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Abstract

Argumentation, as a kind of classroom discourse, is beneficial for establishing mathematical knowledge taking place in classroom conversations. It can enhance learners' development of subject matter knowledge. Hence, this case study was designed to examine the effect of argumentations on developing subject matter knowledge in detail in mathematics. For this purpose, discussions happened in a collective learning environment, designed based on problem-based learning and taking place in a six-week instructional sequence, was investigated focusing on argumentations. 23 preservice middle school mathematics teachers (PMSMT) engaged in tasks designed by the researchers through classroom interaction between them and the instructor in this environment. The PMSMT's mathematical ideas identified through Toulmin's argumentation model were documented in this study. With the help of this model, the mathematical ideas were extracted and the development of PMSMT's subject matter knowledge was analyzed and documented in detail. It was observed that the PMSMT improved their subject matter knowledge by forming mathematical ideas. They constructed new knowledge and revised their previous knowledge through argumentations.

Introduction

Effective teaching in learning environments can be performed by knowledgeable teachers (Van der Sandt and Nieuwoudt 2003). Such teachers can design and organize learning environments by using their knowledge needed for teaching the targeted concepts effectively. Teachers are also among the key factors influencing student learning. Elementary teachers, in particular, are expected to design appropriate learning environments so that they can help their students become successful (Ng 2011). In this respect, it is important for teachers to have deep knowledge of content and use them to enact effective instructions. They can analyze their students' thinking, and enact instructions by making appropriate instructional decisions in their classrooms successfully, as well (Hill and Ball 2004).

Teachers develop rich and deep content knowledge in their preservice years in order to be effective knowledgeable teachers in the future (Chapman 2007; NCTM 2006). In this respect, it is essential to educate teachers by providing them with opportunities to acquire the necessary knowledge through these years. Such opportunities can be provided in courses supported by effective tools and tasks designed and conducted by the instructors in a social learning environment. Hence, preservice teachers can experience rich and useful tasks to learn and teach the concepts through interactions so that they can develop their reasoning and understanding (Henningsen and Stein 1997).

Teacher education programs may be more beneficial when they are supported by discourses based on the view that social interaction is important to encourage reasoning and understanding. In this respect, in the lessons including social interaction, attaining necessary knowledge becomes an important measure to be evaluated (Cobb 2000). In general perspective, by participating in discourse, learners make reasoning aloud and explanations about what and how they think about the concepts (Hufferd-Ackles, Fuson, and Sherin 2004). In the classrooms including discourses, learners can develop knowledge and understanding by thinking and interacting with other people. The learners can provide this improvement by modifying their thinking schemes when the confusions in their thinking through discourse are observed (Steffe and Tzur 1994). These environments illustrate "communication as a process of mutual adaptation wherein individuals negotiate meanings by continually modifying their interpretations" (Cobb and Bauersfeld 1995, p. 8). Moreover, it is clear that there exist positive impacts of communication on attaining knowledge through interactions of teacher-student and students (Lampert and Cobb 2003).

Argumentation as a kind of discourse can increase the communication which is essential in attaining good understanding of mathematics since the research show that teachers have deficiency in their understanding as well as in their skills to communicate (Hershkowitz 1989; Owens and Outhred 2006; Sundberg and Goodman 2005). Argumentation can provide these benefits since it takes role in interactive dialogue of two or more people reasoning together. It is also important to make scientific claims since the people obtained the idea after evaluating alternatives and weighing evidences (Voss and Van Dyke 2001). Also, argumentation encourages conceptual understanding, problem solving, criticizing and justifying the ideas (Abi-El-Mona and Abd-El-Khalick 2011; Duschl and Osborne 2002; Jim´enez-Aleixandre et al. 2000; Jonassen and Kim 2010; Osborne, Erduran, and Simon 2004; Zembal-Saul 2005). In this respect, it is beneficial to use argumentation in mathematics since the teachers having good understanding of mathematics tend to have qualified scientific thinking, articulation of their ideas, and development of clearly structured arguments. Furthermore, argumentation promotes conceptual understanding and learning of the content effectively and deeply (Driver, Newton and Osborne 2000) with the skills of communication and critical reasoning as two significant features of argumentation. In this respect, argumentation van be useful to help preservice teachers become knowledgeable and acquire necessary mathematical knowledge. Based on this view, it was focused on the preservice middle school mathematics teachers' development of subject matter knowledge through argumentations in a designed instructional sequence prepared problem-based learning in the current study.

Theoretical Background

Subject Matter Knowledge

The concept of mathematical subject matter knowledge (SMK), developed by Hill, Ball, and Schilling (2008) represents the mathematical knowledge needed by teachers to teach mathematics effectively. It has an impact on teachers' teaching performance because teachers having deep and rich knowledge and understanding of the mathematical concepts can improve their teaching skills effectively (Hill and Ball 2004). Moreover, SMK is related to other types of knowledge in mathematical knowledge for teaching. Teachers equipped with deep SMK can direct their students to a developmental process by analyzing their students' thinking and organizing the lessons with appropriate instructional decisions (Hill and Ball 2004). SMK, stated as the mathematics teachers' detailed knowledge of common and specialized content knowledge related to their performance of the teaching profession effectively (Hill and Ball 2004). In this respect, SMK represents the conceptual part of mathematics curriculum that teachers teach in their classrooms. Hence, preservice mathematics teachers should become well-prepared and –equipped with necessary knowledge and skills before they graduate from their teacher education programs. Hence, they can cope with difficulty of actual teaching mathematical topics, and improve their understanding of mathematical concepts (Sowder et al. 1998). In this respect, teacher education programs are of critical importance, and it is therefore worthwhile to examine preservice teachers' SMK and the ways of constructing their knowledge. From this perspective, the development of preservice middle school mathematics teachers' SMK through argumentation was investigated in the present study.

