

## Conical Pendulum: Part 2 A Detailed Theoretical and Computational Analysis of the Period, Tension and Centripetal Forces

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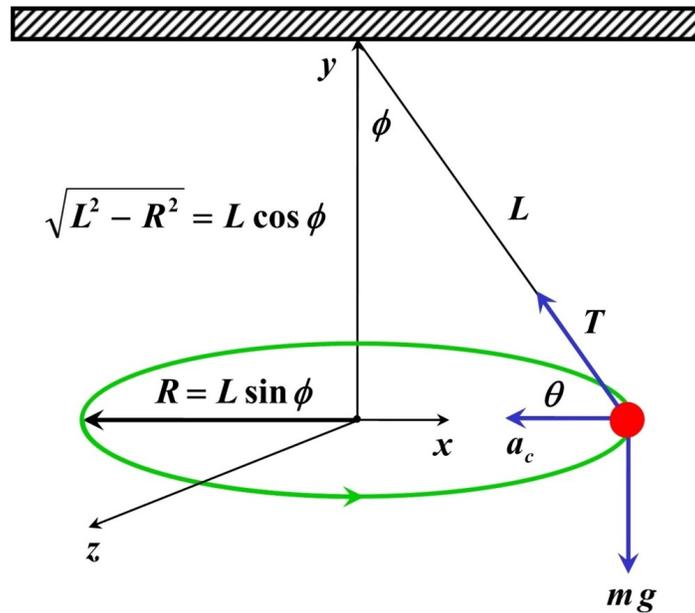
### Abstract

This paper represents a continuation of the theoretical and computational work from an earlier publication, with the present calculations using exactly the same physical values for the lengths  $L$  (0.435 m – 2.130 m) for the conical pendulum, mass  $m = 0.1111$  kg, and with the local value of the acceleration due to gravity  $g = 9.789$  ms<sup>-2</sup>. Equations for the following principal physical parameters were derived and calculated: period  $T$ , angular frequency  $\omega$ , orbital radius  $R$ , apex angle  $\phi$ , tension force  $F_T$  and centripetal force  $F_C$  (additional functions were calculated when required). Calculations were performed over a wide range of values of the apex angle ( $0^\circ \leq \phi \leq 85^\circ$ ), corresponding to a calculated tension force  $F_T$  range of approximately ( $mg \leq F_T \leq 12$  N) or alternatively ( $mg \leq F_T \leq 11$  mg) for the string. A technique is demonstrated to determine an accurate value of an unknown pendulum mass, by using a graphical analysis. Intercepts and asymptotic lines with respect to both the horizontal and vertical axes are described and fully explained. The main emphasis for this paper is to present highly detailed graphical charts for the calculated theoretical functions and appropriate physical parameters. Theoretical analysis is presented in comprehensive detail, showing full mathematical derivations and alternative equations when this approach is considered to be advantageous for both understanding and computational presentation.

**Keywords:** Conical pendulum, theoretical analysis, tension force, centripetal force, period, angular frequency, high precision, computational analysis.

### INTRODUCTION

All that follows is based on and makes reference to the figure below, which was first used in the previous publication (refer to Dean & Mathew, 2017). The figure defines the basic conical pendulum parameters for the subsequent theoretical analysis and clearly illustrates the overall geometry of a typical undergraduate experimental system and the vector nature of the tension and centripetal forces. Basic trigonometric functions are used to express the adjacent side of the Pythagorean triangle in terms of the orbital radius  $R$  and length  $L$  of the conical pendulum. The mathematical analysis contains all relevant sequential derivation steps and clearly shows how the final equation (that is generally numbered in sequence) is obtained. When it is necessary to make specific reference to a previously derived equation, the relevant equation is often reproduced for convenience.



**Figure 1.** Conical Pendulum Schematic

The in-depth physics of a conical pendulum can frequently appear to be mathematically demanding for freshman university students (Czudková & Musilová, 2000). Standard conical pendulum lengths that have been investigated experimentally (and subsequently published) usually range from approximately 0.20 m (Tongaonkar & Khadse, 2011) up to approximately 3.0 m (Mazza et al, 2007). In the typical range of undergraduate laboratory experiments, the conical pendulum is frequently included along with the relevant theory. The conical pendulum can also be used to explain concepts such as: energy and angular momentum conservation (Bambill, Benoto & Garda, 2004), mechanical potential energy (Dupré & Janssen, 1999) as well as more complex rotational dynamic interactions of mechanical systems (Lacunza, 2015).

In a typical experiment, the conical motion is confined to the horizontal plane, however, it is possible to extend the mathematical analysis to three dimensions (Barenboim & Oteo, 2013). In the usual case of horizontal planar motion, this can frequently be observed as elliptical rather than being truly circular. This has been studied in detail in order to determine the physical nature of orbital precession (Deakin, 2012). The string tension force has been measured as a function of the rotational period (or angular frequency) and has been documented (Moses & Adolphi, 1998).

For the analysis and discussion that follows, the five specific parameters are considered to be: the local value of the acceleration due to gravity  $g = 9.789 \text{ ms}^{-2}$  (Ali, M.Y. et al., 2014), the mass  $m = 0.1111 \text{ kg}$  of the conical pendulum, the pendulum length  $L$  (measured to the centre of the spherical mass), the orbital period  $T$  and the tension force  $F_T$  (it is important to note that the angular frequency  $\omega$  is calculated directly from the orbital period  $T$ , so it is therefore regarded as being the sixth fundamental parameter). For the purposes of calculation, the theoretically ideal string is considered to be essentially mass-less (when compared to the conical pendulum mass) and physically inextensible for the present analysis.

Regarding the mathematical representation, when theoretical derivations are presented, all of the analytical steps are provided, in a manner that is considered to be self-explanatory. The arrow

symbol that is shown here within quotation marks “ $\Rightarrow$ ” is used frequently in mathematical derivations and is understood to mean “giving”, or “will give”, or “leads to”. The alternative arrow symbol in equations “ $\rightarrow$ ” is used to mean “approaches”, or “tends to”. When a derivation can be clearly written on a single line using a short-form mathematical layout, this method is used. A preliminary section dealing with the essential physical parameters is intended as a theoretical introduction, and the mathematical complexity has been deliberately minimized, to be readily accessible for readers. If convenient alternative mathematical representations of equations are readily available, these are frequently included in the principal analysis when this is considered to be helpful.

