Pythagoras - Journal of the Association for Mathematics Education of South Africa
ISSN: (Online) 2223-7895, (Print) 1012-2346

Corrigendum: Metacognitive awareness and visualisation in the imagination: The case of the invisible circles

Authors:
Divan Jagals[1]
Marthie van der Walt[2]

Affiliations:
[1]School for Professional Studies in Education, Faculty of Education Sciences, North-West University, South Africa

[2]School for Science and Technology Education, Faculty of Education Sciences, North-West University, South Africa

Corresponding author:
Divan Jagals,
12782890@nwu.ac.za

Date:
Published: 13 Feb. 2019

How to cite this article:
Jagals, D., & Van der Walt, M. (2019). Corrigendum: Metacognitive awareness and visualisation in the imagination: The case of the invisible circles. *Pythagoras*, *40*(1), a464. https://doi.org/10.4102/pythagoras.v40i1.464

In the author list of this article published earlier, Marthie van der Walt's first name and ORCID were unintentionally misprinted as 'Martha' and 'https://orcid.org/0000-0002-6057-8352'. The author's correct first name is 'Marthie' and the ORCID is 'https://orcid.org/0000-0002-0465-6600'. The authors sincerely regret this error and apologises for any inconvenience caused.

Note: DOI of original article: https://doi.org/10.4102/pythagoras.v39i1.396

Pythagoras - Journal of the Association for Mathematics Education of South Africa
ISSN: (Online) 2223-7895, (Print) 1012-2346

Metacognitive awareness and visualisation in the imagination: The case of the invisible circles

Authors:
Divan Jagals[1]
Martha van der Walt[2]

Affiliations:
[1]School for Professional Studies in Education, Faculty of Education Sciences, North-West University, South Africa

[2]School for Science and Technology Education, Faculty of Education Sciences, North-West University, South Africa

Corresponding author:
Divan Jagals,
12782890@nwu.ac.za

Dates:
Received: 14 Sept. 2017
Accepted: 02 July 2018
Published: 13 Aug. 2018

How to cite this article:
Jagals, D., & Van der Walt, M. (2018). Metacognitive awareness and visualisation in the imagination: The case of the invisible circles. *Pythagoras*, *39*(1), a396. https://doi.org/10.4102/pythagoras.v39i1.396

Awareness of one's own strengths and weaknesses during visualisation is often initiated by the imagination – the faculty for intuitively visualising and modelling an object. Towards exploring the role of metacognitive awareness and imagination in facilitating visualisation in solving a mathematics task, four secondary schools in the North West province of South Africa were selected for instrumental case studies. Understanding how mathematical objects are modelled in the mind may explain the transfer of the mathematical ideas between metacognitive awareness and the rigour of the imaginer's mental images. From each school, a top achiever in mathematics was invited to an individual interview (n = 4) and was video-recorded while solving a mathematics word problem. Participants also had to identify metacognitive statements from a sample of statement cards (n = 15) which provided them the necessary vocabulary to express their thinking during the interview. During their attempts, participants were asked questions about what they were thinking, what they did and why they did what they had done. Analysis with *a priori* coding suggests the three types of imagination consistent with the metacognitive awareness and visualisation include initiating, conceiving and transformative imaginations. These results indicate the tenets by which metacognitive awareness and visualisation are conceptually related with the imagination as a faculty of self-directedness. Based on these findings, a renewed understanding of the role of metacognition and imagination in mathematics tasks is revealed and discussed in terms of the tenets of metacognitive awareness and imagination. These tenets advance the rational debate about mathematics to promote a more imaginative mathematics.

Introduction and problem statement

There are not many references to imagination in metacognition research, particularly in the context of mathematics education. There are, however, publications that illuminate the importance of imagination in fields related to mathematics education. Some examples of these fields include physical science and technology education (e.g. Nemirovsky & Ferrara, 2009). As an example of this application, imagine looking up at a water wheel and noticing how it accelerates, or imagine yourself turning a wheel on a bicycle with your hand. Can you sense its acceleration, direction and force with the energy applied?

The scarcity of accepting the applicability of imagination in the field of mathematics could be because imagination received a bad reputation in Western philosophy where some scholars believed that imagination is inferior to reason (e.g. Spinoza's naturalistic theory of imagination) while others (who agree with Hume's theory of imagination) argue that imagination constrains metaphysics and makes way for scientific reasoning. We hold on to Wenger's (1998) view of imagination, stating: imagination is 'a process of expanding our self by transcending our time and space and creating new images of the world and ourselves' (p. 176). Both these schools of thought, however, imply that the understanding of imagination in metacognition and mathematics education literature have, thus, become vague. As a result, a predominant objectivist perspective on school mathematics is held, one that upholds mathematics as a 'rational structure to reality, independent of the beliefs of any particular people' (Murray, 2013, p. 387). Visualisation and imagination are both terms that preserve the spatial layout of an image. The ability to focus your attention on to an object, to imagine being closer or further away from it, and to perceive the greater detail (e.g. at the centre of the object compared to elsewhere) are some examples of the powerful capacity that facilitates the metacognitive skill of visualisation (Gilbert, 2005). However, research by Fried (2014), for example, shows that students prefer algorithmic mathematical problems, emphasising routine memorised procedures, over non-routine problems that require some form of visualisation. In this regard, Reddy et al. (2016) show that, for Grade 9 advanced international benchmark levels, learners are expected to apply and reason in a variety of problem situations, to solve linear equations and make generalisations, all of which requires metacognition (Jagals & Van der Walt, 2016) and visualisation. Visualisation is therefore not only about seeing, but is a process and product of reflecting, thinking deeply and creating illustrations or mental models as meta-representations of what is imagined. In classrooms where routine procedures dominate problem-solving activities, a typical 'talk and chalk', and 'drill and practice' pedagogy reigns, and might explain why South Africa falls short in the Trends in Science and Mathematics Study when compared to the mathematics performance of other countries (Reddy et al., 2016). We therefore argue that, in non-routine problems, the power of metacognition and imagination can make the mathematical concepts visible, tangible and represented in a more accessible form through visualisation. Learners who develop this imaginative power can become more aware of their own understandings, in the sense that they will be able to see the invisible ideas in mathematics, and be equipped to solve problems of a similar nature. Such a holistic view of mathematics problem-solving emphasises a different and more sufficient approach to teaching and learning of visualisation. If we wish to understand the conceptual linkage between imagination and metacognitive awareness, then the reductionist view of mathematics must be revisited to bring to mind and imagine what is unseen about the problem or its approach.

