ACTA DIDACTICA NAPOCENSIA

Volume 12, Number 1, 2019

SWEDISH UPPER SECONDARY STUDENTS' UNDERSTANDING OF LINEAR EQUATIONS: AN ENIGMA?

Paul Andrews and Sofia Öhman

Abstract: Prompted by the evidence of international tests of achievement that Swedish students have poor equation solving skills, this paper presents an interview study of upper secondary students' understanding of the topic. Students, from both vocational and academic tracks in two typical suburban upper secondary schools were invited to explain how a solution to the algebraic equation x + 5 = 4x – 1, presented with no explanation, had been conceptualised by the hidden solver. Analyses showed that students were not only familiar with linear equations but consistently invoked a 'do the same to both sides' procedure, indicating a secure understanding of the principles underpinning the solution of algebraic equations. Such understandings, which seem to defy international tests' outcomes are conceptually beyond a reliance on the inverse operations necessary for arithmetical equations. Explanations for this apparent inconsistency, which go beyond mere maturation, are proposed.

Key words: Sweden; linear equations; upper secondary school

1. Introduction

Algebra in general and linear equations in particular stand as *gatekeepers* between school mathematics and higher education and employment (Knuth, Stephens, McNeil & Alibali, 2006; Capraro & Joffrion, 2006). Linear equations, a topic that straddles the border between mathematics as concrete and inductive and mathematics as abstract and deductive, offers early opportunities for students to connect arithmetic to the symbolism of formal mathematics (Wasserman, 2014). However, despite its importance as a transitional topic in school mathematics, international tests of achievement like TIMSS and PISA have repeatedly indicated that Swedish students are mathematically unsophisticated and, in particular, poor equation solvers. And yet, the the regular competitiveness indices produced by the World Economic Forum present Sweden as one of the most competitive nations in the world, with high levels of innovation, sophistication and technological competence. In other words, evidence from different international assessments suggest, on the one hand, that Swedes lack mathematical sophistication, while, on the other hand, that they are remarkably sophisticated in areas where one would expect to see mathematical sophistication. In this paper, I attempt to explain this apparent contradiction by means of an analysis of Swedish upper secondary students' linear equations-related competence. Unlike international tests, which typically draw on answers recorded as right or wrong, my goal is to go deeper and examine students' underlying conceptions of equations and their solutions to determine whether they have the adaptability and versatility necessary for algebraic success (Sfard & Linchevski, 1994).

2. Research on the teaching and learning of linear equations

Before summarising what research has to say on the teaching and learning of linear equations, I feel I should address an unresolved issue concerning the equations-related vocabulary found in the mathematics education literature. Broadly speaking, a linear equation can be characterised according to whether the unknown exists one or both sides of the equals sign. In the case of the former, an equation like $3x + 1 = 16$ can be simply solved by means of operation reversal (Herscovics & Linchevski, 1994). Admittedly, an equation like $13 – 3x = 4$ is not quite so simply managed, not least because the operation reversal of 'subtract from 13' is the counter-intuitive 'subtract from 13'. That

Received March2019.

Cite as: Andrews, P. & Öhman, S. (2019). Swedish upper secondary students' understanding of linear equations: An enigma?. *Acta Didactica Napocensia, Acta Didactica Napocensia*, 12(1) , 117-129, DOI: 10.24193/adn.12.1.8.

said, all such equations can be solved with no knowledge other than that of arithmetic. With respect to the latter, an equation like $x + 5 = 4x - 1$ cannot be solved in such ways. Of course, there will come a point when such an equation will have been reduced to one in which the unknown is on one side, but the initial equation requires a different approach. Problematically, there seems to be no accepted vocabulary for this distinction. One of the first explicit distinctions was that of Filloy and Rojano (1989), who wrote of arithmetical and non-arithmetical equations. Since then scholars have written of procedural and structural equations (Kieran, 1992), operational and structural (Sfard, 1995), arithmetical and algebraic (Andrews & Sayers, 2012) and manipulation and evaluation (Tall, Lima, & Healy, 2014) respectively. Of these, some resonate more closely with the mathematical knowledge needed to solve the two forms of equations than others. For example, operational, procedural and manipulation may be problematic because any equation solving involves some form or operation, procedure or manipulation, while evaluation invokes no mental image of either equation. The distinction between arithmetical and non-arithmetical offers a clearer sense of the distinction, although I would argue that the labels arithmetical and algebraic offer more obvious indications as to the mathematical underpinnings of each form of equation.

Solving linear equations, particularly algebraic equations, draws on various forms of knowledge. Firstly, equation solvers need to see the equals sign as an assertion of equality between two expressions (Kieran, 2006) and not as an instruction to operate (Falkner, Levi, & Carpenter, 1999; McNeil et al., 2006). That is, successful equation solving necessitates an operational understanding of the equals sign (Alibali, Knuth, Hattikudur, McNeil & Stephens, 2007; Knuth et al., 2006). Secondly, equation solving requires the solver to understand and be able to manipulate the symbols on which equations are based (Huntley, Marcus, Kahan & Miller, 2007). Thirdly, particularly with respect to algebraic equations, solvers should not only “understand that the expressions on both sides of the equals sign are of the same nature” (Filloy & Rojano, 1989, p. 19) but be able to operate on the unknown as an entity and not a number.