## THEORETICAL RESULTS AND ANALYSIS

For the theoretical derivation of the conical pendulum period  $T$  see (Dean & Mathew, 2017); the resulting equation itself is presented below and will be developed further, in order to derive suitable equations for calculating the orbital radius  $R$  and the conical pendulum apex angle  $\phi$ . Additional uncomplicated analysis will follow, providing two exceedingly simple and elegant equations for the tension force  $F_T$  and centripetal force  $F_C$ .

$$T = 2\pi \sqrt{\frac{L \cos \phi}{g}} = 2\pi \sqrt{\frac{\sqrt{L^2 - R^2}}{g}} = \frac{2\pi}{\omega} \Rightarrow \frac{1}{\omega^2} = \frac{\sqrt{L^2 - R^2}}{g} \Rightarrow$$

$$\frac{g^2}{\omega^4} = L^2 - R^2 \Rightarrow R = \sqrt{L^2 - \frac{g^2}{\omega^4}} = \sqrt{L^2 - \left(\frac{g}{\omega^2}\right)^2} \quad (1)$$

The final Equation (1) above shows that the orbital radius  $R$  can be calculated directly since the required parameters are known (namely  $g$ ,  $L$  and  $\omega$ ). The apex angle  $\phi$  is consequently directly obtained from the following Equation (2):

$$R = L \sin \phi \Rightarrow \phi = \sin^{-1} \left( \frac{R}{L} \right) \quad (2)$$

The apex angle  $\phi$  can be calculated from an alternative derivation using the conical pendulum orbital radius  $R$  as the starting point. This alternative approach provides an equation that is only dependent on three of the fundamental conical pendulum physical parameters. In terms of its mathematical usefulness and applicability, the equation that is derived immediately below, is considered to be the most suitable equation for the calculation of the apex angle  $\phi$ . It can be noted that  $\phi$  is readily calculated for any particular pendulum length  $L$  and angular frequency  $\omega$  (calculated from the period  $T$ ). It is also suggested below how an appropriate graphical analysis could be suitably performed, for the accurate determination of the local value of the acceleration due to gravity.

$$R = \sqrt{L^2 - \frac{g^2}{\omega^4}} \Rightarrow R^2 = L^2 - \frac{g^2}{\omega^4} \Rightarrow g = \omega^2 \sqrt{L^2 - R^2}$$

$$g = L \omega^2 \cos \phi \Rightarrow \phi = \cos^{-1} \left( \frac{g}{L \omega^2} \right) \quad (3)$$

If an experiment is suitably designed so the apex angle  $\phi$  can be determined accurately, then the above analysis readily provides a straightforward graphical method for the determination of the local acceleration due to gravity  $g$ , from the slope of the graph with the following axes:

$$\underbrace{\omega^2}_{y\text{-axis}} = g \underbrace{\left( \frac{1}{L \cos \phi} \right)}_{x\text{-axis}} \Rightarrow \text{Slope} = g \quad (4)$$

Multiple pendulum lengths covering a suitably wide range (for example the lengths reported in the present paper) could readily enable an accurate value of  $g$  to be determined experimentally (see also Tongaonkar & Khadse, 2011). By referring to Figure 1, and with a straightforward understanding of the basic physics principals involved in the circular motion of a conical pendulum, the tension force can be shown to have a proportionality with respect to the reciprocal of the orbital period squared and consequently the square of the angular frequency:

$$\left. \begin{aligned} T &= 2 \pi \sqrt{\frac{L \cos \phi}{g}} \\ F_T &= \frac{m g}{\cos \phi} \end{aligned} \right\} \Rightarrow F_T = \frac{4 \pi^2 m L}{T^2} \quad (5)$$

$$F_T = (4 \pi^2 m L) \left( \frac{1}{T^2} \right) \Rightarrow F_T = m L \omega^2 \quad (6)$$

It can be readily seen that Equation (5) is obtained by making a straightforward substitution, using the second equation for the tension force. A direct re-arrangement of Equation (5) enables it to be written in a mathematically more elegant way in terms of the angular frequency squared. In this form, a graph of  $F_T$  plotted on the  $y$ -axis against either  $(L \omega^2)$  or just  $\omega^2$  on the  $x$ -axis, would enable the pendulum mass  $m$  to be calculated from the slopes of each of the resulting straight lines. By using experimental data acquired from multiple pendulum lengths  $L$ , it is suggested that

an accurate value of  $m$  can be readily calculated. From the tension force  $F_T$  and the gravitational force ( $mg$ ), as well as the triangular geometry; the two equations below are readily derived, where Equation (8) is observed to be independent of the pendulum mass  $m$ .

$$F_T = \frac{m g}{\cos \phi} \Rightarrow \phi = \cos^{-1} \left( \frac{m g}{F_T} \right) \quad (7)$$

$$\text{Pythagoras} \Rightarrow \phi = \cos^{-1} \left( \frac{\sqrt{L^2 - R^2}}{L} \right) \quad (8)$$

As a natural physical extension of the preceding analysis concerning the tension force  $F_T$ , the centripetal force  $F_C$  can also be considered in terms of the rotational properties of the pendulum. The short (one line) self-explanatory derivation below (using  $v = R \omega$ ) addresses this issue:

$$a_c = \frac{v^2}{R} \Rightarrow m a_c = \frac{m v^2}{R} = \frac{m R^2 \omega^2}{R} \Rightarrow F_C = m R \omega^2 \quad (9)$$

Equation (9) shows that the centripetal force is directly proportional to the angular frequency squared (see also Tongaonkar & Khadse, 2011). Substitution of Equation (1) for the orbital radius  $R$  enables the centripetal force equation to be written in the following way (by again making use of the short-form single-line mathematical layout):

$$F_C = m R \omega^2 \quad \text{using} \quad R = \sqrt{L^2 - \left( \frac{g}{\omega^2} \right)^2} \Rightarrow F_C = m \sqrt{L^2 \omega^4 - g^2} \quad (10)$$

The resulting Equation (10) above makes it clear that the centripetal force  $F_C$  can therefore be calculated directly without having to calculate a value for the orbital radius  $R$  first. It is observed from the above  $F_C$  equation that the conical pendulum mass  $m$  is specifically required (Mazza et al, 2007). Making reference to Figure 1, and using a Pythagorean triangle analysis, the following elegantly simple mathematical relation between the tension force  $F_T$  and centripetal force  $F_C$  can be immediately obtained (again using the short-form mathematical layout):

$$F_T = \frac{m g}{\cos \phi} \quad \text{and from Figure 1:} \quad F_C = m g \tan \phi \Rightarrow F_C = F_T \sin \phi \quad (11)$$

This particular equation is considered to be especially useful due to its simplicity; however a prior calculation of the orbital radius  $R$  is required before the apex angle  $\phi$  can be calculated. A substitution for  $R$  gives rise to a mathematically more complex equation, which is now seen to be

dependent on four of the primary parameters. It is important to recall that (as stated earlier) the angular frequency  $\omega$  is regarded as being a known parameter, since it is calculated directly from the conical pendulum period  $T$ .

$$\left. \begin{aligned} F_C &= F_T \sin \phi = F_T \frac{R}{L} \\ R &= \sqrt{L^2 - \left(\frac{g}{\omega^2}\right)^2} \end{aligned} \right\} \Rightarrow F_C = F_T \sqrt{1 - \left(\frac{g}{L \omega^2}\right)^2} \quad (12)$$

Despite the increased mathematical complexity, the final Equation (12) for the centripetal force  $F_C$  is directly calculated from the available parameters and does not require either the apex angle  $\phi$  or the orbital radius  $R$  to be calculated in advance. The calculation of  $F_C$  is now mathematically independent of the conical pendulum mass  $m$ , which consequently extends the applicability of the above centripetal force equation.