A holistic view of mathematics aspires to a more imaginative approach to teaching and learning mathematics. In the sections that follow, the inclusion of the imagination as part of the holistic view of mathematics education is explored in terms of the role metacognition and imagination play to facilitate visualisation about a Euclidean geometry task. In contrast to the rational view of mathematics, we suspected that learners who follow such a holistic approach would more likely be able to become aware of their own strengths and weaknesses during problem solving. As part of a larger project (Jagals, 2013), the study reported in this article explored the metacognitive awareness and imagination that emerged from learners' mathematics problem-solving experiences. Specifically, the study sought to answer the question: *What role does metacognitive awareness and imagination play in facilitating visualisation in solving a mathematics task?* The concepts of metacognitive awareness and imagination were aligned against the theoretical framework of embodied, situated and distributed cognition to explore learners' awareness and imagination of their visualisation processes.

The conceptual and theoretical framework on which the study was based and the empirical design of the study that was done follows. The results informed the role of imagination in metacognitive awareness during a mathematics visualisation task. In essence, the findings portray the results of four instrumental case studies and a discussion in view of the theoretical framework. This is concluded by some guidelines for classroom practice and recommendations for future consideration.

Conceptual framework

Imagination can serve as an indispensable tool for unlocking and discovering reality, and the mathematical ideas that surround reality's hidden structures. Without knowing how to 'see' a geometric point, or without awareness of the fact that a geometrical plane has no thickness, or that a series of points about the same axis can form a circle rather than a straight line, a personal mistrust in one's thinking and awareness about reality (Schoenfeld, 2013), and mathematics for that matter, can develop. In Euclidean circle geometry, imagination can therefore serve as a medium for scientific visualisation, as Galileo used for his thought experiments, and may help in understanding how mathematical models are formed, thus encouraging reason. Take for example a situation where learners reflect on problem-solving processes. They tend to look at and sometimes away from their written or sketched work. They, then, do not only think metacognitively about the process, but also bring to mind and imagine what is unseen about the problem or its approach.

Imagination and circle geometry

The circle is probably one of the oldest figures of mathematics (Kasner & Newman, 2001). It has characteristics of a non-straight line and is often regarded as a polygon with an infinite number of sides. Thus, a circle is a limit of the inscribed polygons. There are, in fact, some interesting generalisations of the circle when viewed this way. For example, Greek mathematicians such as Apollonius posed the classical circle problem: *given three fixed circles, find a circle that touches them all*. Teachers and learners could, through visualisation, try to methodologically fit a circle to touch all circles, starting with the smallest possible radius and using trial and error. The link between visualisation and imagination could offer both teachers and learners a more constructive approach to this problem. Imagination, in light of Wenger's (1998) definition, could facilitate metacognitive thinking about the experience, to encourage reasoning towards a new knowledge of reality, or at least of circles. As this example suggests, creative problem-solvers engage in metacognitive thinking about strategies and monitor and evaluate their attempts. During this process the faculty of the imagination is employed to facilitate the visualisation of the problem-solving process. The rationale behind this argument is that mathematical models (in the case of Apollonius's circles) are generated through problem-solving experiences and critical thinking which have become a cornerstone in visualisation. Visualisation then serves as a metacognitive skill (Gilbert, 2005). Yet, problem-solving skills require metacognitive awareness, which, in turn, predicts imagination (Liang, Hsu & Chang, 2013). Learning opportunities that are, therefore, framed around exploring, questioning, understanding and imaginative experiences are exemplary of the role metacognitive awareness plays in the imagination.

Theoretically, Franklin and Graesser (1999) clarify that precision is needed to explain what it means when a mathematical image is brought to mind. The image conjured up about Apollonius's circles (mentioned above) preceded any explanation about radius, foci, area, circumference or diameter. Yet, the image conjured up by the mention of 'circles' is an implicit theory with particular visual and spatial content. Further examples of such an implicit theory of circles include wheel of fortune, circle of life, ring of fire, sphere of philosophy, to name but four. The theory behind these images is that they raise awareness of circle properties (or at least knowledge of its shape). However, when confronted with mathematical ideas, such as the properties of a circle, a learner will typically distinguish between three stages of this imaginative experience. Franklin and Graesser explain these stages as metacognitive awareness of (1) a conjured image, (2) a line drawing not necessarily labelled and (3) attached symbols that serve as a map or a plan with some symbolised spatial relation.

Imagination as visualisation of metacognitive awareness

Visualisation, also called the representational view of the mind (Makina, 2010), integrates the mental processes made up of visual imagery, visual memory, visual processing, visual relationships, visual attention and visual imagination (p. 25). The aim, then, of teaching mathematics, with these visual functions in mind, is to support learners in the construction of mental representations of mathematical phenomena such that they develop an awareness of the underlying notions or concepts that the mathematical ideas develop from. Such an understanding not only brings to mind the imagined visual object (such as a circle), but also creates awareness of the knowledge of the person, task and strategies in order to provide a mental space in which these representations can be regulated. Mainly, visualisation should be considered as a fundamental aspect of mathematics learning and understanding (Makina, 2010) as learners learn to demonstrate their thinking and, thus, become metacognitively aware of the experiences this thinking involves. Such experiences could be reading for a particular purpose to select important words or phrases, describing them, providing proof of the knowledge they are accepting, examining or recognising. As a result, these representations can act as ways in which thinking and reasoning can be visualised, and acted out, or embodied, to extend the knowledge of one's own understanding, the problem or task and the strategies to solve a particular problem. Imagination can therefore serve as a crucial faculty of the metacognitive awareness of one's own and others' mathematical understanding. The awareness, then, transposes to other higher order faculties, such as metacognitively planning, monitoring and evaluating one's attempts in solving a mathematics task, and assists in directing one's learning and problem-solving attempts.

Theoretical framework

The cognitive processes related to the expression of metacognitive awareness are embodied, situated and distributed in nature.