Argumentation in Education

Argumentation explains how students form, interpret and use justifications in communications. It can also be specified as a process including try-outs of an individual with the aim of persuading others about a claim. Learners can form a shared understanding of the concepts by discussing and forming mathematical argumentations. In the process of producing argumentations and shared understandings through discussions, there exist justifications, active negotiation of claims and modifications of concepts, statements, and ideas used in discussions (Forman et al. 1998). Hence, it can be claimed that producing arguments refers to conceptual understanding (Lampert 1990). Through argumentation, learners can attain knowledge about concepts by questioning, discussing and understanding effectively. They can also improve reasoning skills, necessary for learning and understanding. Effective learning can be provided by actively engaging in ideas in problem-solving and developing critical thinking skills so that conceptual change can occur successfully through argumentation (Abi-El-Mona and Abd-El-Khalick 2011; Jonassen and Kim 2010).

Conceptual change and production of scientific claims can occur through argumentation since people form ideas after evaluating alternatives and weighing evidence in interactive dialogue of two or more people reasoning together (Voss and Van Dyke 2001). Also, argumentations encourage problem-solving, criticizing, justifying ideas and especially conceptual understanding (Abi-El-Mona and Abd-El-Khalick 2011; Duschl and Osborne 2002; Jim´enez-Aleixandre et al. 2000; Jonassen and Kim 2010; Osborne, Erduran, and Simon 2004; Zembal-

Saul, 2005). In this respect, it is beneficial to use argumentation in courses in teacher education programs (Nam and Chen 2017; Yaman 2018) since teachers having deep SMK can tend to be qualified to think scientifically, articulate their ideas, and develop clearly structured arguments. Furthermore, argumentation encourages conceptual understanding and attaining knowledge of contents deeply (Driver, Newton and Osborne 2000) with the skills of communication and critical reasoning as its significant features.

Although there are studies on the development of preservice teachers' SMK and understanding of content, this remains a domain needing further investigation since each mathematical concept needs to be analyzed in detail effectively. In light of the explanations above, it seems worthwhile to explore how argumentation emerged in the HLT including the designed tasks, tools, and imagery enhanced the PMSMT's development of SMK. For this purpose, an instructional sequence enacted based on the designed hypothetical learning trajectory and, teaching and learning as it occurred in this classroom was observed. In this sequence, the discussions were analyzed to identify the arguments, and their emergence and the effects on the PMSMT's development of SMK.

Hypothetical Learning Trajectory

Learning trajectories can be stated as "a hypothesized description of successively more sophisticated ways student thinking about an important domain of knowledge or practice develops as children learn about and investigate that domain over an appropriate span of time" (Corcoran, Mosher and Rogat 2009, p. 37). By hypothetical learning trajectory (HLT), teachers can make predictions on student learning and then test them in practice. In this respect, it becomes possible to talk about the hypothetical nature of the learning trajectories as a bridge linking the theory of constructivism to practice (Duncan 2009; Simon 1995). In other words, in the process of the teaching period, the teachers have the opportunity to test the designed HLT and make modifications based on the experiences obtained in this process. HLT as a way of connecting constructivist theory to practice can be defined as ". . . the teacher's prediction as to the path by which learning might proceed. It is hypothetical because the actual learning trajectory is not knowable in advance and it characterizes an expected tendency" (Simon 1995, p. 135). The construct of a HLT can be accepted as a cognitive tool improving mental processes and mathematical learning actions constructed with respect to the philosophy of constructivism (Clements and Sarama 2004). Hence, in the present study, the instructional sequence in which the argumentations emerged was performed by the HLT designed based on problem-based learning about the concept of triangles. The discussions helping the establishment of argumentations in which stating, analyzing, discussing and convincing the PMSMT's ideas were made.

Problem-based Learning

Problem-based learning (PBL) as a teaching strategy activates students to learn by using their prior knowledge and interests, makes connections with the real world (Goodnough 2006) and functions in all grade levels from primary to college levels. In this respect, PBL is identified as "focused, experiential learning organized around the investigation, explanation, and resolution of meaningful problems" (Hmelo-Silver 2004, p. 236). Problems are beneficial to design "an environment for students to reflect their conceptions about the nature of mathematics and develop a relational understanding of mathematics" (Skemp 1978, p.9) with the learning opportunities. When the learners face with the problem, they have cognitive conflict since the situation does not fit their existing knowledge. Then, they start working on it. Through this studying process, they try to make some modifications on their existing knowledge by learning additional ones since "they confirm or redefine their conceptual knowledge, relearn mathematics content and become more open to alternative ways of learning mathematics" (Steele and Widman 1997, p.190) since problem solving is not remembering the memorized facts or using and following well-learned operations or procedures (Lester 1994). In other words, through problem solving, learners attain the skills of organizing their mathematical ideas, participating in the discussions, defending their ideas and convincing others on their ideas. Hence, the learners realize the dynamic nature and structure of mathematics and attain deep insight of mathematics (Manuel 1998; NCTM 2000).

The common definitions of problem solving are "a situation where something is to be found or shown and the way to find or show it is not immediately obvious" (Grouws 1996, p.72), "to have a problem means: to search consciously for some action appropriate to attain a clearly conceived, but not immediately attainable aim" (Polya, 1962 p.117) and "the situation is unfamiliar in some sense to the individual and a clear path from the problem conditions to the solution is not apparent" (Grouws 1996, p.72) benefiting from prior knowledge (Frensch and Funke, 1995) where a problem is defined as "a situation for which one does not have a ready solution" (Henderson and Pingry 1953, p.248). Through these definitions, there are assumptions to be provided for a situation to become a problem. In this respect, it can be stated that it is needed to determine whether a situation is problem since it changes based on individuals and their experiences (Henderson and Pingry 1953; Lester 1980). Therefore, a situation is a problem in case of holding some criteria. These criteria can be explained in a way that an individual must realize the situation and be willing to remove it, and then he cannot directly move on the solution process but he insists on to reach the solution (Lester 1980).