Broadly speaking, successful learners of algebra are versatile and adaptive (Sfard & Linchevski, 1994), where versatility refers to the ability to see an expression as either a process or the product of that process and adaptability refers to the ability to interpret expressions through appropriate lenses. More prosaically, versatility refers to the tools a student has available and the ability to use them, while adaptability refers to the ability to select appropriate tools for the task. Importantly, students who approach equations flexibly are not only more successful equation solvers than those who do not (Rittle-Johnson & Star, 2007) but also more successful learners later across a range of domains (Hästö, Palkki, Tuomela (Star, 2019). In this respect, flexibility comprises two key characteristics (Star & Rittle-Johnson, 2008, p566). Firstly, it draws on a knowledge of multiple strategies and, secondly, it involves an awareness of the contextually-situated efficiency of those strategies. That is, a strategy that is efficient for solving one equation may not be for another. Not least of these would be an attempt to use an operations reversal with an equation with the unknown on both sides of the equals sign.

Historically, two approaches have dominated the didactics of equation solving (Filloy & Rojano, 1989). The first, based on the transposition of terms from one side of the equation to the other, forms the conceptual basis of a ‘swap the side swap the sign’ (SSSS) procedure and dominated the textbooks of the 18th and 19th centuries (Buchbinder, Chazan & Fleming, 2015). The second exploits operations performed simultaneously on both sides of an equation and provides the conceptual underpinning a ‘do the same to both sides’ (DSBS) procedure. The latter has underpinned most equations-related intervention studies, although there have been some critical exceptions (see Pirie & Martin, 1997, Lima & Tall, 2008). Many of these studies have exploited the balance scale to warrant a DSBS procedure and support learners’ relational understanding of the equals sign (see, for example, Araya et al., 2010; Caglayan & Olive, 2010; Vlassis, 2002). Other studies, drawing on what has been described as the canonical approach to equation solving (Buchbinder et al., 2015), have discussed a set of rules or basic transformations. Typical of these are the four transformations of combining like variable or constant terms, using the distributive property, adding or subtracting a constant or variable term to both sides of an equation, and dividing both sides of an equation by a constant (Rittle-Johnson & Star, 2007: Star & Rittle-Johnson, 2008). Interestingly, while researchers in this tradition do not make specific reference to the balance scale, their third and fourth transformations implicitly allude to a DSBS.

Despite scholars' broad enthusiasm for the didactical advantages of the balance scale-related consensus, assessments of students' equation solving have typically found either a guess and check approach (Chazan, 1996) or an SSSS procedure (Huntley et al., 2007; Linsell, 2009). This latter approach, reflecting a rote-learnt and arbitrary transposition whereby the unknown finishes on the left-hand side and a value on the right (Filloy & Rojano, 1989), not only perpetuates an operational conception of the equals sign but fails to support students' understanding that such movement does not change the equation's equality (Perso, 1996). It masks mathematical understanding (Star & Seifert, 2006), and frequently leads to a variety of later difficulties (Capraro & Joffrion, 2006; Kieran, 1992). It is, for many students, a 'magical' (Nogueira de Lima & Tall, 2008) procedure that frequently reduces students "to performing meaningless operations on symbols they do not understand" (Herscovics & Linchevski, 1994, p. 60).

In sum, the literature on recommended approaches to equation solving seems to be at odds with the findings of students' preferred approaches to equation solving. Moreover, if, as the different international tests indicate, Swedish students are unsophisticated equation solvers then they may be at a profound disadvantage in relation to their peers elsewhere. Consequently, this paper aims to shed light on both issues.

3. Data collection and analysis

Drawing on the earlier work of Wasserman (2014), in which a single task had proved successful in eliciting students' understanding of linear equations, the task used in this study exploited one used earlier with beginning primary teacher education students (Andrews & Xenofontos, 2017, Xenofontos & Andrews, 2017). It comprises a single equation and a solution, presented on paper, with no annotations to suggest the hidden solver's solution process.

$$x + 5 = 4x - 1$$

$$5 = 3x - 1$$

$$6 = 3x$$

$$2 = x.$$

Oral instructions asked students to imagine, first, that they had a friend who had been absent when their class had been shown how to solve such an equation and, second, to consider what they would say to help their friend understand the given solution. Additional oral instructions confirmed that the friend in question related to when participants had first learnt about such equations at school, that is, linear equations with the unknown on both sides. Providing oral explanations is a familiar process for Swedish students for whom an oral component is an integral part of all national mathematics assessments. Moreover, explaining, particularly to another person, whether real or fictitious, facilitates the development and demonstration of both understanding (Lachner & Neuburg, 2019) and competence (Denancé & Somat, 2015), particularly from the perspective of mathematical content knowledge (O'Neil, Chung, Kerr, Vendlinski, Buschang & Mayer, 2014) and problem solving (Wetzstein & Hacker, 2004).