Embodied, situated and distributed cognition

The theory of embodied, situated and distributed cognition is applied as a lens to explore the role of metacognitive awareness, imagination and visualisation. This theory suggests we belong within a reality through actual engagement with that reality, and in the activities we engage in we develop the power of our imagination (Murray, 2011), for instance solving a mathematics task. In terms of visualisation, this means that learners might imagine themselves drawing a circle as part of an activity, becoming aware of an (often similar) experience in order to gain knowledge (or cognition) from that experience needed to solve the problem or create a mathematical model. According to theorists of imagination (Murray, 2013; Yueh, Jiang & Liang, 2014) and embodied, situated and distributed cognition (Clark, 1998; Nemirovsky & Ferrara, 2009), cognition becomes embodied when the mental mathematical imaginations (e.g. of a circle) are expressed physically, that is, outside of the mind. This can be through body poise, facial expressions, gestures, utterances or any other motor activity, like drawing a circle, which resembles some form of mathematical intuition.

This embodiment of mathematics can be harnessed in situated specific contexts. O'Connor and Aardema (2005), for example, explain that cognition is imagined interaction with the imaginary world. It is in this world that senses, acts and ideas are shaped to visualise mental mathematical models. There are, however, some doubts as to the applicability of situated cognition to imagination (Jansen, 2013). Reasons for this include the offline nature of metacognition and imagination where the object or model is absent, and needs to be conjured up in the mind, and therefore does not define any particular situation. As an example of the situatedness, Jagals and Van der Walt (2016) developed a mathematics task, based on an example by Fortunato and Hecht (1991), during which the image of circles needs to be imagined and then visualised in the context situation of the problem. Then there is also the argument that situatedness deals with the conscious rather than the imagination which requires a phenomenal consciousness (awareness about a particular phenomenon). Imagination therefore deals with the paradigmatic of higher cognitive levels, beyond awareness, towards intuition. It therefore seems that imagination also associates with the distribution of cognition. In this sense, knowledge of circles' (mathematical) properties can be applied in calculations involving glass mirrors or Sacrobosco's sphere, thereby distributing the knowledge from one domain to another. In another thread, an argument can be made that a circle, as a mathematical object, can be used to show the definitions and logic, thoughts and other non-mathematical ideas (as in the case of Venn diagrams), and therefore plays a role, in terms of its imagined image in the metacognitive awareness.

In developing an empirically categorised imagination scale Liang et al. (2013) identified initiating, conceiving and transforming as three types of imagination. In this study, however, to explore the role of metacognitive awareness and imagination in facilitating visualisation in solving mathematics tasks, these types of imagination are aligned against the theoretical framework of embodied, situated and distributed cognition (Clark, 1998). The theoretical framework thereby serves to inform a more conciliatory image of the role of metacognitive awareness and imagination in facilitating visualisation and the consequences for metacognition and mathematics education research.

Tenets of metacognitive awareness and imagination

Metacognitive awareness refers to reflecting on understanding and regulating knowledge (Schraw & Moshman, 1995). When the mathematical model is not immediately tangible and accessible (like the conjured image of a circle) it can be connected with through the power of the imagination (Murray, 2011). This connection, according to Wenger (1998) 'needs an opening. It needs the willingness, freedom, energy, and time to expose ourselves to the exotic, move around' (p. 185). When learners engage and experiment with reality, they develop the power of imagination. Wenger's comments indicate a strong parallel between imagination and metacognitive awareness.

Initiating imagination involves the imaginative capability to explore and produce new, unfamiliar and unique ideas (Lin, Hsu & Liang, 2014). Learners who reflect deeply on themselves are more metacognitively aware. Perhaps this explains why new and novel ideas are often generated in solitude (Lin et al., 2014).

Conceiving imagination is the imaginative capability to understand mathematical ideas through personal intuition and using one's senses (Lin et al., 2014). Typically, individuals who have a strong sense of conceiving imagination are capable to understand and conceptualise mathematical ideas through rehearsed concentration (and thereby memorise a formula or strategy) and follow logical steps or apply appropriate strategies through metacognitive awareness of the knowledge of these strategies.

Transforming imagination is the imaginative capability to identify a pattern, and to transform it into an abstract idea to distribute and apply the conceptualised understanding across different situations (Lin et al., 2014). Individuals who have a strong transforming imagination therefore tend to reflect on mathematical ideas from past experiences, become aware of these knowledge and strategies, and mimic the route of memorised strategies they employed in different situations. They can also transform one abstract idea into another by manipulating the abstract pattern, typically as a learner will manipulate the formula for the area of a circle ($A = \pi r^2$) to calculate the circle's radius.

In other words, the role of metacognition and the imagination becomes powerful in an environment which provides learners with personal mathematical independence. When reflecting on a problem-solving experience the imaginer can become metacognitively aware of the imaginings and can express these imaginings in the form of a drawing or written steps (embodying their cognition) as strategies are applied (Nemirovsky & Ferrara, 2009). Utterances such as drawings or written steps allow us to explore the processes whereby learners express their imaginations during visualisation. To do this, these imaginations through metacognitive awareness were explored across a collective case study based on a mathematics word problem in the case of the invisible circle.

Empirical investigation

In order to explore the role of metacognitive awareness and imagination in facilitating visualisation in solving a mathematics task, a predominantly qualitative approach with a collective case study design was followed. An interpretivist perspective allowed us to explore the role of metacognition and imagination to facilitate visualisation, by means of the tenets of metacognitive awareness. The tenets show how the conceptual and theoretical frameworks of this study are intertwined and the empirical investigation that follows explores the significance of these tenets further.

Research design

Four instrumental case studies were conducted to explore the role of metacognition and imagination, to facilitate visualisation, towards embedded, situated and distributed cognition during visualisation of elementary Euclidean circle geometry. Each case study facilitated an understanding of the construction of metacognitive awareness on three levels, to be triangulated.

Sampling of the participants

Four participants were purposively and conveniently invited to take part in an individual interview and included the top achievers from Grade 8 and Grade 9 in the senior phase having mathematics as a compulsory school subject. Since this was not a comparative study, we wanted to know how academically strong learners think and do mathematics and therefore asked teachers to identify participants who achieved 70% or higher during the previous year for mathematics. In addition to this criterion, teachers were asked to identify those top achievers who would be willing and comfortable to communicate about their thinking. According to literature, learners who perform well in mathematics can provide more accurate information about their thinking and problem-solving behaviour. If learners were identified with poor academic achievement and who could not explain their thinking and reasoning, then it would have limited the opportunity to collect data regarding metacognitive awareness. Teachers at the three participating schools identified these participants mainly because of their achievement in mathematics at the end of the previous school year. To enhance the trustworthiness of this small case study, one main criterion for sampling these learners was that their teachers had to identify them as learners who were not too shy to provide information about themselves, particularly about their thinking. The assumption was that having a high achievement in mathematics suggests they had the necessary knowledge and understanding of mathematical ideas to solve open ended and non-routine mathematics word problems.