Based on explanations about the nature of PBL and the characteristics of problem, it can be related to argumentation since argumentation is performed through discourses including producing claims, supporting them using evidences and reasons, explaining them to the others and criticizing the ideas to reach consensus about the accuracy of the claim (Driver, Newton & Osborne, 2000). These process of performing argumentations can be performed with the help of problems since problems provide opportunities to produce claims about its solution and content, and evidence and reason can be provided by solution strategies. Also, through solving problems, learners can share their ideas, criticize others' ideas, try convince others about their own ideas and reach a consensus about the solution. Hence, PBL can be beneficial for the emergence of arguments, developing the PMSMT's SMK. In line with this idea, the HLT of the current study was designed by the PBL.

Purpose of the Study

Because the research in the literature of mathematics education show that teachers and preservice teachers may have deficiency in their understanding of mathematics as well as in their skills to communicate it, argumentation can enhance the formation of the communication encouraging to attain good understanding of mathematical concepts (Hershkowitz 1989; Owens and Outhred 2006; Sundberg and Goodman 2005). Argumentation can provide these benefits by taking role in interactive dialogue of two or more people reasoning together. It is also important to make scientific claims since the people obtained the idea after evaluating alternatives and weighing evidences (Voss and Van Dyke 2001). Through these activities, it can become possible to develop knowledge of mathematics connected with knowing mathematics and doing mathematics, applying mathematical procedures and possessing mathematical knowledge (Ball, Hill and Bass 2005).

In the light of the explanations, the present study focused on argumentations as a kind of classroom discourse affecting preservice teachers' knowledge about mathematical concepts. More specifically, the development of preservice middle school mathematics teachers'(PMSMT) subject matter knowledge in a designed social learning environment including argumentations was investigated. In other words, the answer to the research question of "How do the argumentations support the development of subject matter knowledge of preservice middle school mathematics teachers?" was examined by the current study.

Method

The case study design is used in the current study because its characteristic of viewing, finding and stating the holistic and meaningful aspects of real-life phenomena can provide answering research question (Yin 2003). Because the instructional sequence enacted based the designed the HLT can be examined by holistic and analytic perspectives, the ways of emergence and the effects of argumentation of the development of the PMSMT's SMK can be analyzed by this design effectively in the current study. The particularistic or intrinsic case studies as a kind of case study research that the case is selected with respect to the researchers' interest and willingness to understand the phenomena (Stake 1995) was selected for the present study with the aim of in-depth investigation of the case. The case of the development of the PMSMT's SMK about the mathematical concept of triangles was selected by the researchers.

Participants

The participants were composed of 23 PMSMT (14 female, 9 male) enrolled in elementary mathematics education program at a university in the northern part of Turkey. The junior PMSMT were specifically selected based on the criterion sampling technique as a kind of purposive sampling strategy. The criterion in selecting the participants was being familiar with the mathematical concept of triangles and main theorems related to it. Also, they were selected because they completed Geometry and Analytic Geometry courses in previous semesters.

Data Collection

Data were collected through video recordings of classroom sessions, audio recordings of the small group work and research team discussions, and field notes taken by the instructor and learners' works such as worksheets in the context of an instructional sequence that addressed the concept of triangles. Video and audio recordings were mostly used in order to clearly identify argumentation logs and analyze the emergence of mathematical practices. Worksheets were also used to analyze the formation of the argumentation logs and mathematical practices. Field notes were mostly used in order to criticize the benefits of activities and problems through research team discussions. Also, different ideas to be used in whole class discussions that could enhance the formation of argumentation were written on the field notes. Initially, an HLT was designed including problems mostly about geometric constructions activities. These activities were created by focusing on constructing triangles in different ways, and needing exploration and mathematical justifications for these ways. Data collection period took place during the six-week instructional sequence performed based on the HLT (see in Table 1). This HLT was designed by Author (2016) and illustrated in Table 1. By the HLT including tasks, topics of discussion, tools and imagery, the instructional sequence was organized and performed enhancing the emergence of the arguments. By the first learning goal, it was aimed that the PMSMT would not only form triangles, but also evaluate different contexts about the formation of triangles. Hence, they were engaged in the activity designed for the objective about the formation of triangles focusing on the definitions of types of triangles and the classification of them. In the second activity about the first learning goal, there were problems examining the possibility of formation of triangles based on some known elements which were main and auxiliary elements. There were problems in the activity sheets such "Classify the different types of triangles based on their definitions." and "When we know the measures of ha and a and $m(BAC) = 90^0$ in the triangle of ABC, is it possible to draw/construct this triangle? How?". By the second learning goal of the HLT, it was aimed that the PMSMT would not only construct these auxiliary elements (median, altitude, perpendicular bisector, and angle bisector), but also attain deep knowledge about properties of them and formation of critical points formed by them such as centroid as the concurrence point of the medians, orthocenter as the concurrence point of the altitudes. The activity sheets included questions such "Construct a triangle and its perpendicular bisector by compass and straight edge.", after observing their concurrency "Can you justify that they are concurrent? How?" and "What can you say about this concurrency point?". In the last learning goal, it was aimed to help the PMSMT acquire basic ideas necessary to develop deep conceptual understanding of congruence and similarity of triangles. They examined the congruence and similarity by related concepts such as geometric transformations by constructing congruent and similar triangles. They discussed about the problems such as "How can you show that two triangles are congruent/similar?". Through engaging in activities and problems prepared based on these learning goals, the PMSMT benefited from geometric constructions and they participated in whole class discussions.

