999) of the above square:

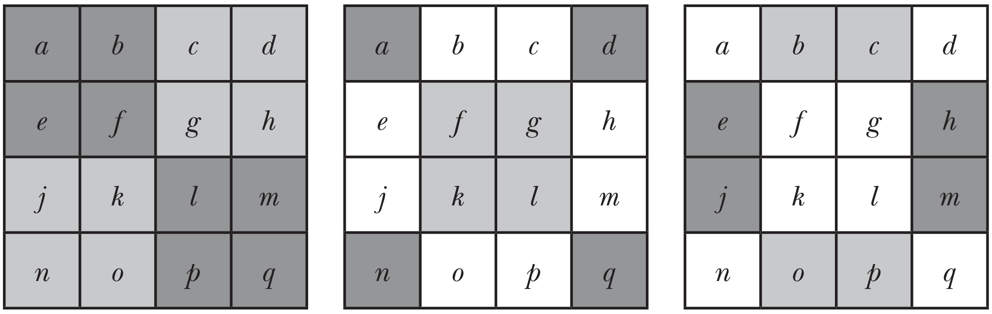
- Rows: $816^2 + 357^2 + 492^2 = 618^2 + 753^2 + 294^2$
- Columns: $834^2 + 159^2 + 672^2 = 438^2 + 951^2 + 376^2$.

4 × 4 magic square

Let us say we want to create a 4 × 4 magic square. Each row, column, diagonal will have the same sum, which we call S. According to previous discussion, let's say $S = 34$. There are 16 variables, but only 4 rows, 4 columns and 2 diagonals, totalling 10 equations. And some equations may even be redundant.

So, we may create some other 'special areas' that keep the same sum. See the areas below. There are eight new areas: each 'quadrant', an area composed by the corners a, d, n, q, an area with four 'centre' numbers f, g, k, l, plus two rectangles.

Students need to check the number of variables and the total number of equations. Also, they need to check the rank of the matrix A. And check whether they need to start giving values to different cells.



Class experience

There was a three-week period reserved for this experiment. Since class meets twice a week for 75 minutes each time, there were a total of only six meetings. That is how it worked:

First day

On the very first day of classes, the 26 students are divided into six groups, each with 4 or 5 members. The instructor goes over the syllabus and explains that the first few weeks would have a different structure: the active learning method. He also informally assesses students' level of familiarity with classification of matrices, and the basic operations (sum, difference and product). All topics are well-known, but some students show deficiency in matrix product. A homework on these operations is then assigned. Also, students are asked to watch TED talk by Arthur Benjamin where he shows a 4 by 4 magic square based on a birth date.

Two students are required to go to the tutoring centre to work on matrix multiplication. Both go.

Second day

It starts with groups solving systems of equations with two variables and two equations. Different groups solve a different system, and members choose to solve problems via either 'elimination process' or 'substitution'.

Once all groups show a relative knowledge on both methods, instructor decides to present the systems as a matrix multiplication $A \cdot X = B$. Each group also presents their own problems as matrix multiplication.

Then instructor shows WolframAlpha website, and taught students how to type matrices, how to solve the system by $A^{-1} \cdot B$, and how to read the results. Each group sends a representative to redo their own problems at the instructor's computer. All answers match.

Instructor asks each student to download a 'matrix app' on their cell phones and try to use them. A student downloads a wrong app (it just computes determinants), a student has no battery left, and another has no memory available. All others download good ones. A class discussion emerges on how to enter data. They realise that different programs have different forms of entering rows.

Class ends with students solving systems of three equation with three variables.

Homework involves some systems. It also included a two-page handout for solvable systems with less equations then variables. On the handout, it simply says that the solution exists, but it would not be unique. By attributing values to some variables, different solutions would be found. Students are encouraged to watch other magic square videos, including the one on Numberphile channel. Students who were unable to install an app are required to do so until next class.