The two equations for the tension force  $F_T$  and the centripetal force  $F_C$ , that are expressed in trigonometric terms of the apex angle  $\phi$  can be re-expressed in the following algebraic way:

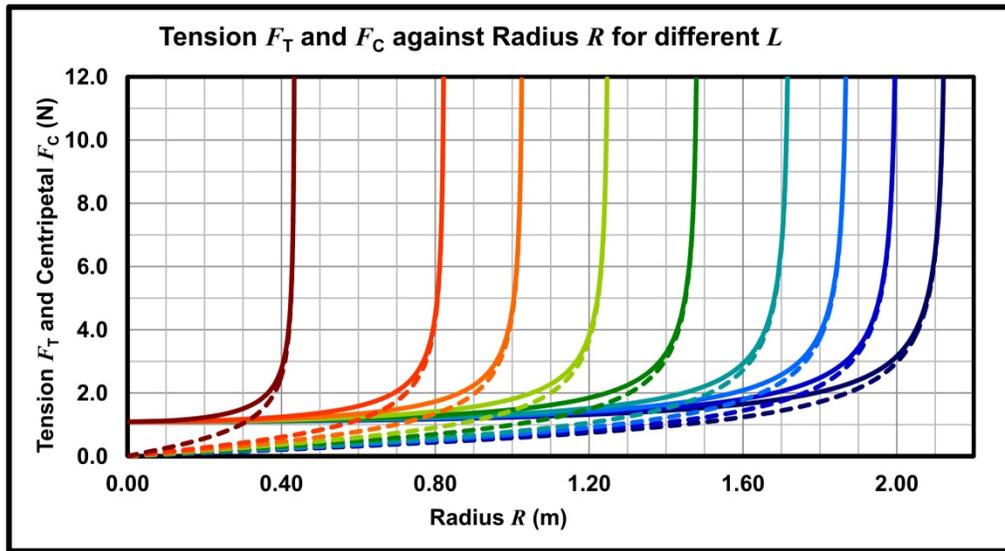
$$F_T = \frac{m g}{\cos \phi} = \frac{m g L}{\sqrt{L^2 - R^2}} \quad (13)$$

$$F_C = m g \tan \phi = \frac{m g R}{\sqrt{L^2 - R^2}} \quad (14)$$

The  $F_T$  and  $F_C$  equations can be used to investigate their dependence on the pendulum orbital radius  $R$ , as shown in the chart below, which clearly demonstrates the complex manner in which  $F_T$  and  $F_C$  depend on  $R$ , for a wide range of conical pendulum lengths  $L$ . It is of considerable interest to make direct reference to the functional dependence of the conical pendulum period  $T$  on the orbital radius, through the equation below, which is referred to as Equation (1) in the paper by (Dean & Mathew, 2017) and reproduced below for convenient reference:

$$T = 2 \pi \sqrt{\frac{\sqrt{L^2 - R^2}}{g}}$$

The two forces  $F_T$  and  $F_C$  depend on  $R$  in a very different way (although mathematical similarities exist in the form of the equations), which will be considered below.



*Chart 1.  $F_T$  and  $F_C$  dependence on Orbital Radius for nine different  $L$  values*

In order to gain a detailed understanding of the physical information displayed in the chart, it is advisable to make reference to the Appendix, which contains the colour-coded chart legend and provides an identification of the lines (solid and dashed) as well as the solid data markers. The shapes of the  $F_T$  and  $F_C$  lines in Chart 1 can be readily explained by considering the algebraic (rather than the trigonometric) form of the appropriate equations, because the orbital radius  $R$  is specifically included. Since  $0 \leq R \leq L$ , it is of significant interest to examine the end-points of the two graphical plots. The two extreme values of  $R$  will be considered below, starting with the situation where  $R = 0$ :

$$\underbrace{F_T}_{R=0} = \frac{m g L}{\underbrace{\sqrt{L^2 - R^2}}_{R=0}} = m g \quad (15)$$

$$\underbrace{F_C}_{R=0} = \frac{m g R}{\underbrace{\sqrt{L^2 - R^2}}_{R=0}} = 0 \quad (16)$$

The nine calculated curves that are shown on the chart above, clearly demonstrate that in the asymptotic limit that arises when the conical pendulum orbital radius decreases progressively to zero, then the tension force becomes equal to  $mg$ . The physical interpretation of this specific limit is that the conical pendulum string would be hanging, such that when projected in one of the two vertical planes ( $x$ - $y$  or  $x$ - $z$ ) the mass would appear to be vertically downwards. A zero orbital radius also requires that the centripetal force becomes asymptotically zero, since Equation (9) clearly shows this to be the case.

When the apex angle  $\phi$  is large and approaching (but physically never reaching)  $90^\circ$  it is appropriate to consider the mathematical upper orbital radius limit  $R = L$  giving the following:

$$\underbrace{F_T}_{R=L} = \frac{m g L}{\underbrace{\sqrt{L^2 - R^2}}_{R=L}} = \infty \quad (17)$$

$$\underbrace{F_C}_{R=L} = \frac{m g R}{\underbrace{\sqrt{L^2 - R^2}}_{R=L}} = \infty \quad (18)$$

Assuming the conical pendulum mass  $m$  remains constant, it can be seen that when the orbital radius  $R = 0$ , the vertical axis intercepts of Chart 1 will all have exactly the same tension force  $F_T$ , namely,  $(mg)$  the gravitational force (Chart 3 demonstrates the situation for different masses). The simple analysis above also shows that the centripetal force  $F_C$  will always start at the coordinate origin and initially appear to be independent of the pendulum mass.

As the orbital radius  $R$  increases from zero, then the apex angle  $\phi$  correspondingly increases, according to  $0 \leq \phi \leq 90^\circ$  and consequently the trigonometric functions become:

$$\underbrace{F_C}_{\phi \rightarrow 90^\circ} = \frac{m g \sin \phi}{\underbrace{\cos \phi}_{\phi \rightarrow 90^\circ}} \rightarrow \frac{m g (\sin 90^\circ)}{\underbrace{\cos \phi}_{\phi \rightarrow 90^\circ}} = \underbrace{F_T}_{\phi \rightarrow 90^\circ} = \infty \quad (19)$$

Chart 1 unquestionably shows that the centripetal force  $F_C$  becomes progressively closer to  $F_T$  as the orbital radius increases, corresponding to the progressive increase in the apex angle  $\phi$ . Both forces are asymptotic (mathematically) to infinity on the vertical force axis; this would correspond to the conical pendulum being at  $\phi = 90^\circ$ , therefore horizontal and consequently physically unattainable. However, a straightforward projection of the respective asymptotic forces downwards onto the horizontal radius axis, occurs at  $R = L$  for all of the conical pendulum lengths  $L$  that have been investigated computationally. Theoretically from Equations (13 & 14), the downwards projection limit  $R = L$  must apply for all possible conical pendulum lengths when both  $F_T \rightarrow \infty$  and  $F_C \rightarrow \infty$ . An inspection of Chart 1 clearly supports this interpretation of the theoretical chart plots.