The HLT helped the instructor organize the instructional sequence and discussions including mathematical arguments. Six-week instructional sequence was enacted based on the HLT in 3-h class time per week. The argumentation emerged through the application of the HLT prepared based on PBL was focused on the study. By this HLT, the tasks and activity sheets were organized. In the classrooms, the PMSMT studied the problems on the worksheets with their peers. Then, they participated in whole class discussions about their solutions and explanations for the problems under the guidance of the instructor. In this process, the PMSMT were not informed about the structure of Toulmin's argumentation model. They were encouraged to share their ideas by supporting the reasons, criticize them and reach a consensus about the accuracy of the ideas and claims. It was provided that these actions became the norms of the classroom. In this respect, the PMSMT became familiar with performing the process and actions of producing argumentation without knowing they made argumentation. At that point, the instructor used the technique of questioning to help the PMSMT produce their arguments in whole class discussions since teacher questioning was considered as an effective way to emerge mathematical argumentation through developing the learners conceptual understanding as suggested by Chen, Hand and Norton-Meier (2017). Peer group and whole class discussions were recorded by video cameras and audio recordings. Through this process, the instructor took field notes in order to use in the whole class discussion, team discussion and analysis processes. In the process of the discussions, the instructor provided the occurrence of the norms that the PMSMT shared, listened and criticized ideas, made reasoning, and explained and justified their solutions. From the data emerged through the instructional sequence guided by the HLT, the ideas related to the PMSMT's SMK about the mathematical concept of triangles formed through the argumentations were extracted. These ideas represented specific SMK about a particular case. Hence, the development of PMSMT's SMK was analyzed and represented by the changes and corrections on the mathematical ideas discussed by the PMSMT.

Table 1. Hypothetical Learning Trajectory

Learning Goal	Concepts	Supporting Tasks	Tools and Imagery
Evaluating the formation of triangles	Definitions of triangles Formation of triangles	Classification of triangles Basic drawings of triangles by construction	Diagrams Compass and straight edge
Reasoning on auxiliary elements of triangles	Auxiliary elements of triangles	Definitions, constructions and properties of auxiliary elements	Compass and straight edge Drawings
Reasoning on congruence and similarity	Transformation geometry Congruence & Similarity	Formation of images Comparing triangles and their images	Compass and straight edge Dot paper

Data Analysis

The data were analyzed through constant comparative data analysis technique that the data itself collected in the same week were compared with the data gathered across different weeks. The meanings of the obtained categories and themes were interpreted by reflecting personally on the impact of the findings and on the literature. Initially, the arguments representing the SMK of the PMSMT about triangles were identified and then the emergence and the changes on them were examined in other arguments.

Toulmin's argumentation model (1969) was used in order to illustrate the structure of the arguments and determine the mathematical ideas emerged in the current study. This model is composed of four parts: claim, data, warrant, and backing. The first part, the claim, is composed of the opinions proposed as true by the learners. They are also conclusions of the discussions and easy parts of this model to be identified since they may be answers to a problem or a mathematical statement to be questioned. Data, as the second part of the model, are the expressions encouraging claims. They provide evidence for the claims in a way that the learners participating in the argumentation show the truth of the claims. Moreover, they can be mathematical procedures, methods, relationships, facts, theorems or definitions leading to the claims. The third part, the warrant, makes the connection between the data and the claim. They provide this connection by benefiting from the implications of the data. They explain how the data encourage the claim by justifying the reasons that the data lead to the claim. The last part of the model is backing. A backing expresses the reasons for acceptance of an argument by increasing the validity of the claim. In this process, six themes are emerged and named as mathematical ideas. In the current study, the elements of Toulmin's model of argumentation were identified as explained above after transcribing the data. Then, each argumentation log was examined in order to identify the mathematical ideas representing the PMSMT's SMK about the concept of triangles. In line with this action, the themes representing the mathematical ideas, used for naming the related argumentation logs, and categories illustrating the topics or subjects of each argumentation log, of the data analysis process illustrated in Table 2.

Table 2. Themes and Categories

Themes (Mathematical Ideas)	Categories (Argumentation Topics)
Reasoning on the definition and classification of triangles	Definition of different types of triangles such as obtuse, right, isosceles etc. Classification of them based on angle or edge
Reasoning on the construction of triangles	Construction of triangles by some of main or auxiliary elements
Reasoning on the construction of auxiliary elements of triangles	Construction of median, angle bisector, altitude, perpendicular bisector
Reasoning on the concurrence of auxiliary elements of triangles	Concurrency of medians, angle bisectors, altitude, perpendicular bisector Proof of these concurrent points
Reasoning on the names of concurrent points of auxiliary elements of triangles and their places	The names of concurrency points of auxiliary elements Their places for different types of triangles
Reasoning on congruence and similarity	Congruent triangles by translation, rotation, reflection Similar triangles by dilation Congruence/similarity criteria

Results

In the present study, six mathematical ideas emerged in the six-week instructional sequence based on the HLT designed by PBL strategy about the concept of triangles as illustrated in Table 2. The mathematical ideas were explained using direct quotations from the transcripts and argumentation logs and each PMSMT participating in the current study was illustrated by S_n (where n is a number). Through six-week instructional sequence, a lot of argumentation logs about the engagement of 23 PMSMT in the tasks were formed but some of them produced by some of the participants were reported in this section.

Mathematical Idea 1: Reasoning on the Definition and Classification of Triangles

For this mathematical idea, the definitions of different types of triangles (obtuse, right, isosceles etc.) were formed through argumentations. The following explanations represent a part of the discussion period about the definitions of right triangles as a type of triangle. The participants discussed to define the types of triangles in order to produce the accurate definition of a right triangle as follows:

S_1: We know that a triangle is composed of three non-linear points. For three non-linear points, two equidistant points to a specific point refer to a right triangle when they are combined with line segments.