With respect to investigating in depth students' construal of linear equations, conventional tests are limited in the detail they can elicit, while clinical interviews (Ginsburg, 1981) may have created too artificial an environment. Consequently, we were led to qualitative interviews in general and focus groups in particular (Kvale & Brinkmann, 2014). Focus group interviews, which allow topics to be introduced in ways that facilitate relaxed discussion and the emergence of desired insights, were undertaken with between two and five friends, typically from the same track. Focus groups optimise participation, legitimate an authentic uncensored language, reduce the power distance between participants and are time efficient (Parker & Tritter, 2006).

Twelve interviews were conducted with 39 students from two upper secondary schools in different parts of Stockholm. In the context of Sweden, compulsory school finishes at the end of grade nine and this is typically followed by three years of post-compulsory study in which students follow one of several academic or vocational tracks. In this study, 24 students derived from academic tracks and 15 from vocational. Typically, around 90% of students progress to upper secondary school (OECD, 2015). It is also important to note that interviews took place during the second semester of students' first year of upper secondary school, which means they were interviewed around a year later than would have been the case if they had been involved in PISA. Moreover, all interviewed students would have experienced some equations-related instruction in the semester before their interviews, in much the same way that students who sat PISA would have experienced something similar in the semester before their participation in PISA. In other words, while the interviewed students were typically one year older than those who sat PISA, both cohorts would have experienced similar equations-related instruction in the semester prior to the their being involved in the respective projects.

In any study of this nature, it is important to consider how many interviews should be undertaken. One reason is because there is an ethical responsibility not to collect data that will make no contribution to the analysis. Another, which informs the first, is the number of interviews needed to reach thematic saturation, or the point after which no new ideas are generated by the analyses (O'Reilly and Parker 2013). In this respect a recent focus group study of Mexican upper secondary students' emotional responses to mathematics, Martínez-Sierra and García-González (2017) achieved thematic saturation after nine interviews. In a study unrelated to mathematics education, Guest, Bunce and Johnson (2006) found not only that thematic saturation occurred within the first twelve but also that the basic structural elements had become apparent after six. In light of such findings, twelve interviews seemed an appropriate goal.

From the procedural perspective, interviews, which were undertaken at the students' school at times of their choosing, were video-recorded on a laptop computer. Video-recordings, which capture not only what is said, including tone of voice, but also body language and facial expressions, facilitate transcription. Moreover, since laptops are familiar objects in any modern classroom, students were thought to be less intimidated by its presence than a video camera and other equipment. All informants had given written consent for participation and all, in return, had been promised the right to withdraw at any point and for no reason. All students were over 15 years of age so parental permissions were unnecessary (Vetenskapsrådet, 2011). Finally, all students were assured that they, their teachers and their schools would remain anonymous. Interviews were transcribed and pseudonyms introduced for each student. Transcripts were read and re-read to elicit not only how students interpreted the solution but how they would explain that solution to a novice equation solver.

4. Results

In all twelve interviews students discussed DSBS, an approach indicative of a relational view of the equals sign. In eight interviews DSBS emerged as students' preferred approach to explaining the solution presented to them. In three other interviews students alluded to a 'change the side change the sign' rule, before it was superseded by DSBS invocations. Finally, in one case DSBS appeared tacitly as part of a rather confused account. The following is structured by these three different perspectives on DSBS.

4.1. Doing the same thing to both sides as students' preferred approach

DSBS emerged as students' preferred approach in eight interviews. Of these, five yielded evidence of students with confident perspectives on a DSBS approach to equation solving. For example, Mikael and Johan offered very clear and unambiguous perspectives on the equation before them;

Mikael: *Yes, the most important thing is to start thinking about the equals sign, it shows in fact that it should be equal on both sides… and then we'd love to get x alone and by doing it… so we add or subtract or divide or what it says on this page. So, in this case, then you can of course take what they have done. Yes, they have…*

Johan: *They've added one to both sides*

Mikael: *They have added one to both sides, exactly. And then, they have subtracted x from both sides. So, then you get three x by itself and four, no six, on the other side, and then it takes only division by three to get x alone. And six divided by three is two.*

Much can be inferred from these three utterances. Firstly, Mikael's initial statement alludes to a relational understanding of the equals sign; 'it should be equal on both sides'. Secondly, Mikael indicates, albeit tacitly, a procedural objective of getting *x* alone. Thirdly, he hints at DSBS, which Johan confirms by inferring, when other inferences would have been possible, that the hidden equation solver had done the same to both sides. Fourthly, Mikael observes that *x* has been subtracted from both sides before dividing both sides by three, further highlighting both his preference for DSBS and his understanding of inverse operations. Later, in the same interview, Johan added, in response to an interviewer question about the hypothetical students' understanding of the unknown that:

> *If they do not know what x is, that's okay; x is when something's, we do not know yet, we'll find out. And yes, so they should match, this equal sign. And then, one strives to have the x on one side otherwise it will be very difficult to figure it out. So then, they have done so they have added one on both sides, for it is minus one, where, in order to work after to get x on one side. Then they took the minus x on both sides, it is important to do it on both sides for it to be equal. Then you would get when six is equal to three x, and then you just have to share three if you want to find out what the x's worth and not three x.*

In this lengthy utterance, Johan confirms that solving an equation means finding a missing value represented by *x*. Then he repeats Mikael's view on the importance of separating the unknown from the numbers and reiterates the whole solution process, including DSBS. In short, both Johan and Mikael seem to have very strong and warranted stories as to how equations should be solved. Similarly, with respect to confident assertions of DSBS, Fredrik and Christian offered the following:

Fredrik: *Okay. I would probably explain it as, there is a number hiding behind this symbol [pointing to x]. In this case, an x, but you can rewrite it as a smiley or whatever, it really has no relevance. And for that, because I can imagine that this person will know how an equals sign works, it needs to be as much on this side as this side, and then you need to know what value is hiding behind this symbol. And to do that you need to make sure that there is as much on each side, and then you need to get x to stand alone. What you can do then is to add one on this side and one on this side or that when you have done this then you can do the opposite so that one takes. . . or no wait [inaudible] that was not it. The thing to do then…*

Christian: *[inaudible]*

Fredrik: *… and then you take care of the minus x, just remove x from this side and from that side, so that you then get six is equal to three x and then I divide it by three time to get this symbol alone and then it will be done with the equation.*

In the above, while Christian remains largely passive, Fredrik's first comments offer clear articulations of the aims and processes of equation solving. These include comments about the arbitrary nature of the unknown, 'you can rewrite it as a smiley or whatever', reference to the equals sign and the need to keep both sides of the equation equal, as well as the objective of identifying the 'value hiding behind this symbol' through getting it on its own. Finally, with respect to his first utterance, Fredrik spoke of adding one to both sides, at which point he became confused because, while being correct, his suggestion did not match the first stage of solution presented to them. After Christian's imperceptible comment, Fredrik regained his composure and completed his solution, albeit in a different order from that presented to him, by 'removing' an *x* from both sides and undertaking an appropriate inverse operation to eliminate the coefficient of *x*. In short, Fredrik offered a very complete account, an account that differed from the one he had been given on paper, that employed a clear understanding of DSBS based on an understanding of inverse operations.

At the less confident end of the spectrum were three interview groups in which one or two students seemed clear as to what they were doing and others less so. For example, in their interview Anna and Ida offered the following:

Ida: *You do that by either adding or taking away or divide by, so you get away all the time. It shortens …*

Anna: *The only thing is…*

Ida: *… the expression until you have the unknown on one side…*

Anna: *The unknown on one side and the answer on the other side. The key is to do the same to both sides because it is like a balance-scale; they have to be equal all the time. If you take away too much of one the other becomes greater…*

In this first excerpt, Ida offers some general statements of intent that seem focused on the broader objectives of equation solving; to get the unknown on one side by means of various arithmetical operations. However, there is nothing in her statements to indicate that she understands how these goals would be achieved. It is left to Anna to articulate the principles invoked when solving equations. She confirms Ida's statement about transforming the equation until the unknown on one side and the answer on the other, before adding that to achieve this 'the key is to do the same to both sides because it is like a balance scale. Their interview continued…

Anna: *On the one side we have four x minus one and on the other we have x plus five. You start by getting over the unknown… carry over all x to one side and transfer the numbers to the other side…*

Ida: *One can start by taking a plus*

Anna: *Yes, add one to the other side*

Ida: *So, then it becomes the x plus six…*

Anna: *Or really takes the plus on both sides, because one must do the same to both sides. But since we have a minus here, so it will be plus one to give zero. And then it will be a plus one here, and then we have x plus six equals four x…*

Ida: *And then, we can divide by x [uncertain]*

Anna: *No, then we have to remove x, because we still have an x here [in the right-hand side]. So, then we must remove the x from the other side. Then we get six is equal to three x.*

In Anna's first comment, phrases like 'getting over the unknown' and 'carry over the all x to one side' hint at a 'change the side change the sign' rule. Moreover, following Ida's initial and rather imprecise suggestion, Anna attempts to clarify the situation by suggesting that they should 'add one to the other side', which, while being precise continues to hint at the possibility of her using a 'change the side change the sign' rule. However, in her third statement she asserts very clearly that 'one must do the same to both sides' before completing the first line of her solution. At this point, Ida interjects with a tentative and largely unhelpful suggestion of dividing by x, which Anna instantly rejects, before completing the solution according to the principle she had asserted twice before. In sum, while Ida's contributions seemed uncertain and at times, incorrect, Anna showed confidence in her understanding of the equation and the principles involved in solving it.