Learner A, a 15-year-old girl, has a seemingly quiet nature. She had an average of 70% for mathematics at the end of Grade 8 and took 9 min and 20 s to solve the given word problem. Learner B was 15 years old and from an all-girls school. She smiles a lot and seemed to enjoy the discussion and questions in the interviews. She averaged 80% for mathematics at the end of Grade 7 and solved the word problem in 16 min and 58 s. Learner C was 14 years old. She seemed more interested in the study than were the other participants. She asked questions about why the study was done and whether she was allowed to ask questions during the interview. This learner had obtained an average of 76% at the end of Grade 7 and solved the word problem in the shortest amount of time, only 4 min. Learner D was a 13-year-old boy, the youngest, and from an all-boys school. His teachers considered him a top student, achieving 96% at the end of Grade 8. He took 9 min and 30 s to solve the problem, almost equal to Learner A.

Research instruments

The four instrumental case studies, each instrumental towards the collective case study, were conducted using: (1) a word problem based on the area of circles to transgress the mathematical ideas from invisible to visible, (2) metacognitive statement cards to elicit metacognitive awareness of the mathematical ideas underpinning the visualisation of circles, (3) observations of the utterances and gestures that pose the imaginative capabilities of the participants during visualisation, and (4) a collection of the images of participants' own conjured mathematical models of the invisible circles on which the word problem was based. A brief outline on each of these instruments now follows.

The task of the invisible circles

The idea of the task was initiated by Jagals (2013), published for the first time in Jagals and Van der Walt (2016), and read as follows:

> Suppose a circle's diameter is 20 cm. This is also four times the radius of a second circle. Calculate the area of a third circle if the third circle's radius is half of the second one. (p. 158)

The task was inspired by a similar activity by Fortunato and Hecht (1991) and adheres to the characteristics of a rich mathematical task identified by Boston and Wolf (2006), including the requirement of multiple approaches and representation, engagement, curiosity and creativity and has the potential to relate and extend to other tasks. Participants had no time restriction to complete the task and were allowed to use a non-scientific calculator. No images of circles were given as this allowed the participants to exhibit their own visualisation of the problem, and to elicit the mathematical ideas from invisible to visible. After the problem was solved, the participants were interviewed individually.

Metacognitive statement cards

Before solving the problem, each participant received a set of 15 metacognitive statement cards, taken from the idea by Wilson (2002) and Van der Walt, Maree and Ellis (2008), on which metacognitive awareness capabilities were spelt out. Participants also received blank cards in case they wanted to write some thoughts that were not already stated. No participant made use of the blank cards. Participants were asked to focus on their thoughts while solving a mathematics task. The statements had a bearing on the embodied, situated and distributed cognitive processes identified by Clark (1998). Examples of the metacognitive statements include: I drew an image to better understand the question (embodied), I thought about something that I had done in the past that was helpful (situated), I thought about a different way to solve the problem (distributed). The procedure with the metacognitive statement cards followed after the invisible circle problem was solved and served as a starting point before interview questions were asked. After the task was solved, participants had to reflect on what they did and indicated which thoughts represented on the cards were thoughts that they were thinking. Participants had to reflect on the problem-solving experience and select only the cards with the statements that they are aware of and can agree to have used. These cards were then placed in the order they had been used and numbered consecutively to obtain a detailed account of the cognitive processes followed. The metacognitive statement cards were collected to ensure that the best possible data of participants' reflections and reporting on their experiences could be obtained.

Observations of the utterances and gestures that pose imaginative capabilities

Careful observation was used to note any utterances and gestures or movements with hands and arms that can indicate the embodiment of mathematical ideas. Mainly, gestures of a rest position, stroke, pointing, preparation or movement about the work place, retraction and movement of hands, or movement of the lips and other relevant body language which represented the embodiment of cognition of mathematical ideas served as criteria for noting a gesture or utterance.

Images of participants' own conjured mathematical models

Since the word problem was meant to elicit imaginative capabilities, it was anticipated that participants will conjure images of the elements of a circles (e.g. radius, chord, circumference and diameter) which, at the end, served as a mathematical model for the word problem. Each participant's conjured images served as evidence of the situated and distributed cognition instigated by the information provided in the word problem.

Trustworthiness issues

Participants continuously referred back to the video recordings and reflected on their actions and the statements they made. In this sense we consider their statements as valid and trustworthy. The metacognitive statement cards also validated interview responses. To ensure trustworthiness the guidelines by Elliott, Fischer and Rennie (1999) were followed. These included: checking the interpretations of participants' utterances with the participants and cross-checking the findings with the conceptual-theoretical framework of this study. In line with Curtin and Fossey (2007), we also offer a thick description of the empirical investigation and findings, and employ within-method triangulation where selection of metacognitive statement cards, the individual interviews on the problem-solving experience, observations, and images serve as different methods within the same methodological approach.

Ethical considerations

Permission and ethical clearance was sought and obtained from the Department of Basic Education, and the university's ethics committee (reference NWU-00043-11-A2) in which the initiated study was proposed; learners' parents and school principals also gave consent. Participants were informed about the aim of the investigation and could have withdrawn at any moment, although none did. Participants' identities were protected by using pseudonyms (e.g. Learner A).

Data collection procedures

Data were collected for each instrumental case study through four research instruments. First, inductive analysis was conducted regarding the selection of metacognitive statement cards after problem solving. *A priori* analysis was then conducted on the transcriptions at interview level with a focus on what the learners were thinking, and why they did what they had done. This was followed by identifying the utterances or gestures that relate to imaginative capabilities as observed through video playback. Together with these recordings, we also collected particular 'screen shots' of the learners' mathematical models to represent the circle images of learners' imaginations, to showcase how they have conjured and imagined distributed mathematical ideas regarding the elements of a circle. Together, the following instrumental cases were chosen because they are thought to be instrumentally useful towards the aim of this investigation.

Data processing procedures

Each statement card represented a metacognitive awareness statement relating to either embodied, situated or distributed cognition. The video recording was played back to identify the metacognitive statements that were used during particular moments where mathematical ideas were expressed through gestures and utterances. Imaginative capabilities and metacognitive awareness were identified by visible (observable) regulatory actions such as starting the next step, getting an answer on the calculator or rereading the question.