When S_1's definition of a right triangle was examined, her definition was unnecessary and insufficient since she did not emphasize the perpendicularity. Through discussion, the participants realized the unnecessary usage of equal length of lines and the necessity of perpendicular lines.

S_7: We can form an isosceles triangle whose two edges are equal in length when we focus on two equidistant points to a point. This situation does not provide perpendicularity.

Afterward, the instructor guided them to think about this inappropriate definition and identify the critical attributes of a right triangle to compose the necessary definition.

S_3: We can define right triangles based on angles and edges which are main elements of triangles ... we must emphasize three non-linear points necessary for the formation of a triangle. Right triangles are triangles whose two of the edges intersect perpendicularly at a corner.
S_5: We can say that the right triangles are triangles whose one of interior angle measure is 90^o.

Through the argumentation guided by the instructor, S_1 realized that the property of having equal length of edges was not necessary and perpendicularity was needed to define the right triangles. To conclude, the participants produced the necessary and sufficient definition of a right triangle based on the evidence of the definitions of S_3 and S_5 at the end of the discussion. Also, they removed their inappropriate knowledge and formed the correct one through argumentation.

S_1 used the definition of a triangle as the data for this argumentation. In other words, it was observed that the definition of triangles served as data for the argument about the definition of right triangles. However, she used the warrant for the claim in an incorrect way. Then, S_3 and S_5 provided the correct data and warrant by defining right triangles accurately. They benefited from angles and edges which were main elements of triangles as it was explained in the definition of triangles. The Toulmin's model of argumentation for some parts of this debate is shown in Figure 1.

DATA
S_1: We know that a triangle is composed of three non-linear points… We need to emphasize the perpendicularity.

CLAIM
S_3: Right triangles are triangles whose two of the edges intersects perpendicularly at a corner.
S_5: We can say that right triangles are triangles whose angle measure of one of interior angles is 90^0.

WARRANT
S_3: We can define right triangles based on angles and edges which are main elements of triangles as we have done to define triangles…

Figure 1. Toulmin's model of argumentation for the first idea

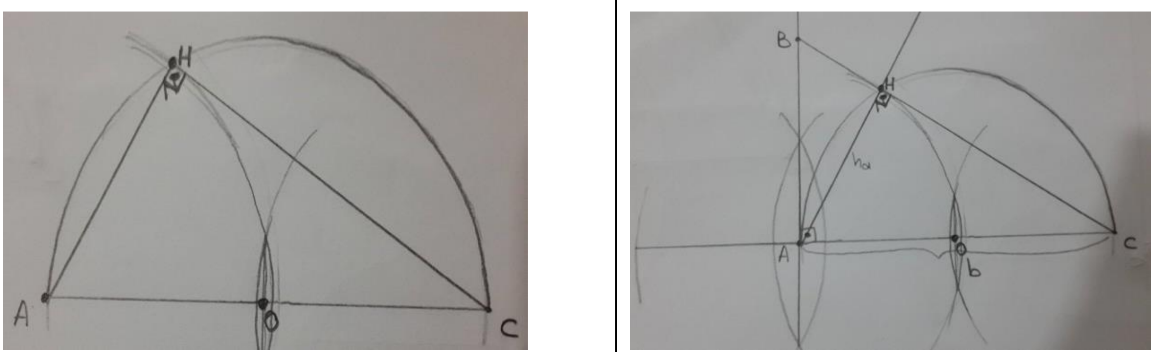
This mathematical idea was used by the participants in the tasks of the following weeks. They used the knowledge about the definition of right triangles attained by the participants as data and warrant in their arguments in the first, third, fifth and sixth weeks of the instructional sequence without necessitating backings. Firstly, the participants used this knowledge in order to examine the possibility of the formation of triangles when the measures of some of their elements were known as in the following mathematical idea. Secondly, they produced the claim about the definition and construction of the altitudes using the definition of a right triangle as the data and warrant of the argument emerged in the third week.

Mathematical Idea 2: Reasoning on the Construction of Triangles

After defining the triangles focusing on the critical elements of them, the PMSMT focused on the construction of triangles using some of their main or auxiliary elements and examined the possibility of construction of triangles under particular circumstances. To line with this view, they examined the construction of triangles by knowing some of their main or auxiliary elements. In other words, the participants were asked to investigate if it was possible to construct a triangle by connecting some of the elements and knowing their measures. In these problems, they investigated whether they were able to form explained triangles by reasoning differently. This mathematical idea can be exemplified by the following discussion taking place in a part of the argumentation:

Instructor: When we know the measures of ha and b and m(BAC) = 90^0 in the triangle of ABC, is it possible to draw/construct this triangle? How?

For the solution of this problem, S_5 made computations by Pythagorean Theorem and Euclidean Theorem using these values to find the measures of necessary unknown elements. Then, S_{10} explained that these theorems showed that they were able to construct a triangle by these values and added: "Is it possible to determine the type of triangles constructed by these values?". By this question, argumentation was directed to another solution for this problem. Hence, the data and warrant of the argument were provided. Also, they used geometric construction steps for another solution strategy of the problem under the guidance of the instructor as the backing of the argument. S_8 initially constructed the right triangle of AHC as in Figure 2.a with the hypotenuse having the length of b benefiting from the property that the inscribed angle opposing of the diameter had the angle measure of 90^0. Then, he constructed a right angle whose one of the rays was the edge of AC as in Figure 2.b. Afterwards, he extended the line segment passing through the points of H and C by providing that it intersected the other ray of the right angle on the vertex of A. This intersection point was named as the vertex of B so that the triangle of ABC was constructed. Hence, the claim about the possibility of the formation of this triangle was produced. Also, S_{11} added, "isosceles or scalene triangles can be constructed based on the case of b>ha or b=ha".