Similar issues emerged from the interview with vocational students, Christoffer, Jakob and Linus. Their discussion, prompted by an interviewer question asking what had happened between the first two rows of the solution, began as follows:

Christoffer: *You have taken away x (from the left-hand side) for suddenly there is three x*

Jakob: *No, one adds one to both sides ---*

Linus *Yes, plus one to both sides*

In this brief initial excerpt, Christoffer offers what seems to be an observation of fact, 'you have taken away (removed) *x*'. At this point, indicating that both Jakob and Linus see 'doing the same thing to both sides' as important, Jakob seems to correct Christoffer and asserts that one has been added to both sides, which Linus reiterates. The conversation continues:

Christoffer: *And so, the four has disappeared here, so there will be three x, as I said… Oh, what the hell, Linus, you know this stuff…*

Linus: *From the first to the second, you've just…*

Christoffer: *Taken away…*

Linus: *Yes, you remove x from both sides, therefore, on both sides, then, you've got, then you want to have x on its own, that is your goal with this task, and then you add one to each side to remove the minus one, and then you take (…)*

Jakob: *It was divided by three. Six divided by three, x equals two.*

As the discussion progresses, Christoffer's confusion persists, as shown in his exasperated 'oh, what the hell' plea to Linus. Consequently, Linus redeems the situation by repeating a set of operations based on DSBS, whether subtracting x or adding one. Finally, acknowledging a moment of hesitation in Linus' solution, Jakob finishes the task by introducing division by three. In sum, while Christoffer seems keen not to be a passive member of the interview, his utterances indicate a lack of conviction, which both Jakob and Linus address repeatedly.

4.2. Accounts in which 'change the side change the sign' was superseded

In three cases, students began by describing a 'change the side change the sign' procedure before deciding that DSBS would be more productive and didactically helpful. For example, Alfred, Frans and Ivan, students following an academic track, delegated Omar, the fourth member of the group, to start. He said,

> *Me? Alright. So here we have x plus five on one side of the equals sign, on the left side, and we have four x minus one on the right side of the equal sign, and what you want to do then is to have x and the ordinary numbers on different sides of the equal sign. So you want to get x alone, and the ordinary numbers alone. So, they (the anonymous solvers of the equation) started by moving the x from the left side of the equal sign to the right side, and since it is positive on the left side so when you move over it becomes negative. So, four x minus x becomes three x.*

In Omar's comments, is an explicit articulation of the 'change the side change the sign' rule. Moreover, from his comment that 'you want to get *x* alone, and the ordinary numbers alone', can be inferred a clear objective concerning the separation of the unknowns from the 'ordinary numbers'. The conversation continued:

Ivan: *Okay, now you have five on the left side of the equal signs, three x minus one on the other side… you want to remove this minus one; the only thing to do then is to add one for then you get three x and then you have to add one on the other side of the equal sign. So then, there will be six equals three x.*

Alfred: *Then six equals three x and then we want to get x alone, so then we take the six, split into three or three divided into both sides of the equal sign. Then the x is alone.*

Interviewer: *Okay, (looking at Frans) since you have not said anything… is there anything you want to say…*

Frans: *The first thing I would like to say is to explain that you can do what you want in principle, just to do it on both sides. So, if we want a variable, x, alone, we can subtract the x from one side, but then you also have to do it to the other side.*

Omar's initial 'change the side change the sign' suggestion was superseded by explicit DSBS invocations, as seen in Ivan's adding one to both sides and Frans' more general comment that whatever one does to one side of the equation, 'you also have to do it to the other side'. In addition, Frans' comment that 'you can do what you want in principle' highlights an understanding that DSBS is a flexible process to be adapted as the user wishes. Finally, Alfred's comments indicate his understanding of the inverse operations embedded in eliminating the coefficient of the unknown. In sum, the various contributions of these four students present a robust and flexible understanding of the equation solving process in which actions are based on clear principles. It is also interesting to note that while none of Ivan, Alfred or Frans commented on Omar's approach, they implicitly seemed to reject it.

Similar perspectives could be found in the interview undertaken with vocational students Pedram, Isak and Michael, who spent a few seconds looking at the solution before offering the following suggestions:

Pedram: I do not remember clearly how you do this now, but I would have moved over the *minus one so that it becomes zero over here. But I do not know if it is possible, then there would be six over here, x plus six equals four x, but then oh, it will not work... Oh, wait, one takes the minus x over here…*

Isak: *Yes, you can subtract the x there so it becomes three x and then you also have six, then you come here, if you move over, then you have six equals three x and then it will be much easier to solve; three times something will be six, yes two, so it is solved…*

In these two comments are indications of differing conceptions of the equation solving process. On the one hand, Pedram's, 'I would have moved over the minus one' indicates a tacit use of the change the side change the sign mantra, albeit with no mention of changing the sign. However, his conclusion, 'then there would be six over here' indicates his having seen the move as making the negative positive. On the other hand, Isak's 'you can subtract the x there so it becomes three x' seemed to show an awareness that subtracting the x from one side of the equation was reciprocated on the other side. Later, when asked about these two approaches, the following transpired

Interviewer: *On the one hand you said 'move over the minus one' and on the other you said 'subtract x', what is the difference between these two ways?*

Pedram: *Do you think we meant the same thing, or…?*

Interviewer: *You can either say that you add one to both sides or it's also common to say that you move the one over to the other side and change the sign. Is there something about the two ways that you think is better or worse?*

Isak: *I'm used to the first one, adding one to both sides*

Michael: *I am also used to that, so I figured that one adds one on both the sides, and…*

Isak: *I learn better,*

Pedram: *Me too*

These comments suggest, despite Pedram's earlier proposal, that all three students were not only familiar with DSBS but that it would seem to be their preferred approach on the grounds that it helps them 'learn better'. The reasons for Pedram's 'move over the minus one' are not known but it is not inconceivable that it was a response to his interpretation of the solution presented to him. Importantly, it seems clear that the students were not unfamiliar with the approach but, when asked, indicated a preference for the alternative.