When the tension force  $F_T$  and centripetal force  $F_C$  are, both plotted against the apex angle  $\phi$ , the calculated chart lines for all selected pendulum lengths  $L$ , falls on only two  $F_T$  and  $F_C$  lines as shown on Chart 2 below. It is important to note that this chart contains all of the theoretical lines for the nine values of  $L$  that were used computationally. The theoretical lines are superimposed where they overlap when plotted, hence the appearance of only two theoretical lines:

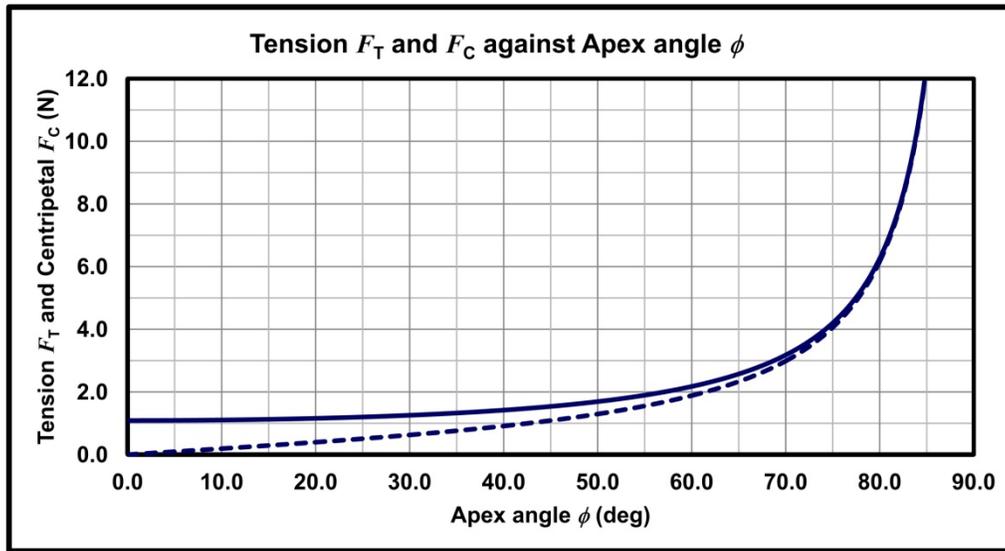


Chart 2.  $F_T$  and  $F_C$  dependence on Apex angle  $\phi$  for a constant mass

This is explained by making reference to Figure 1, where the apex angle  $\phi$  can be seen to be given by Equation (20) below. When the apex angle  $\phi$  is held constant at any specific value, there are an infinite set of paired values for the orbital radius  $R$  and pendulum length  $L$  that will satisfy the arcsine equation for  $\phi$ . Consequently, provided the pendulum mass remains constant, all values of  $F_T$  and  $F_C$  for any pendulum length  $L$  must be positioned on only two lines (one each for the tension force  $F_T$  and the centripetal force  $F_C$ ) as shown on Chart 2 above.

$$\left. \begin{aligned} R_1 &= L_1 \sin \phi \\ R_2 &= L_2 \sin \phi \end{aligned} \right\} \Rightarrow \phi = \sin^{-1} \left( \frac{R_i}{L_i} \right) \quad (\text{where } i = \text{integer}) \quad (20)$$

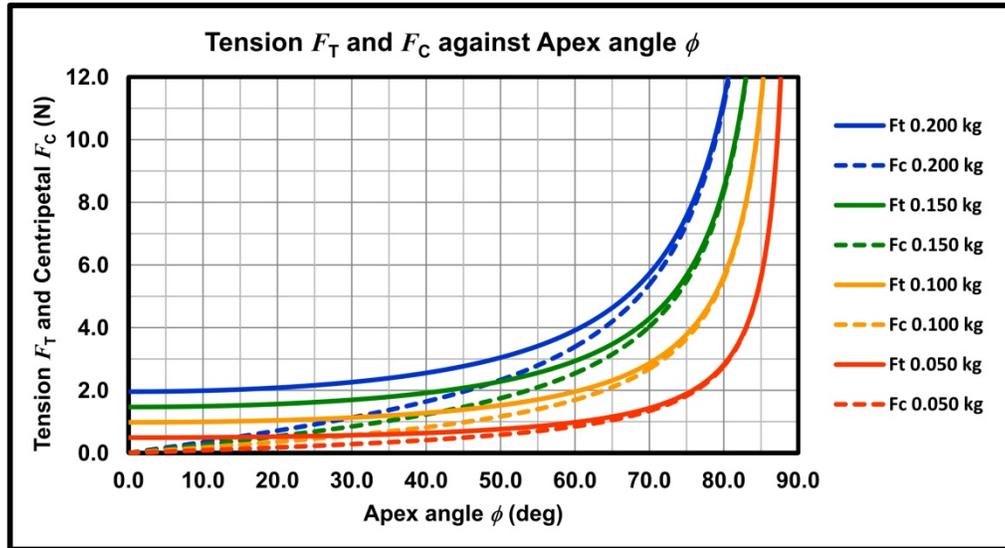
It is noted that from the earlier analysis (Equation 11),  $F_C$  is proportional to  $F_T$  as:

$$F_C = m g \tan \phi = F_T \sin \phi$$

As the apex angle  $\phi$  progressively increases towards  $90^\circ$  then the sine term correspondingly increases towards the mathematically limiting value of +1. Consequently the centripetal force progressively approaches (but physically can never actually equal) the tension force, which is observed as the asymptotic behavior on the above chart.

The fact that the tension force  $F_T$  and the centripetal force  $F_C$  are specifically mass-dependent is demonstrated by the theoretically calculated chart shown below. It is essential to note that this particular chart is unique in one respect; the self-explanatory colour-coded lines refer to different masses as indicated in the included chart legend, for the  $F_T$  and  $F_C$  lines that are plotted. All the

remaining charts that are presented in this paper possess the same colour code, which refers only to the individual pendulum lengths  $L$  (refer to the Appendix for details).



**Chart 3.**  $F_T$  and  $F_C$  dependence on Apex angle  $\phi$  with four different mass values

Attention is directed to the vertical axis intercepts for the tension force  $F_T$  solid lines that clearly indicate the initial values of this force are equal to the gravitational force ( $mg$ ), which is expected. The initial angular situation is taken to be  $\phi = 0$ , corresponding to the pendulum string being vertical. The dashed lines on Chart 3, which apply to the centripetal force  $F_C$  all start from the coordinate origin, as expected from the derived Equations (11) presented earlier. Therefore, this particular chart helps to confirm the validity of the theoretical analysis, with respect to the functional dependence of the two forces  $F_T$  and  $F_C$  on the apex angle  $\phi$ .

From an earlier analysis, it was explained that the tension force  $F_T$  and centripetal force  $F_C$  are related through the above Equation (11), where it is clear that  $F_C$  depends on  $F_T$  as:

$$F_C = F_T \sin \phi$$

It is a mathematical point of interest to draw attention to the fact that as the orbital radius  $R$  progressively increases, then the apex angle  $\phi$  correspondingly increases towards the maximum possible value of  $90^\circ$  (although it is not physically possible for  $\phi = 90^\circ$ ). Therefore, from the above equation it is readily observed that:

$$F_C \rightarrow F_T$$

Chart 4 below shows this relationship theoretically up to approximately 12 N (which corresponds to an apex angle  $\phi \approx 85^\circ$  for a conical pendulum mass  $m = 0.1111$  kg). This is regarded as the maximum feasible tension force that is attainable under reasonable conditions and has therefore been selected for computational purposes.