a. Construction of the triangle of AHC formed by S_8

b. Construction of the triangle of ABC formed by S_8

Figure 2. The triangles formed by S_8 and S_{11}

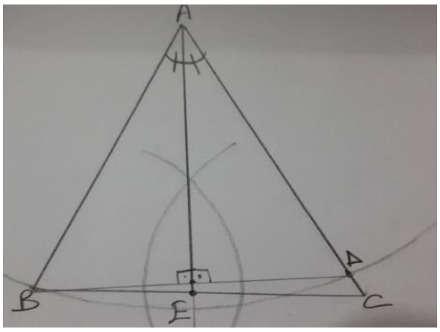
Through argumentation, the PMSMT formed different solution strategies for the problem and complete the correct idea together. Then, they used these strategies for different contexts and problems. This idea served as the data in the arguments about the similarity and congruence content since it was used in order to determine whether they were congruent/similar triangles. This idea was also observed as the data and warrant in different argumentations.

Mathematical Idea 3: Reasoning on the Construction of Auxiliary Elements of Triangles

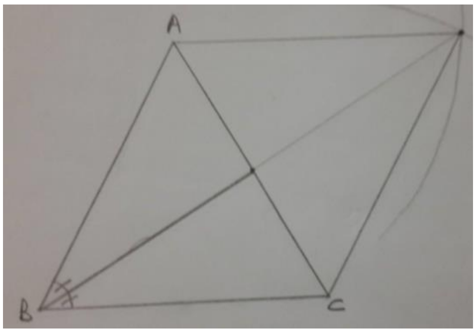
The PMSMT constructed the auxiliary elements of triangles through geometric constructions and then they discussed their solutions. This mathematical idea includes the argumentation logs about the construction of median, angle bisector, altitude, perpendicular bisector discussed by the PMSMT. This idea can be exemplified by the whole class discussion about angle bisector as follows...

All of the participants used the definition of angle bisector of a triangle as the data in order to construct it. Then, two different ways representing different construction steps of an angle bisector were produced. One of them was provided as the warrant and the other way was stated as the backing of the argument. In the warrant, they constructed the angle bisector by forming an isosceles triangle since angle bisector of it was the median of the opposing edge as in Figure 3.a. below is …

S_9: Initially, I form an isosceles triangle. By drawing an arc passing through the vertex of B, the intersection point of this arc on the other edge is identified. When this intersection point is combined with the vertex of B with a line segment, we form the isosceles triangle of ABD… (Figure 3.a)



a. By isosceles triangle



b. By a parallelogram

Figure 3. Different constructions of angle bisector

Also, in the backing in Figure 3.b, S_6 formed a parallelogram with its diagonals as the angle bisectors of the interior angles of it. In this construction process, the possibility of construction of an angle bisector of a triangle was discussed as follows:

S_7: … I adjust compass width in the length of the edge of BC and I draw an arc without changing this width by placing compass on the vertex of A. Then, I adjust compass width in the length of the edge of AB and I draw an arc placing the compass on the vertex of C. I combine the intersection point of arcs with the vertices …

The PMSMT formed the argument and idea collectively by discussing and analyzing their solutions. They also formed different solution strategies through argumentations. Based on the knowledge about isosceles triangles and parallelograms, the participants reasoned successfully about the geometric construction of an angle bisector by providing accurate mathematical justification. Moreover, the formation of this idea and its contributions to other argumentations happened for other auxiliary elements in similar ways.

Mathematical Idea 4: Reasoning on the Concurrence of Auxiliary Elements of Triangles

The PMSMT investigated the concurrency of auxiliary elements of triangles. They performed the tasks about the concurrency of medians, angle bisectors, altitudes, perpendicular bisectors of a triangle focusing on the type of triangle. The formation of an argumentation log in the mathematical idea can be exemplified using the discussions about angle bisector as follows…

In a task about angle bisectors, S_{11} claimed the concurrency of angle bisectors of a triangle and provided the data by the geometric construction of angle bisector as explained in the previous mathematical idea. Also, the warrant was illustrated by constructing all angle bisectors of the triangle and showing their concurrency. Then, S_8 provided explanations about the concurrency of angle bisectors using Ceva and Menelaus theorems as the backing of the argument. He assumed that they intersected at exactly one point and then by showing the applicability of these theorems for the concurrency of angle bisectors, their concurrency was showed and justified in a mathematically correct way. Afterwards, the instructor asked another way and S_{13} provided the backing by angle bisector theorem that a point placed on an angle bisector was far away in equal distances from the rays of an angle. She formed perpendicular lines to the edges of the triangle as in Figure 4 so that she determined the line segments in equal length. Hence, the backing represented the necessary mathematical justification for the argumentation. In order to show that the third angle bisector belonged to the angle of C passed through the intersection point of other two angle bisectors, she represented |CK| = |CH| and |OK| = |OH|. Through the process, she drew the line segments to combine the point of K with H, and C with O so that a deltoid was formed with its diagonals. Based on the property that one of the diagonals of the deltoid separated it into two isosceles triangles (isosceles triangle of OHK and CHK) and the other diagonal divided it into two congruent triangles (congruent triangles of OHC and OHK). Therefore, these diagonals were also angle bisectors of the interior angles of the deltoid. The diagonal passing through the points of O and C was the angle bisector of the angle on the vertex of C so that the concurrency of angle bisectors of the triangle of ABC was showed accurately and necessarily. At the end of her explanation, the instructor summarized the reasoning process and emphasized the important parts.