4.3. Ambivalent use of two rules

Finally, in one of the twelve interviews, students seemed to lack certainty with respect to how they would address the task given them. In this respect, Andreas and Max, during their interview with their friend Dennis, who does not figure in the following, commented that:

Andreas: *You have an x on one side and four x on the other, then one can just easily remove the x from one side and remove the x from the four x. Then you have the five equals three x minus one.*

Max: *And then one should of course get x on its own so that one can divide it by three*

Andreas: *Yes, and because it's a minus, you can just as easily move it and add it to the others…*

Interviewer: *What does it means to move it and add to the others?*

Max: *Because it is minus one, it will be the same as if one were to add it to the other side; it is the same thing really.*

Andreas' first comment, that 'one can just as easily remove the *x* from one side and remove the *x* from the four *x*' hints at a 'doing the same thing to both sides'. Also, Max's first comment is indicative of his goal, to get *x* on its own, as well as, as manifested in his comment about dividing by three, an understanding of inverse operations. However, this latter suggestion, while correct in both aim and process, was unlikely to prove helpful, as it would have yielded fractions likely to cloud the issue. Andreas, seeming to have noticed this, immediately suggested, with respect to the negative one, that 'because it's a minus, you can just as easily move it and add it to the others'. His suggestion seemed to imply a 'change the side change the sign' perspective, which after drawing a clarifying question from the interviewer, he confirmed. Finally, while these three students seemed inconsistent in their choice of approach, implying between them both common approaches, it is interesting that Andreas, spoke repeatedly of the need for equation solvers to understand the role of the unknown in the process. For example, at different stages he was heard to say:

Andreas: *Yes, it is well understood that x should be a number or the like and then you should also figure out what is the number that is behind x*

Andreas: *Yes, you need the knowledge to understand not only that x is a letter but also a number behind the x, not just a letter*

Andreas: *One needs well someone who can explain what x is and what to do with it and that this little thing that you have a figure in front so you know how many x is.*

In short, while he was clear as to the objective and wanted the absent student to know it, he was considerably less confident in his articulation of the equation solving process.

5. Discussion

This paper was motivated by concerns, identified by international assessments of Swedish students' equations-related competence. The data of this study, albeit limited and derived a year or two after these assessments, indicate that students, two thirds of the way through their first year of upper secondary school, are not only familiar with linear equations but have a secure understanding of the principles that underpin the solution processes of algebraic equations. That is, these students seemed to have strong anchoring conceptions (Arcavi, 2004) based in convincingly argued DSBS statements.

This commitment to DSBS is interesting because the balance scale, from which such arguments typically derive (Araya et al., 2010; Caglayan & Olive, 2010; Vlassis, 2002), emerged unprompted only once, in the interview with Anna and Ida. In no other interview was there spontaneous mention of the balance scale, and yet it is difficult to imagine, considering the strength of students' arguments for the cognitive transparency of DSBS, how such beliefs could have developed independently of it. Moreover, due to the nature of the task posed to them, students were often conscious of the relationship between how they construed the processes undertaken by the hidden equation solver and their desire to facilitate others' understanding of it. That is, when given the freedom to choose whatever method they wished, students elected to DSBS, not only because it supported their own understanding of equation solving but also because they believed it would best support the hypothetical student who had missed the lesson in which equations had been taught. They could have spoken about SSSS' (Kieran, 1992) but typically did not. Indeed, while SSSS was mentioned in three interviews, it was superseded by DSBS in two of them and part of a rather confused juxtaposition of the two rules in the interview with Andreas and Max.

A key contributor to the quality of equation solving lies in the extent to which students have a relational understanding of the equals sign (Alibali et al., 2007; Knuth et al., 2006). The evidence of all the interviews, in both implicit and explicit ways, confirmed that students see an equation as an object on which other operations may be performed (Kieran, 2006). There was no evidence of the equals sign being construed as an instruction to operate (Falkner, Levi, & Carpenter, 1999; McNeil et al., 2005). Of course, it is reasonable to assume that students at age 16 going on 17 would have outgrown any instinct to see the equals sign as an instruction to operate, but it is reasonable to assume that students who know little about equations, as international tests indicate, would hold naïve conceptions of the equals sign.