Interviews were first transcribed verbatim, then entered into the computer as a Word document and saved using pseudonyms for the participants (Learners A, B, C and D). Second, *a priori* codes were identified based on the conceptual-theoretical framework and categorised in the transcriptions. We read and reread the transcriptions and labelled related sentences or words according to the *a priori* codes. Particular sentences, words or phrases were then cross tabulated according to the *a priori* codes and were subsequently compared to determine if there was a pattern or structure in participants' visualisation, metacognitive awareness or imaginative capabilities. Codes represented in the transcriptions (in terms of responses and phrases) were summarised and then tabled in four comparing columns. The columns included the identified codes for each participant. Patterns emerging from the data were compared between participants' overall visualisation. Care was taken to identify which metacognitive awareness and imaginative capabilities the participants exhibited. Similar patterns for the different themes were joined together and comparisons were made between all participants and relevant themes.

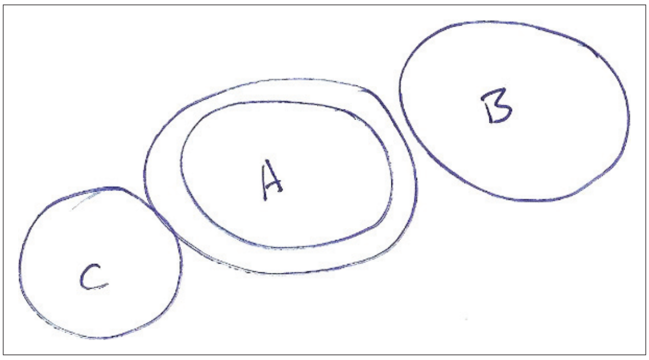
Findings

Following is a narrative account on each of the cases' findings, flowing from the four instrumental case studies. In each case, tenets of metacognitive awareness (by Franklin & Graesser, 1999), imagination (by Liang et al., 2013; Lin et al., 2014) as well as the embodied, situated and distributed cognitive processes (by Clark, 1998) were identified. The main findings of the instrumental cases follow.

Instrumental case 1 – Learner A

Learner A solved the mathematics problem in 9 min and 20 s. She drew three circles as illustrated in Figure 1, and labelled them in the centers as A, B and C. She circled the middle circle, labelled A, again and reread the question before making a gesture by pointing with her pen from one side of circle A to the other, as if drawing a line through A, and in doing so embodied the mathematical idea of the circle's diameter.

The image that Learner A conjured portrays her mathematical model for the three circles. Even though she labelled them clearly, she did not attach any symbols to suggest possible mathematical ideas about each circle. She read and reread the problem again, stopped and then drew her own version of



Source: Jagals, D. (2013). *An exploration of reflection and mathematics confidence during problem solving in senior phase mathematics*. Unpublished master's thesis, North-West University, Potchefstroom, South Africa (p. 127). Retrieved from http://hdl.handle.net/10394/9067

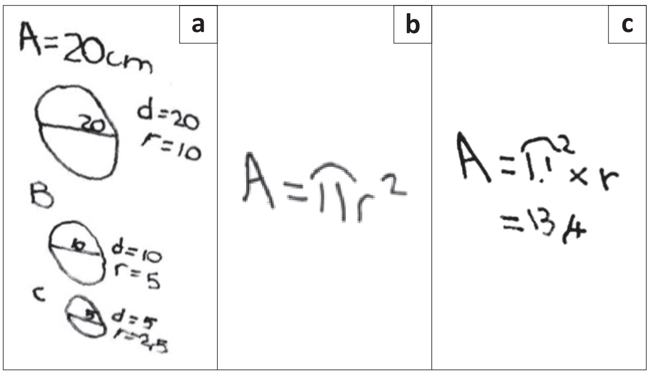
FIGURE 1: Learner A's mathematical model for the three circles.

what she had read through her conceiving imaginative capability. She also applied appropriate strategies by rereading the question, concentrating more and reading longer, while looking up at her written work and drawings. She paused looking at her work and then looked away from her work, staring in the air. She pointed with a pen towards her written and sketched work while reading the word problem. After every sketch or step, she paused and scanned her page from top to bottom. A seemingly major challenge was when she became aware that she 'couldn't remember the formula' to calculate the area of a circle (Jagals, 2013, p. 127). Trying three different strategies, she reflected on her knowledge and practices from experiences. The three strategies included: trying half of the diameter, then drawing a big circle with a shaded area, and eventually writing the word area on a piece of paper and underlining it. This provided the situated cognition through which she then remembered the formula and wrote it down. She explains that 'I think that's why I underlined the word area. … I then remembered the formula for the area' (p. 127). Almost immediately after getting the answer the learner made another attempt to distribute her cognition by using the formula known to calculate the circumference of a circle. She compared the answers and where deemed necessary tried another way to solve the problem. She identified the formula as a pattern and adapted it to calculate the area. She then wrote her final answer for the area of circle C as 78,5 cm^2 (Jagals, 2013, p. 127).

Instrumental case 2 – Learner B

Learner B solved the problem in 16 min and 58 s. Unlike Learner A, this participant first read the whole problem three times before drawing the circles as illustrated in Figure 2a. She seemed to have a calm and relaxed approach, doing what she does slowly and double-checking her work regularly. She first drew circle A and then labelled it. Then she drew circles B and C and labelled them.

After drawing each circle, she read the question again and attached symbols to the circles by using information. She appeared to read some parts more and longer, stopped,



Source: Jagals, D. (2013). *An exploration of reflection and mathematics confidence during problem solving in senior phase mathematics*. Unpublished master's thesis, North-West University, Potchefstroom, South Africa (pp. 133–134). Retrieved from http://hdl.handle.net/10394/9067

FIGURE 2: (a) Learner B's mathematical model for the three circles, (b) Learner B's mathematical formula to calculate the area of a circle and (c) Learner B's mathematical formula to calculate the area of a circle after changes (made to Figure 2b).

read again. When the video recording was played back, the researcher asked at this stage what the learner was doing, and she commented: 'I was putting like into a picture format, yes, like individual, like a, b, c' (Jagals, 2013, p. 133). She moved closer to the question on the desk. She read it, pointing at some parts of the question and looking up at her drawings, then back to the question. She moved the calculator slightly out of the way and began writing A =. She paused while looking at circle C and then finished the first step with a substituted value for π. When the researcher asked the learner why she was pausing, she said:

> I wanted to write my formula so I went back to check what are we doing, what area, perimeter or volume, so then I paused to double check; okay what is the area, so I specifically went to look for, if it's area. (Jagals, 2013, p. 133)

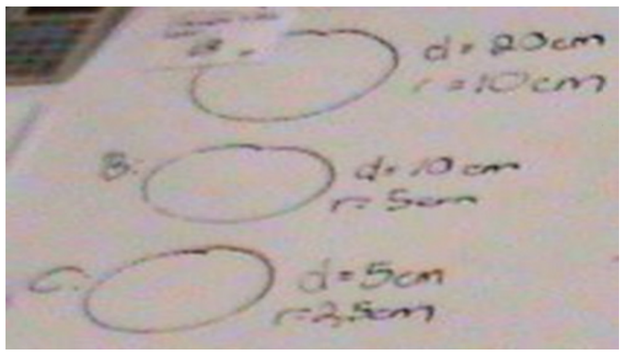
This learner's completed visual formula is depicted in Figure 2b.