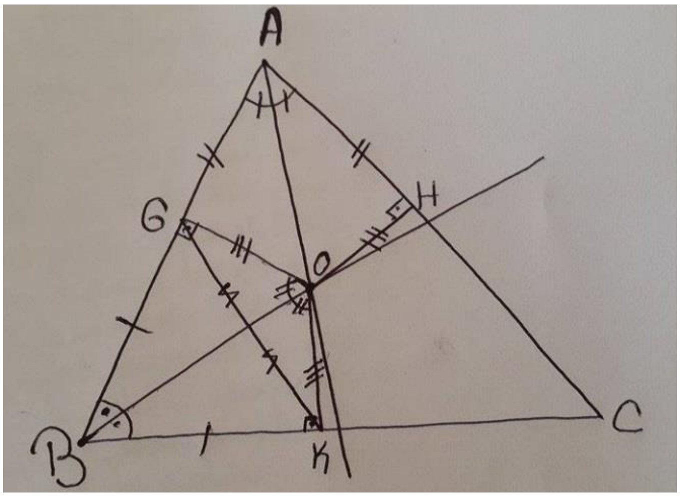


Figure 4. The concurrence by angle bisector theorem

It was observed that the PMSMT justified the concurrency of angle bisectors through argumentations. They formed the idea by sharing their reasoning. On Week 6, it was illustrated that the mathematical arguments about the concurrency of angle bisectors were used in different arguments as data and warrant such as incenter and its place in different types of triangles. Also, it was used to show and to justify the concurrency of perpendicular bisectors mathematically, and to solve the problems. Arguments for ideas about other auxiliary elements and their contributions to following discussions happened in similar ways.

Mathematical Idea 5: Reasoning on the Names of Concurrent Points of Auxiliary Elements of Triangles and their Places

The PMSMT investigated the concurrency of auxiliary elements of triangles such as centroid for medians, incenter for angle bisectors, circumcenter for perpendicular bisectors and orthocenter for altitudes respectively. The PMSMT participated in the discussion to identify the names of these points of concurrency, and whether the place of these points changed based on different types of triangles. For example, S_{10} first made the claim about circumcenter as concurrency point of perpendicular bisectors. Then, the instructor challenged him to explain how this point became circumcenter. He provided the data that three perpendicular bisectors of a triangle concurred at a point as claimed and formed in the previous mathematical idea. He also added the warrant that there were three isosceles triangles of AOB, BOC, and AOC where two of their edges' lengths were equal as in Figure 5. Also, the vertices of the triangle were equidistant from the concurrency point of perpendicular bisectors so that a circle could be formed based on the definition of a circle combining these vertices by the arcs with the center on this point. Then, this case was justified mathematically in a way that the perpendicular line segments passing through the center bisected the chords in a circle as in Figure 5. As it was observed, the concurrency of perpendicular bisectors and the process of showing their concurrence were used to provide the warrant accurately and necessarily.

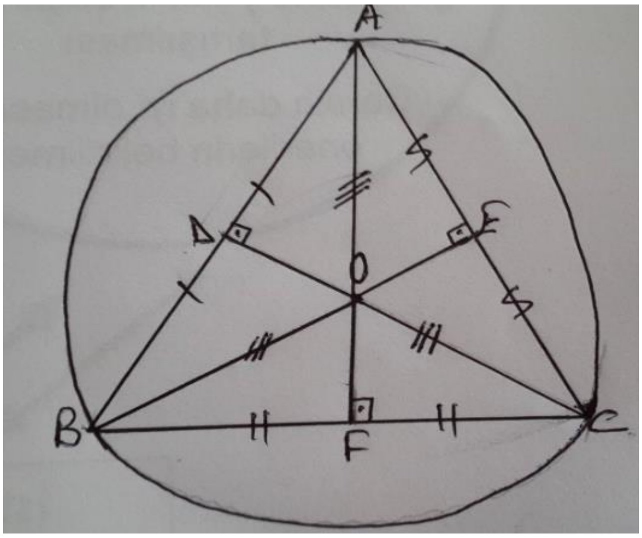


Figure 5. The circumcenter by perpendicular bisectors.

Through argumentation, the PMSMT formed the idea together and this idea was used as data and warrant of different arguments as it happened in the discussion initiated with the questions of “Does the place of the circumcenter change based on the types of triangles? How?”. In this debate, the participants used this mathematical idea as data and warrant in the debates made in order to determine whether the place of the point representing the circumcenter changed for obtuse and right triangles. Moreover, this idea was used in another discussion made about congruence/similarity on Week 5. In this debate, the topic was that the radius of circumcircles of congruent triangles was sometimes in equal length and the distances of circumcenter to the edges were sometimes equal.

Mathematical Idea 6: Reasoning on Congruence and Similarity

The PMSMT examined congruent triangles based on the knowledge that all of the properties of shapes except for their orientation were preserved through rigid motions. Also, dilation was used to form similar triangles since the image was drawn proportionally. The participants were asked to find the image of triangles by geometric constructions through geometric transformations. After forming images, they determined whether triangles and their images were congruent or similar by argumentations. For example, S_7 initially made the claim that the triangle and its image were congruent correctly. They benefited from the definitions of translation and triangle as the data of the argument. S_2 explained the process of construction of the image through translation as the warrant accurately. In her explanation, S_2 stated that the distances between parallel lines were preserved and the vectors represented the line segments having magnitude and direction.

In this respect, the edges of the triangle were moved by using parallel lines preserving the angles between the edges and the lengths of them. Hence, the lengths of the edges and the measures of the interior angles of the image were equal to the pre-image. Hence, S_2 and S_9 provided sufficient and appropriate backings for this mathematical idea. Then, the instructor directed the discussion to talk about the congruence criteria necessarily since they needed the criteria in order to represent and justify the congruence of these triangles. Afterward, S_9 provided the backing that two triangles were congruent. By the cosine formula, it was shown that the lengths of corresponding edges of these triangles were equal since corresponding angles were in equal measure. The known measures of the triangles were written on the formula in order to show the equality of the angle measures opposite of the edges in equal length. When the process was repeated for all opposing angles of the triangles, the similar results were obtained. The Toulmin’s model of argumentation for some parts of this debate can be represented as in Figure 6.