Interestingly, there was no evidence of any graphical interpretation in any students' proposed solutions; all invoked only algebraic approaches. This may have been due to the way in which the task was presented, but it is equally likely to be due to students' conditioning; Swedish textbooks tend not to emphasise the connection between algebraic and graphical solutions in the manner of, say, the Finnish texts discussed by Yang and Lin (2015). Another interpretation is that the students of this study lacked versatility, having only a limited set of tools available to them (Sfard & Linchevski, 1994). My view is that this is unlikely. For example, Fredrik's assertion that the unknown could be represented by a smiley was, I would argue, a profound insight. It was, acknowledging the didactical emphasis explicit in the task, indicative of someone with both flexibility and, in Fredrik's case, adaptability; he was able to draw on a range of tools appropriate for the task he had been given (Sfard & Linchevski, 1994).

So, can the divergence between the findings of this study and international tests of achievement be explained? One possibility could simply be maturation, not least because the students involved in this study were between one and two years older than those who would have taken PISA or TIMSS. During such a time it is feasible, due to their having been exposed to equations-related teaching during the first mathematics course of upper secondary school, that students' understanding of the topic had been transformed. However, the cognitive growth between knowing almost nothing, despite having received equation-related teaching during the final years of compulsory school, and the confident accounts reported here seems too great to be credible. A more likely explanation lies in the effort Swedish students are prepared to make with respect to such tests. For example, the OECD has employed a measure of student effort in two PISA iterations, 2003 and 2012, which have been reported by the Swedish education agency for education. Its report shows that Swedish students' effort score fell by nearly five per cent between the two interations to one of the lowest in the world. In the same period Swedish students' mathematics achievement fell by around over six per cent (Skolverket, 2015). Similar results have been found with respect to TIMSS. For example, 76% Swedish grade 8 students claimed they had been motivated to participate in TIMSS 2003, with a higher proportion (90%) claiming they would try their best (Eklöf, 2007). However, five years later, Swedish year 12 students participating in TIMSS Advanced 2008 reported very different perspectives, with 76% claimed to have spent less effort on this test than other tests undertaken in school, while barely a third felt motivated to do their best (Eklöf, Pavešič & Grønmo, 2014). In short, Swedish students' motivation to participate in international tests seems to be diminishing at a rate comparable with the decline of their achievement.

In sum, the title of this paper asks whether Swedish students' equation solving competence is an enigma. The evidence of this study and international evaluations of students' test effort suggest not. Swedish students have demonstrated that international tests are of little relevance to them; they are able to distinguish between different types of test and make appropriate efforts when necessary. They are, in essence, the independent thinking and mathematically sophisticated students tests like PISA aim to identify. Thus, in closing, one is drawn to ask why would the Swedish authorities persist in participating in international tests them when evaluations undertaken later provide more meaningful data on the system's performance?

6. References

Alibali, M., Knuth, E., Hattikudur, S., McNeil, N., & Stephens, A. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning, 9*(3), 221-247.

Andrews, P., & Sayers, J. (2012). Teaching linear equations: Case studies from Finland, Flanders and Hungary. *The Journal of Mathematical Behavior, 31*(4), 476-488.

Andrews, P., & Xenofontos, C. (2017). Beginning teachers' perspectives on linear equations: A pilot quantitative comparison of Greek and Cypriot students. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 1594-1601). Dublin: Institute of Education, Dublin City University.

Araya, R., Calfucura, P., Jiménez, A., Aguirre, C., Palavicino, A., Lacourly, N., Soto-Andrade, J., & Dartnell, P. (2010). The effect of analogies on learning to solve algebraic equations. *Pedagogies: An International Journal, 5*(3), 216-232.

Arcavi, A. (2004). Solving linear equations: Why, how and when? *For the Learning of Mathematics, 24*(3), 25-28.

Buchbinder, O., Chazan, D., & Fleming, E. (2015). Insights into the school mathematics tradition from solving linear equations. *For the Learning of Mathematics, 35*(2), 2-8.

Caglayan, G., & Olive, J. (2010). Eighth grade students' representations of linear equations based on a cups and tiles model. *Educational Studies in Mathematics, 74*(2), 143-162.

Capraro, M, & Joffrion, H. (2006). Algebraic equations: Can middle-school students meaningfully translate from words to mathematical symbols? *Reading Psychology, 27*(2-3), 147-164.

Chazan, D. (1996). Algebra for all students? *The Journal of Mathematical Behavior, 15*(4), 455-477.

Denancé, V., & Somat, A. (2015). Learning by explaining: Impacts of explanations on the development of a competence. *European Review of Applied Psychology, 65*(6), 307-315.

Eklöf, H. (2007). Test-taking motivation and mathematics performance in TIMSS 2003. *International Journal of Testing, 7*(3), 311-326.

Eklöf, H., Pavešič, B., & Grønmo, L. (2014). A cross-national comparison of reported effort and mathematics performance in TIMSS Advanced. *Applied Measurement in Education, 27*(1), 31-45.

Falkner, K., Levi, L., & Carpenter, T. (1999). Children's understanding of equality: A foundation for algebra. *Mathematics Teaching in the Middle School, 6*(4), 232-236.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics, 9*(2), 19-25.

Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: aims, rationales, techniques. *For the Learning of Mathematics*, *1*(3), 4-11.

Guest, G., Bunce, A., & Johnson, L. (2006). How many interviews are enough?: An experiment with data saturation and variability. *Field Methods, 18*(1), 59-82.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, *27*(1), 59-78.

Huntley, M., Marcus, R., Kahan, J., & Miller, J. (2007). Investigating high-school students' reasoning strategies when they solve linear equations. *The Journal of Mathematical Behavior, 26*(2), 115-139.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.

Kieran, C. (2006). Research on the teaching and learning of algebra. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 11-49). Rotterdam: Sense.

Knuth, E., Stephens, A., McNeil, N., & Alibali, M. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297-312.

Kvale, S. & Brinkmann, S. 2014. Den kvalitativa forskningsintervjun. Lund: Studentlitteratur.

Lachner, A., & Neuburg, C. (2019). Learning by writing explanations: computer-based feedback about the explanatory cohesion enhances students' transfer. *Instructional Science*, *47*(1), 19-37.

Lima, R., & Tall, D. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics, 67*(3), 3-18.

Linsell, C. (2009). Students' knowledge and strategies for solving equations. In K. Hannah (Ed.), *Findings from the Secondary Numeracy Project 2008* (pp. 29-43). Wellington: New Zealand Ministry of Education.

Martínez-Sierra, G., & García-González, M. d. S. (2017). Students' emotions in the high school mathematical class: Appraisals in terms of a structure of goals. *International Journal of Science and Mathematics Education*, *15*(2), 349-369.

McNeil, N., Grandau, L., Knuth, E., Alibali, M., Stephens, A., Hattikudur, S., & Krill, D. (2006). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition and Instruction, 24*(3), 367-385.

O'Neil, H., Chung, G., Kerr, D., Vendlinski, T., Buschang, R., & Mayer, R. (2014). Adding self-explanation prompts to an educational computer game. *Computers in Human Behavior, 30*, 23-28.

O'Reilly, M., & Parker, N. (2013). 'Unsatisfactory Saturation': a critical exploration of the notion of saturated sample sizes in qualitative research. *Qualitative Research, 13*(2), 190-197.

OECD. (2015). *Improving schools in Sweden: An OECD perspective*. Paris: OECD.

Parker, A., & Tritter, J. (2006). Focus group method and methodology: Current practice and recent debate. *International Journal of Research & Method in Education, 29*(1), 23-37.

Perso, T. (1996). Teaching equation solving - Again? *The Australian Mathematics Teacher, 52*(1), 19-21.

Pirie, S., & Martin, L. (1997). The equation, the whole equation and nothing but the equation! One approach to the teaching of linear equations. *Educational Studies in Mathematics, 34*(2), 159-181.

Rittle-Johnson, B., & Star, J. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561-574.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *The Journal of Mathematical Behavior, 14*(1), 15-39.

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification: The case of algebra. *Educational Studies in Mathematics, 26*(3), 191-228.

Skolverket. (2015). *To respond or not to respond: The motivation of Swedish students in taking the PISA test*. Stockholm: Skolverket.

Star, J. R., & Seifert, C. (2006). The development of flexibility in equation solving. *Contemporary Educational Psychology, 31*(3), 280-300.

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction, 18*(6), 565-579.

Tall, D., Lima, R., & Healy, L. (2014). Evolving a three-world framework for solving algebraic equations in the light of what a student has met before. *The Journal of Mathematical Behavior, 34*(1), 1-13.

Vetenskapsrådet (Swedish Research Council). (2011). *Good research practice*. Stockholm: Vetenskapsrådet.

Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics, 49 (3),* 341-359.

Wasserman, N. (2014). Introducing algebraic structures through solving equations: Vertical content knowledge for K-12 mathematics teachers. *PRIMUS, 24*(3), 191-214.

Wetzstein, A., & Hacker, W. (2004). Reflective verbalization improves solutions—the effects of question-based reflection in design problem solving. *Applied Cognitive Psychology, 18*(2), 145-156.

Xenofontos, C., & Andrews, P. (2017). Explanations as tools for evaluating content knowledge for teaching: A cross-national pilot study in Cyprus and Greece. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 1666-1673). Dublin: Institute of Education, Dublin City University.

Yang, D.-C., & Lin, Y.-C. (2015). Examining the differences of linear systems between Finnish and Taiwanese textbooks. *Eurasia Journal of Mathematics, Science & Technology Education, 11*(6), 1265-1281.

Hästö, P., Palkki, R., Tuomela, D., & Star, J. (2019). Relationship between mathematical flexibility and success in national examinations. *European Journal of Science and Mathematics Education*, *7*(1), 1-13.

Authors

Paul Andrews, Stockholm University, Stockholm (Sweden). E-mail: paul.andrews@mnd.su.se

Sofia Öhman, Campus Manilla Gymnasium, Stockholm. E-mail: sofia.ohman@campusmanilla.se