She paused, sat back in her chair and then erased the exponent (2) as well as the radius (r) in the formula (refer to Figure 2b), wrote times r, hesitated (paused) and placed the square at π as illustrated in Figure 2c.

At this stage, the learner commented: 'I was very uncertain there'. She paused again, pen between the lips, and after some time corrected the formula. After substituting the value for π and the radius into the second step and using a calculator to get the answer, the learner evaluated her answer. She recalculated using other strategies and became aware that π was wrongly substituted. She corrected this and wrote the final answer as 19,625 cm^2 (Jagals, 2013, p. 134).

Instrumental case 3 – Learner C

Learner C solved the problem effortlessly and, almost, mechanically, in 4 min. She read the question only partly, concentrating on some particular aspects at a given moment. The steps were taken without doubting the formula, units or substituted values – unlike Learner A or Learner B. After reading the question, she drew three circles of similar sizes as depicted in Figure 3.



Source: Jagals, D. (2013). *An exploration of reflection and mathematics confidence during problem solving in senior phase mathematics*. Unpublished master's thesis, North-West University, Potchefstroom, South Africa (p. 140). Retrieved from http://hdl.handle.net/10394/9067

FIGURE 3: Learner C's mathematical model for the three circles.

Without showing lines to picture radius or diameter, she wrote down the given and mentally calculated information next to the matching circles. The learner calculated or deduced the information for circle B and circle C mentally, not showing any written work for her conclusions about the diameter or radii. She read information every time before she wrote something, thus breaking the question up into smaller manageable parts. While referring to the video recording afterwards, the learner had difficulty describing what she had done. She kept her answers short and sounded uncertain. The dialogue between the learner and researcher explains this:

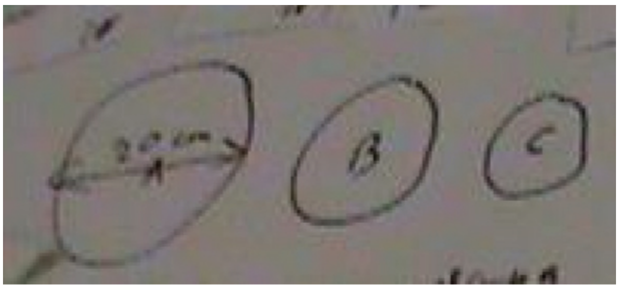
Researcher: Why did you read the question again?
Learner C: I was looking at this [*pointing with her finger towards her solution*].
Researcher: What were you doing there?
(Jagals, 2013, p. 141)

Her response had a futile motive. She commented on a particular step: 'I was writing and then I closed the pen and then I wanted to work out the sum'. Learner C was not aware of what she had done nor could she provide clear reasons for doing what she did. She monitored her work less often than the other participants did, and only reread the question once. She also did not evaluate her answer. She did, however, explain that 'it was bothering me that there were no labels' suggesting that she was metacognitively aware of the imaginative capability that the problem requires of her.

Instrumental case 4 – Learner D

Learner D started drawing the circles almost immediately after being given the word problem as illustrated in Figure 4. Although all four participants drew the three circles and filled in information about each circle, Learner D only included information for circle A. He also did not make use of algorithms or so-called steps; instead he conjured a picture, wrote descriptive words as labels and did mental calculations.

Circles B and C were just drawn and labelled but no symbols were attached inside or outside these circles. He claims that: 'I drew it, what they said it is, I drew it on. So if it is a big circle, I draw a big circle if it is a small circle I draw a small circle'. After drawing the circles, he started calculating the area by first making sure he understood the given information,



Source: Jagals, D. (2013). *An exploration of reflection and mathematics confidence during problem solving in senior phase mathematics*. Unpublished master's thesis, North-West University, Potchefstroom, South Africa (p. 145). Retrieved from http://hdl.handle.net/10394/9067

FIGURE 4: Learner D's mathematical model for the three circles.

which was not written routinely. He wrote the formula for the area of a circle and continued to substitute the values for π as well as the predetermined mentally calculated value for the radius of circle C. Learner D seemed surprised when he wrote down the answer as shown on the calculator's screen. This fraction led him to solve the problem, manually, by doing long division.

Discussion

The purpose of this research was to explore the role of metacognitive awareness and imagination in facilitating visualisation in solving a mathematics task. References to metacognitive awareness in mathematics education research often fail to acknowledge the imagination as a key role player in the cognitive processes during visualisation. Some studies have shown that teachers' metacognition is not adequate to model metacognitive awareness and undertones the vagueness in our understanding of the imagination as a faculty of self-directed learning, as it is rarely promoted in mathematics classrooms (Van der Walt et al., 2008). The typical objectivist perspective on mathematics is, perhaps, the result of this understanding, and impacts on teachers' approaches to, and education philosophy of, mathematics. In this regard we offer here an understanding of the conceptual linkage between the imagination and metacognitive awareness, to advance this debate about the rational view of mathematics, and to promote a greater imaginative mathematics.

We therefore contemplated the visualisation activities learners engage in can prompt their imaginative capabilities. Originating from this investigation, and in line with the model by Lin et al. (2014), learners expressed three types of imaginative capabilities as initiating, conceiving and transformative imaginations. To understand the cognitive processes embedded in these imaginations we aligned Lin et al.'s (2014) model against the embodied, situated and distributed cognition, as the conceptual-theoretical framework borrowed from Clark (1998), and produced the notion that the imagination becomes a powerful construct in the cognitive processes during visualisation. It provides learners with the opportunity to foster metacognitive awareness. Becoming metacognitively aware of the imaginings of mathematical ideas not only creates awareness of the underlying mathematical models, of elementary Euclidean circle geometry, for example, but suggests learners can express these imaginings in the form of a drawing or written steps and, in doing so, depict and become aware of their cognitive processes. The result is that initiating, conceiving and transformative imaginings can become embodied, situated and distributed in the visualisation of elementary Euclidean circle geometry. Moreover, this understanding hints at the imagination as a faculty of self-directed learning and a, philosophically speaking, underlying construct of the elements of a circle in circle theory.