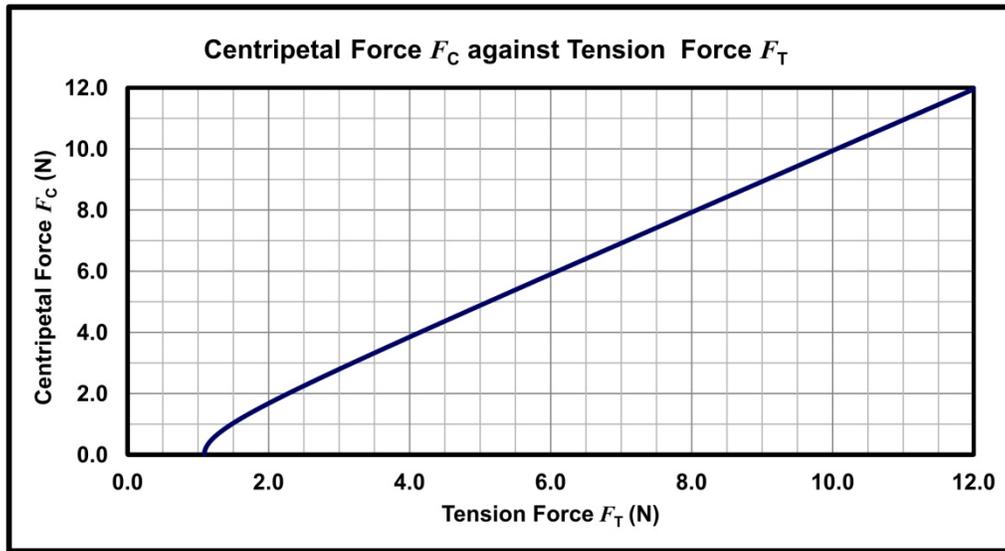


Chart 4.  $F_C$  as a function of  $F_T$  up to 12.0 N

The appearance of the above chart line requires some explanation; from the basic physics equations for the angular dependence of the tension force  $F_T$ , the minimum possible value for  $F_T$  has previously been shown as being equal to the gravitational force ( $mg$ ), which accounts for the starting position of the line along the horizontal axis. At the point where  $F_T = (mg)$  then  $F_C = 0$ , (and the conical pendulum string is hanging vertically down so  $\phi = 0$ ). It is appropriate at this juncture to once again refer to Equations (11), which are reproduced below for convenience:

$$F_T = \frac{m g}{\cos \phi} \quad \text{and from Figure 1: } F_C = m g \tan \phi \Rightarrow F_C = F_T \sin \phi$$

From these  $F_T$  and  $F_C$  equations, it is apparent that as the apex angle  $\phi$  increases from zero, the centripetal force progressively becomes closer to the tension force (i.e. more linear), although always remaining lower in value. The table presented below (which is calculated from the above equations) clearly indicates the very rapid transition of both forces towards linearity, over the scale of their respective Chart 4 axes. Since the lowest possible value of  $F_T$  is  $mg \approx 1.1$  N, the region from the coordinate origin to  $(mg)$  is mathematically undefined and therefore physically unattainable experimentally. Attention is drawn to the fact that although this paper is essentially theoretical in nature; any obvious physical limitations are taken into consideration.

The below table indicates that the initial value of the tension force  $F_T \approx 1.1$  N (which is  $mg$ ) and the centripetal force  $F_C = 0$ ; when the tension has increased to  $F_T = 1.5$  N, then  $F_C = 1.0$  N which corresponds to the apex angle having a value of  $\phi = 43.5^\circ$ . This relatively large and rapid angular change (starting from  $\phi = 0$ ) occurs within the small curved section of the line shown on Chart 4. The initially curved (vertically projecting) line progressively becomes more linear as the two forces continue to increase and become asymptotically equal in magnitude. In a mathematical sense, this can be interpreted as the positive first derivative asymptotically approaching the value of (+1).

**Table 1.** Forces  $F_T$  and  $F_C$  with reciprocals and corresponding Apex angle  $\phi$

Forces, Reciprocals and corresponding Apex angles				
$F_T$ (N)	$F_C$ (N)	$1 / F_T$ (N) <sup>-1</sup>	$1 / F_C$ (N) <sup>-1</sup>	$\phi$ (deg)
1.1	0.0	0.9195	$\infty$	0.0
1.5	1.0	0.6667	0.9680	43.5
2.0	1.7	0.5000	0.5958	57.1
2.5	2.3	0.4000	0.4442	64.2
3.0	2.8	0.3333	0.3577	68.7
3.5	3.3	0.2857	0.3006	71.9
4.0	3.8	0.2500	0.2598	74.2
4.5	4.4	0.2222	0.2290	76.0
5.0	4.9	0.2000	0.2049	77.4
5.5	5.4	0.1818	0.1855	78.6
6.0	5.9	0.1667	0.1695	79.6
6.5	6.4	0.1538	0.1560	80.4
7.0	6.9	0.1429	0.1446	81.1
7.5	7.4	0.1333	0.1348	81.7
8.0	7.9	0.1250	0.1262	82.2
8.5	8.4	0.1176	0.1186	82.6
9.0	8.9	0.1111	0.1119	83.1
9.5	9.4	0.1053	0.1060	83.4
10.0	9.9	0.1000	0.1006	83.8
10.5	10.4	0.0952	0.0958	84.1
11.0	10.9	0.0909	0.0914	84.3
11.5	11.4	0.0870	0.0873	84.6
12.0	12.0	0.0833	0.0837	84.8

It is of interest to analytically explore the variation (and physical dependence) of two primary physical parameters that form the basis of this paper. Namely, the conical pendulum tension force  $F_T$  and orbital period  $T$ . The principal equation that will be used for this parametric analysis, namely Equation (5) is reproduced below for convenient reference:

$$F_T = \frac{4 \pi^2 m L}{T^2}$$

Inspection of the basic mathematical format of the above equation clearly reveals the familiar symbolic format of an inverse-square law. For this particular case, the parametric dependence of the tension force is the inverse square of the conical pendulum period. This is an extremely familiar graphical line-shape that is taught in all standard physics courses and appears in many analyses of different types of force laws.

The basic theoretical analysis can be visualized by closely examining the chart below, where the  $x$ -axis and  $y$ -axis are essentially taken to be the two relevant computationally determined values of  $F_T$  ( $y$ -axis) and  $T$  ( $x$ -axis) and follow the mathematical dependence implied above (as an inverse square law).