Third day

Class starts with a system with three variables and two equations. Students are asked to find "a" solution. Most students try to use their apps, and conclude that it would be "impossible." A student, let's call him student A1, proposes a correct solution. He decides to make x equal to zero, and solves a system with two equations and two variables. The instructor does not confirm whether it would be valid. Students debate and conclude that *it makes sense, but that it assumes information that was not given.*

Another student from other group, student B1, raises the question of whether he could have considered that y and not x is zero. Instructor encourages the class to see if that would be possible. Another solution was found. Student C1 answers the system once $z = 0$.

Instructor asks why students were simply fixing on the possibility of assigning zero to a variable. He asks about what would happen if other values were assigned. Different groups have different values, and even within each group, different members would assign the same value, but for different variables. Several new solutions are created. Student A1 conjectures that there would be infinitely-many solutions. Everyone seems to accept without any proof.

Instructor, then, decides to write a problem with four variables and three equations. Groups debate to see if they can solve. Student leaders emerge in every group, leaving some students without much participation. After a while, two groups were able to almost simultaneously solve the problem. Instructor divides the class into two halves, and a representative of these two groups lead each half.

Class is almost over, when instructor assigns the homework. It is solving a system of five equations with seven variables. However, and students are not aware, there were only four independent equations, since the last one was a linear combination of the others.

Fourth day

Most of the students were unable to solve the only problem on the homework. Instructor then asks them to type the matrix of coefficients A into their apps, and asks them to compute its 'rank'. At least one person per group is able to do so. The rank was 4. Instructor then shows that the fifth equation is a linear combination of the first four. He briefly talks about excluding the fifth equation since it does not add new piece of information. Students take few minutes to come up with the solution.

Group D gives values to three variables and solve the system. However, Group C only gives values to two variable and struggles. Meanwhile, Group A tries to give value to four variables and finds an inconsistency. Instructor refrains from any input in the discussion. Students conjecture that the difference between the rank and the number of variables has anything to do with the amount of new data to be included. Instructor gives another problem: six equations and eight variables.

Finally, instructor introduces a generic 3 by 3 magic square and asks problems 1 and 2 (see previous sections):

a	b	c
d	e	f
g	h	i

Class answers the problems relatively easily. Instructor assigns as homework an YouTube about rank of matrices, and gives each student an option:

1. Give a through solution for a system of equations that would summarise the magic square, with discussion on rank.
2. Do a list with ten systems of equations.

Fifth day

About half of the class chose each option. Instructor decides to let students explain their own solutions to students who had not chosen their options. That took longer than anticipated.

Instructor explains the relationship of determinant and existence of an inverse. Then, with previous examples, he shows that the square matrices whose rank were less than number of rows all had zero determinant.

Instructor shows a generic 4 by 4 magic square, and by selecting the 'magic areas' they come up with the number of equations. Their apps help them identify the rank, or the number of different pieces of information.

Due to the increased number of variables and equations, it is hard to identify which equation is redundant. Instructor reviews linear independence and assigns homework problems and two linear algebra videos.

Sixth day

Instructor shows the entire development of the problem of using the concepts to solve a 5 by 5 magic square. Since the problem of identifying 'magic areas' to the problem of typing too many equations with too many variables. Some apps limit the amount of characters per equation. Instructor had a long preparation, since he tested each and every app that were being used.

That class, which was the last one on the three-week period that was reserved for the material, also served as a review of the concepts, and examples of additional applications. Homework on word problems was assigned.

Other days

When other concepts were later explained, instructor did his best to relate to magic square. For instance:

- the sum of two magic squares of the same order is also a magic square;
- the multiple of a magic square is also a magic square, of same order;
- hence the space of magic squares is a vector space;
- the space of all magic squares of a certain order whose magic number is zero is a vector subspace;
- the transpose of a magic space is also a magic square;
- the eigenvalue of a magic space is the magic number;
- a vector with only ones is an eigenvector.

Students seemed particularly excited with all these references to magic squares.

Conclusions and future work

With the perceived success of this strategy, the next step is trying to create a whole semester course solely based on active learning.

Also, students seem to enjoy motivation from different areas. When it comes to inspiring students via magic, there are several examples that highlights good uses. For instance, Teixeira (2017) teaches a nice card trick while relating probability concepts. Also, Lesser and Glickman (2009) use magic to teach mathematics (probability and statistics). We conclude that we should include more magic tricks in this and other courses.

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