The four instrumental case studies collectively reflect this theoretical stance and support the role which the tenets of metacognitive awareness and visualisation, through embodied, situated and distributed cognition in the imagination, plays in facilitating visualisation in solving a mathematics task. To elucidate these findings, Table 1 reflects the tenets of metacognitive awareness and visualisation in the imagination.

Based on these tenets, it seems reasonable to suggest that metacognitive awareness and imaginative capabilities can serve as guiding principles embedded in visualisation tasks. To do so, the tenets need to be incorporated in the development of appropriate activities (such as open-ended non-routine word problems) to elicit the necessary mental images that will conjure metacognitive awareness. Each of these underlying tenets seems to advance, collectively, the rational debate about mathematics, promoting a more imaginative mathematics.

In respect of the role of metacognitive awareness of the learners, all cases were joined and overlapping with Franklin and Graesser (1999)'s stages as metacognitive awareness of mental images. In this sense, each learner conjured an image, drew lines that were not necessarily labelled and attached symbols with spatial relation to their images. These images served as the expressions of their embodied mathematical ideas of the three circles, represented as a mathematical model with circle properties. The possibility that one had more or less metacognitive awareness of the mathematical idea of a circle was not evident. Instead, each case raised awareness of a deliberate conscious attempt to decipher the task and, within the perimeters of elementary circle geometry properties, conjure a circle based on what they have read and reread, although they were not instructed to do so. The conjured image was embodied, situated and then distributed even though the task did not expect learners to draw an image. Their visualisation, then, appeared obvious and evident, even if they did not, in all cases, attach symbols to the image as Franklin and Graesser (1999) proposed. This obviousness of conjuring an image based on the metacognitive awareness of the mathematical ideas captured in the task reflects a radical constraint to the view of mathematics as a rational structure to reality. It is this independent belief, of what the task requires, that separates these learners' attempts from one another. It is therefore not the imaginative capability, of what the mathematical model looks like, or what strategies they used to calculate the area, that distinguishes their metacognitive awareness from one another; perhaps this as well, but the fact is that each participant portrayed a conjured image, a formula and a series of steps (be it correct or not) which reflects in all cases the tenets of embodied, situated and distributed cognition. In so doing, the mathematical models that these learners made explicit show that the role of metacognitive awareness was largely emphasised by the mathematical ideas that the task required (e.g. Learner B illustrated this in Figure 2 as ratio of proportion of the elements of the circles). What the ground is for this intuitive mathematical idea remains in the imagination.

Conclusion and future directions

Visualisation research has recently received increasing attention with a focus on metacognitive awareness and visualisations, including imagination. Since imagination is a scarce topic in metacognition research, a recent emphasis in the South African school mathematics curriculum prompts mental images, predictions and visualising of thoughts and decisions to promote self-directed learning as imaginative states of mind. The conceptual link, therefore, between these imaginative states of mind and the cognitive structures that

TABLE 1: Summary of findings that reflect the tenets of metacognitive awareness and imagination.

Metacognitive awareness of:	Examples of metacognitive statement cards	Link with theory	Examples of indicators of imagination through utterances and gestures	Learner
Metacognitive knowledge and imagination				
Knowledge of the person/self	I thought I cannot do it	Embodied	I went back to the question to read the second part again.	A, B
Knowledge of the task	I thought I know this sort of problem	Situated	They did not give you the fact, I had to make it up for myself.	A, C, D
	I thought about what I already know	Distributed	…and then I put a question mark.	A, B, C, D
	I tried to remember if I had solved a problem like this before	Distributed	I think I looked at that because it looked similar.	A, B, C, D
Knowledge of the skills and/or strategies	I thought I know what to do	Situated	I was not sure about the whole thing.	A, C, D
	I thought about something I had done in the past that was helpful	Distributed	I could not remember what was the difference between circumference and area, and then I remembered that area has a squared at the end.	A, C
	I thought about a different way to solve the problem	Distributed	I tried to make other formulas… and do other things as well.	A
Metacognitive regulation and imagination				
Regulation of understanding	I read the question more than once	Embodied	I read that like three times.	A, B, C, D
	I drew a diagram to better understand the questions	Embodied	If it is a big circle, I draw a big circle and if it is a small circle, I draw a small circle.	A, B, C, D
Regulation of planning	I changed the way I was working	Situated	I then used plan B to do long multiplication.	A, B, D
	I thought about what I would do next	Embodied	…and that gave me a fraction sign so I was confused, I was like what? And then I kept looking for a decimal.	A, B, C
Regulation of monitoring	I thought about how I was doing	Embodied	I was checking if I am right because I wrote it down.	A, B, C
	I checked my answer as I was working	Embodied	Here I wondered if this is right or am I doing something wrong.	A, D
	I thought about whether what I was doing was working	Embodied	I was making a dot in the middle. I do not know why I did that, maybe just to make sure that I was right.	A, B, C
Regulation of evaluation	I thought, is this right?	Embodied	I paused … I was checking if I was on the right track.	A, C, D

Source: Jagals, D. (2013). *An exploration of reflection and mathematics confidence during problem solving in senior phase mathematics*. Unpublished master's thesis, North-West University, Potchefstroom, South Africa (p. 151). Retrieved from http://hdl.handle.net/10394/9067 draw on mathematical ideas seem to relate to a demand for exploring the conceptual understanding of the role of metacognitive awareness and imagination in facilitating visualisation in solving a mathematics task. The significance of this study is the close connection between metacognition and imagination as portrayed through the tenets of metacognitive awareness to illustrate the role it plays during problem-solving tasks. Educators need to design learning environments that support learners' metacognitive development and encourage them to engage their imaginations in the learning process. In mainstream mathematics, the holder of a rational view will need to reflect on the cognitive demands of the problem and the associated tenets of the imaginative capabilities to solve circle geometry problems that are non-routine and open-ended in nature. The beholder of a more imaginative and holistic view, however, should reflect on at least two kinds of imagination: *What do I imagine myself knowing*? and *How do I imagine this*? Even in this sense, metacognitive awareness of oneself and of the mathematical ideas that govern reality needs further contemplating.