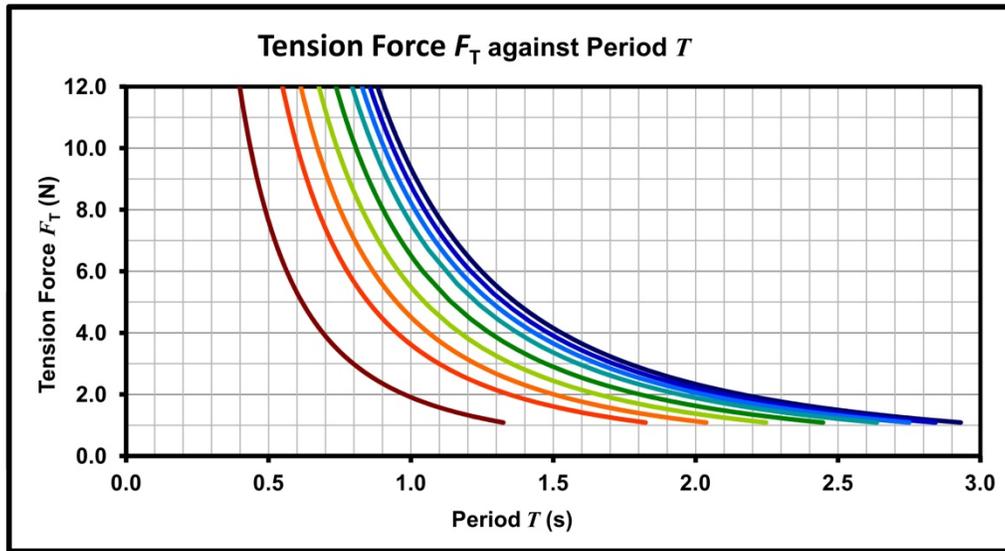


Chart 5.  $F_T$  as a function of Period  $T$

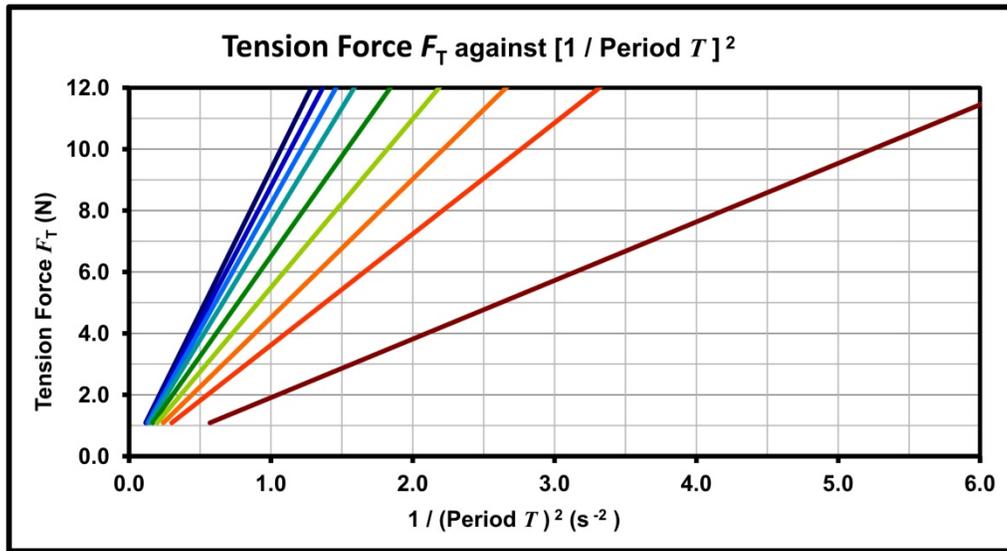
It is observed that the apparent horizontal asymptotes abruptly terminate at well-defined values of the period  $T$ ; this corresponds to the period that is appropriate for a simple pendulum of the same length  $L$  (this is the longest possible period and occurs for when  $R = 0$ ). A straightforward analysis will be helpful with understanding the physical reason for the behavior shown above. In what immediately follows, the subscripts  $SP$  and  $CP$  refer to the simple pendulum and conical pendulum respectively:

$$\left. \begin{aligned} T_{SP}^2 &= \frac{4\pi^2}{g} L \\ T_{CP}^2 &= \frac{4\pi^2}{g} \sqrt{L^2 - R^2} \end{aligned} \right\} \Rightarrow T_{CP}^2 = \frac{\sqrt{g^2 T_{SP}^4 - 16\pi^4 R^2}}{g}$$

$$R = 0 \Rightarrow T_{CP}^2 = T_{SP}^2 \Rightarrow T_{CP} = T_{SP} \quad (21)$$

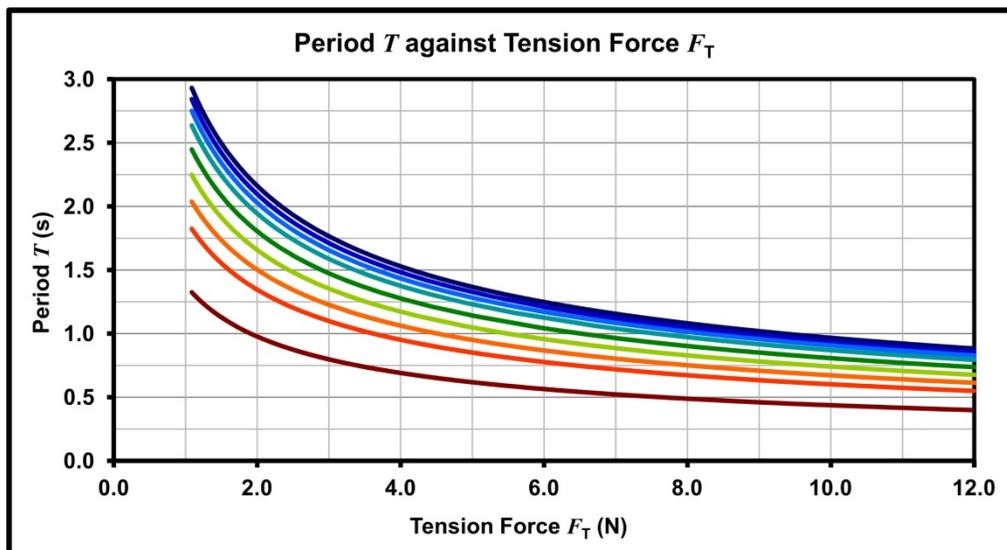
The above concise analysis therefore confirms the behavior observed in Chart 5, for when the tension force  $F_T$  has the minimum possible value ( $F_T = mg$ ). A consideration of the above chart with respect to Equation (5) directly suggests an alternative functional scaling for the horizontal axis and is shown in the chart below.

$$F_T = (4\pi^2 m L) \left( \frac{1}{T^2} \right)$$



*Chart 6.  $F_T$  as a function of  $(1/\text{Period } T)^2$*

In this linear form, the slopes of the straight lines ( $4 \pi^2 m L$ ) can provide a straightforward graphical technique to determine an unknown conical pendulum mass  $m$  experimentally. As shown above, the straight lines cannot physically pass through the coordinate origin because the longest period must be equal to the period of a simple pendulum of the same length (this defines the smallest attainable value for the reciprocal of the conical pendulum period squared). The second reason for the above observation is due to the fact that the tension force must have a minimum non-zero value on the vertical axis ( $F_T = mg$ ). When the two axes are interchanged in Chart 5 the following alternative presentation is obtained:



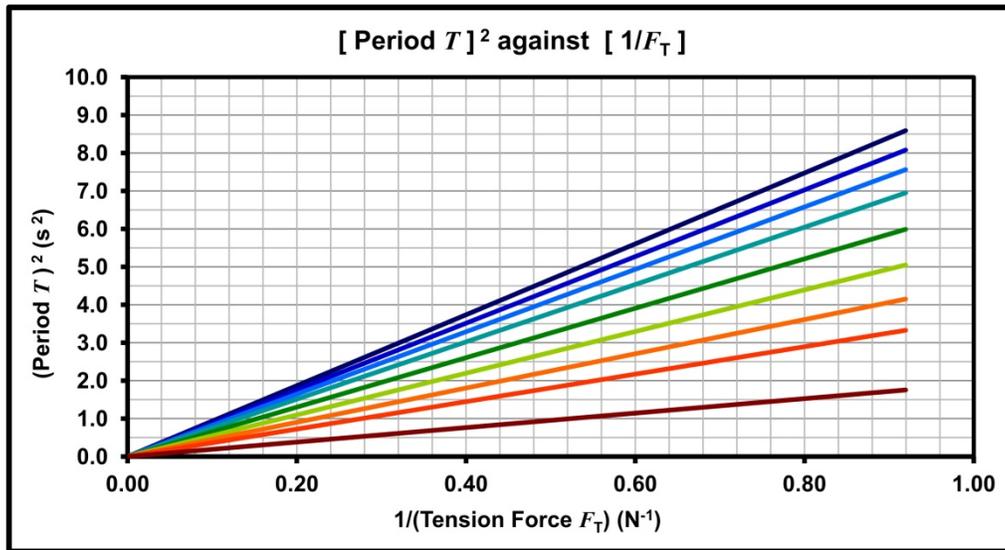
*Chart 7. Period  $T$  as a function of  $F_T$*

This selection of the chart axes implies that the period  $T$  can be considered to be dependent on the tension force  $F_T$ , while the earlier chart presentation implied the reverse. The selection of parametric axis generally follows the convention where the “independent” variable is plotted along the horizontal axis ( $x$ -axis) and the “dependent” variable is plotted vertically ( $y$ -axis). A classic example of when this format is usually not adhered to is the well-known equation  $V = I R$  often referred to as Ohm’s Law ( $V = I R \Rightarrow y = m x$  with the straight line of slope  $R \equiv m$  passing exactly through the coordinate origin). It is clearly apparent from the laws of Physics that the potential  $V$  does not depend on the current  $I$  flowing through the resistance  $R$ , however, the inverse is true (the current  $I$  depends on the potential  $V$  across the resistance). When to follow the axes convention can frequently be determined from the equation that relates the “dependent” and “independent” parameters. In the present case, it is assumed that the rotational period  $T$  is in fact dependent on the string tension force  $F_T$  even though the usual representative convention has not been applied inflexibly.

By referring to Equation (5), it can be readily observed that the two principal theoretical parameters can be re-arranged to yield Equation (22) below:

$$T^2 = (4 \pi^2 m L) \left( \frac{1}{F_T} \right) \tag{22}$$

Using this equation form, the chart appears as shown below:



**Chart 8.** Period  $T^2$  as a function of  $(1/F_T)$

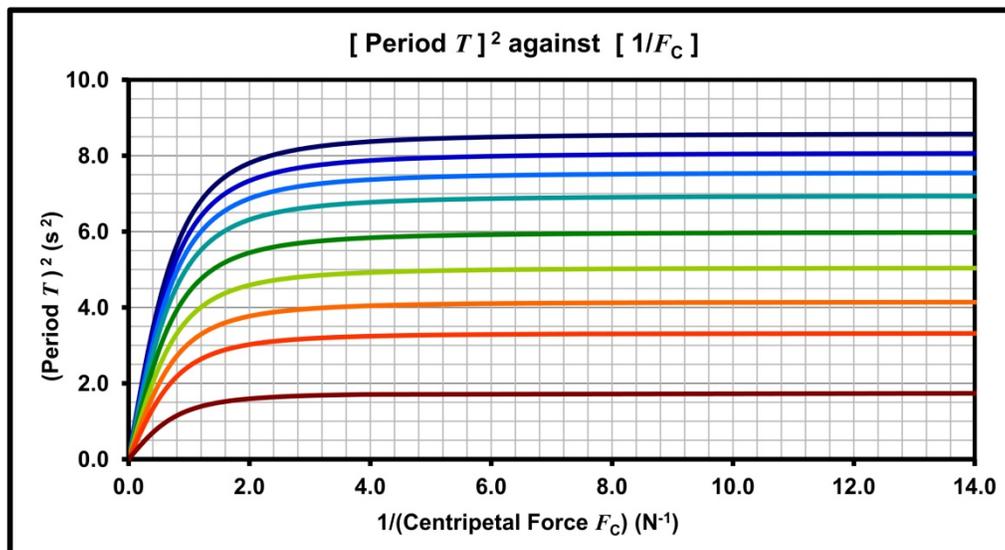
With respect to this particular chart axes representation, the straight lines can mathematically pass through the coordinate origin, although the situation cannot be physically achieved. Once again, just as for Chart 6, the slopes of the straight lines ( $4 \pi^2 m L$ ) can be used to determine an unknown conical pendulum mass  $m$ . It is of interest to note that the right-side of every line stops

abruptly along the horizontal axis at a calculated value of  $(1/F_T) = 0.9195$ , corresponding to the minimum possible value of the tension force ( $F_T = mg$ ), with the mass hanging vertically down.

As an extension of the above chart and analysis, it is appropriate to consider the variation of the period squared with respect to the centripetal force  $F_C$  that is directly responsible for the horizontal circular motion of the conical pendulum. Equations (9 & 22) provide the following:

$$T^2 = \left(4 \pi^2 m R\right) \left(\frac{1}{F_C}\right) \quad (23)$$

The chart below shows the above equation for each of the nine values of the length  $L$ , where it is noted from Equation (23) that due to the non-constancy of the orbital radius  $R$ , the plotted lines will not be linear:



**Chart 9.** Period  $T^2$  as a function of  $(1/F_C)$

This chart can be explained by noting that the initially curved lines progressively become linear as the reciprocal of the centripetal force ( $1/F_C$ ) increases to the right (the centripetal force  $F_C$  itself increases towards the left), which corresponds to a progressive decrease in  $F_C$  itself. The square of the rotational periods (on the  $y$ -axis) initially increase as expected, however, they subsequently become asymptotically horizontal with values of the period squared corresponding to the period squared appropriate for a simple pendulum of the same length  $L$ . The rapidity with which the lines become horizontal is due to the non-linear reciprocal scale of the horizontal axis, as shown in the appropriate column of Table 1 above.

Theoretical analysis presented in earlier sections of this paper detailed the derivation of two concise equations for the tension force  $F_T$  and centripetal force  $F_C$  namely Equations (6 & 9) as reproduced below for convenient reference.

$$F_T = m L \omega^2 \quad \text{and} \quad F_C = m R \omega^2$$

It is of interest to consider the simultaneous variation of these two forces as a function of the square of the angular frequency, as shown in the chart below:

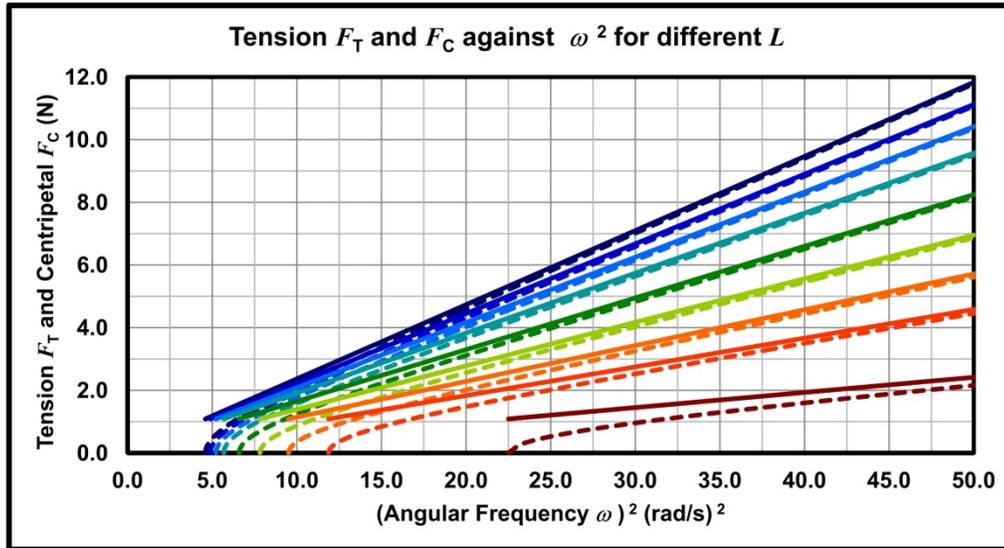


Chart 10.  $F_T$  and  $F_C$  as a function of  $\omega^2$

With reference to this figure and the Appendix, it is of importance to note that the solid lines and dashed lines represent  $F_T$  and  $F_C$  respectively. As expected from earlier discussion, the  $F_T$  lines all start from the minimum tension force value of  $F_T = (mg)$  on the vertical ( $y$ -axis) and with a horizontal displacement corresponding to the minimum value of the square of the angular frequency. This corresponds to the largest possible value of the orbital period squared, which occurs for a simple pendulum (subscript  $SP$ ) of the same length  $L$  in Equation (24) below:

$$\omega^2 = \frac{4 \pi^2}{T^2} \Rightarrow \omega_{min}^2 = \frac{4 \pi^2}{T_{max}^2} = \frac{4 \pi^2}{T_{SP}^2} \quad (24)$$

The centripetal force  $F_C$  (dashed lines) must also start from the same minimum value of the angular frequency along the horizontal axis, but from  $F_C = 0$  on the vertical ( $y$ -axis), meaning the conical pendulum string would consequently be hanging vertically downwards. It can be observed from Chart 10 that the plotted tension force  $F_T$  lines are all straight and would all pass directly through the coordinate origin if this was physically possible, with slopes of value  $(m L)$  for each length  $L$ .

Reference to Equation (3) provides the following:

$$\phi = \cos^{-1} \left( \frac{g}{L \omega^2} \right) \Rightarrow 0 \leq \frac{g}{L \omega^2} \leq 1$$

There are clearly two extreme mathematical limits to be considered as shown below:

$$\frac{g}{L \omega^2} = \begin{cases} 0 & \Rightarrow \phi = 90^\circ \\ 1 & \Rightarrow \omega^2 = \frac{g}{L} \end{cases} \quad (25)$$

The second limit (for which the Apex angle  $\phi = 0$ ) must correspond to the minimum possible value for the square of the angular frequency, which therefore immediately provides the equation shown below. From Equations (24 & 25) it is clear that there must be a minimum threshold (or critical) angular velocity before the centripetal force becomes present and the conical pendulum can start to perform uniform planar circular motion (see also Klostergaard, 1976).

$$\omega_{\min}^2 = \frac{g}{L} = \frac{4 \pi^2}{T_{\max}^2} = \frac{4 \pi^2}{T_{SP}^2}$$

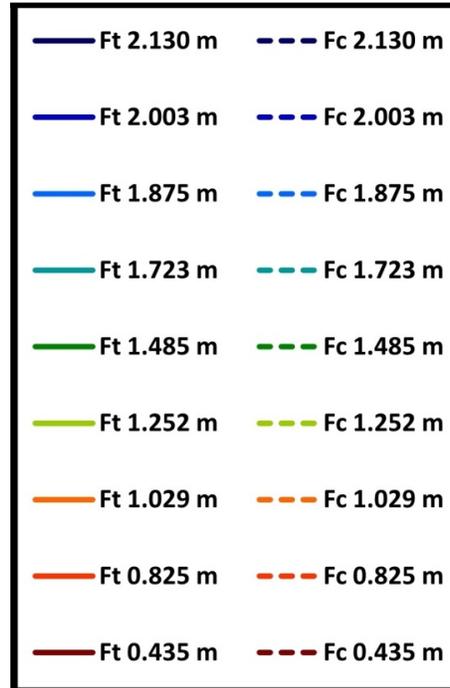
## DISCUSSION OF RESULTS AND CONCLUSIONS

It can be readily seen from the charts presented in this paper, that the theoretical analysis is wide-ranging and continuous, over the complete range of conical pendulum period, tension force, centripetal force, orbital radius and apex angle values that were computationally studied. With reference to the charts and the accompanying theoretical equations, the overall shapes of plotted lines, their initial points, end-points and any unmistakably observable axial asymptotes ( $x$ -axis and  $y$ -axis) were clearly explained. It is proved theoretically that when the tension force and centripetal force are both plotted on the vertical  $y$ -axis against the apex angle along the horizontal  $x$ -axis, there will be only two lines (despite the two forces being determined for nine different pendulum lengths). The asymptotic nature of these two forces becoming progressively more equal as the apex angle increases is also explained; as is the effect of using different masses for the conical pendulum. When the tension force is plotted against the period, the abrupt termination of the horizontal asymptotes at well defined values of the period is explained in terms of the period of a simple pendulum having the same string length as the conical pendulum. When the square of the conical pendulum period is plotted as a function of the reciprocal of the tension force, the abrupt termination of the calculated chart lines along the horizontal axis is explained as being unequivocally due to the minimum value of the tension force. The effect that using different mass values has on the computational results were also demonstrated.

## APPENDIX - CHART LEGEND FOR THEORY LINES

The chart legend used to identify the theoretical lines is presented below (adapted from Dean & Mathew, 2017), from the largest pendulum length ( $L = 2.130$  m) to the shortest ( $L = 0.435$  m).

Solid lines are used for the tension force and dashed lines represent the centripetal force, when these are required to illustrate the theoretical calculations.



**Figure 2.** Chart Legend

*Note:* if there is either a single solid dark-blue line, or a solid dark-blue line and a dashed line of the same colour, plotted on any particular chart, the accompanying section of text will make it clear that the lines are from a theoretical calculation. It will be clear from the theoretical analysis that lines on the chart are independent of the pendulum length  $L$ , by being superimposed.

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