Acknowledgements

The authors acknowledge funding from the National Research Foundation. Views expressed are not necessarily those of the National Research Foundation.

Competing interests

The authors declare that no significant competing financial, professional or personal interests might have influenced the performance or presentation of the work described in this manuscript.

Authors' contributions

D.J. conceived of the presented idea, developed the theory and performed data collection and analysis. M.v.d.W. verified the analytical methods, and the tenets that emerged from them, and supervised the findings of this work. Both authors discussed the findings and contributed to the final manuscript.

References

Clark, A. (1998). Embodied, situated, and distributed cognition. In W. Bechtel & G. Graham (Eds.), *A companion to cognitive science* (pp. 506–517). Malden, MA: Blackwell. Retrieved from https://onlinelibrary.wiley.com/doi/pdf/10.1002/9781405164535.ch39

Curtin, M., & Fossey, E. (2007). Appraising the trustworthiness of qualitative studies: Guidelines for occupational therapists. *Australian Occupational Therapy Journal*, *54*(2), 88–94. https://doi.org/10.1111/j.1440-1630.2007.00661.x

Elliott, R., Fischer, C.T., & Rennie, D.L. (1999). Evolving guidelines for publication of qualitative research studies in psychology and related fields. *British Journal of Clinical Psychology*, *38*(3), 215–229. https://doi.org/10.1348/014466599162782

Fortunato, I., & Hecht, D. (1991). Metacognition and problem solving. *The Arithmetic Teacher*, *39*(4), 38–40. Retrieved from http://www.jstor.org.ez.sun.ac.za/stable/41195004

Franklin, S., & Graesser, A. (1999). A software agent model of consciousness. *Consciousness and Cognition*, *8*(3), 285–301. https://doi.org/10.1006/ccog.1999.0391

Fried, M.N. (2014). Mathematics & mathematics education: Searching for common ground. In M.N. Fried & T. Dreyfus (Eds.), *Mathematics & mathematics education: Searching for common ground* (pp. 3–22). Dordrecht: Springer. https://doi.org/10.1007/978-94-007-7473-5

Gilbert, J.K. (2005). Visualization: A metacognitive skill in science and science education. In J.K. Gilbert (Ed.), *Visualization in science education. Models and modeling in science education* (Vol 1, pp. 9-27). Dordrecht: Springer. https://doi.org/10.1007/1-4020-3613-2_2

Jagals, D. (2013). *An exploration of reflection and mathematics confidence during problem solving in senior phase mathematics*. Unpublished master's thesis, North-West University, Potchefstroom, South Africa. Retrieved from http://hdl.handle.net/10394/9067

Jagals, D., & Van der Walt, M. (2016). Enabling metacognitive skills for mathematics problem solving: A collective case study of metacognitive reflection and awareness. *African Journal of Research in Mathematics, Science and Technology Education*, *20*(2), 154–164. https://doi.org/10.1080/18117295.2016.1192239

Jansen, J. (2013). Imagination, embodiment and situatedness: Using Husserl to dispel (some) notions of 'off-line thinking'. In R. Jensen & D. Moran (Eds.), *The phenomenology of embodied subjectivity. Contributions to phenomenology* (pp. 63–79). Heidelberg: Springer. https://doi.org/10.1007/978-3-319-01616-0_4

Kasner, E., & Newman, J.R. (2001). *Mathematics and the imagination*. New York, NY: Courier Corporation.

Liang, C., Hsu, Y., & Chang, C.C. (2013). Intrinsic motivation as a mediator on imaginative capability development. *Thinking Skills and Creativity*, *8*, 109–119. https://doi.org/10.1016/j.tsc.2012.09.001

Lin, W.S., Hsu, Y., & Liang, C. (2014). The mediator effects of conceiving imagination on academic performance of design students. *International Journal of Technology and Design Education*, *24*(1), 73–89. https://doi.org/10.1007/s10798-013-9244-x

Makina, A. (2010). The role of visualisation in developing critical thinking in mathematics. *Perspectives in Education*, *28*(1), 24–33. Retrieved from http://hdl.handle.net/10520/EJC87564

Murray, G. (2011). Imagination, metacognition and the L2 self in a self-access learning environment. In G. Murray, X. Gao, & T. Lamb (Eds.), *Identity, motivation and autonomy in language learning* (pp. 75–91). Bristol: Multilingual Matters.

Murray, G. (2013). Pedagogy of the possible: Imagination, autonomy, and space. *Studies in Second Language Learning and Teaching*, *3*(3), 377–396. https://doi.org/10.14746/ssllt.2013.3.3.4

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, *70*(2), 159–174. https://doi.org/10.1007/s10649-008-9150-4

O'Connor, K.P., & Aardema, F. (2005). The imagination: Cognitive, pre-cognitive, and meta-cognitive aspects. *Consciousness and Cognition*, *14*(2), 233–256. https://doi.org/10.1016/j.concog.2004.07.005

Reddy, V., Visser, M., Winnaar, L., Arends, F., Juan, A., Prinsloo, C., & Isdale, K. (2016). *TIMSS 2015. Highlights of Mathematics and Science achievement of Grade 9 South African Learners*. Cape Town: HSRC Press. Retrieved from http://www.hsrc.ac.za/en/research-data/ktree-doc/17642

Schoenfeld, A.H. (2013). Reflections on problem solving theory and practice. *The Mathematics Enthusiast*, *10*(1), 9–33. Retrieved from http://scholarworks.umt.edu/tme/vol10/iss1/3

Schraw, G., & Moshman, D. (1995). Metacognitive theories. *Educational Psychology Review*, *7*(4), 351–371. https://doi.org/10.1007/BF02212307

Van der Walt, M., Maree, K., & Ellis, S. (2008). A mathematics vocabulary questionnaire for use in the intermediate phase. *South African Journal of Education*, *28*(4), 489–504. Retrieved from http://www.scielo.org.za/pdf/saje/v28n4/a03v28n4.pdf

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. New York: Cambridge University Press. https://doi.org/10.1017/CBO9780511803932

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin & Review*, *9*(4), 625–636. https://doi.org/10.3758/BF03196322

Yueh, H.P., Jiang, B.C., & Liang, C. (2014). How does human aggregate moderate the effect of inspiration through action on the imagination of engineering majors? *International Journal of Engineering Education*, *30*(1), 128–135.