The PMSMT formed the idea by sharing their reasoning as the elements of the argumentation. They used this knowledge in the debates in order to decide whether the triangles and their images for other rigid motions were congruent. They provided the warrant and backing using the construction of images through other motions. Moreover, this mathematical idea was used as data and warrant in different argumentations formed about the problems related to congruence and similarity of triangles.

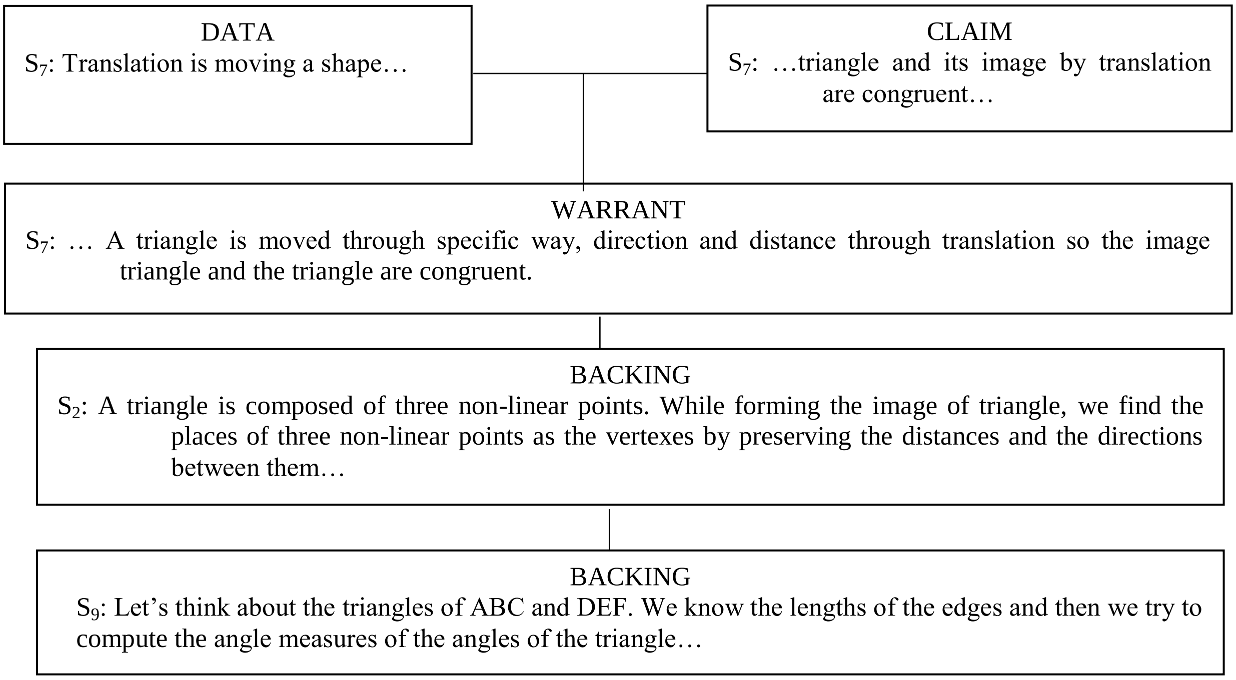


Figure 6. Toulmin's model of argumentation for the sixth idea

Discussion and Conclusion

By the current study, information about the effects of instructional sequence guided by the HLT supported by argumentations and PBL to enhance the development of the PMSMT's SMK is provided. Six mathematical ideas, acquired through their engagement in the tasks and argumentations, were illustrated to represent the development of the PMSMT's SMK. It was observed that they had limited knowledge about triangles. For example, they did not have sufficient knowledge about definitions of triangles, justifying the concurrence of auxiliary elements, proving congruence, and similarity of triangles and their criteria. On the other hand, their understanding of them was encouraged through argumentations. To line with this view, it was observed that the PMSMT could revise and construct their SMK through criticizing with the help of argumentation. Also, the increase on their familiarity with the content of triangles and development of SMK could improve the quality of argumentation formed through whole class discussion. Hence, it can be stated that the quality of argumentation formed in the classroom was related to the development and familiarity with SMK as explained in previous research (Cavagnetto and Kurtz, 2016; Chen, Park, and Hand, 2016; Grooms, Sampson, and Enderle, 2018). The period of acquiring necessary SMK of triangles was represented by the mathematical ideas in the study. Their development of SMK about triangles was also provided by argumentations. By applying the Toulmin's model of argumentation to the discussions, the argumentation logs were determined and then argumentations representing the mathematical ideas referring to the SMK of the PMSMT were identified. Through instructional sequence guided by the HLT, the PMSMT formed their ideas by revising their incorrect and missing knowledge, and developed their SMK appropriately.

In the HLT, the problems were designed mostly on geometric constructions. For example, the PMSMT examined the formation and definition of triangles through geometric constructions. The geometric constructions can be accepted as problems since they hold criteria of being problem that an individual must realize the situation and be willing to remove it, and then he cannot directly move on the solution process but he insists on to reach the solution (Lester, 1980). The geometric constructions are solutions of a problem because the learners do not decide easily how to start constructing the shapes at first glance and then they have challenge to complete the constructions (Erduran and Yeşildere, 2010). In this respect, geometric constructions represented problems for PMSMT and they had challenge to solve these problems. The findings show that PBL enhanced the emergence of argumentations and mathematical ideas since the instructional sequence provided the opportunities of forming, sharing, reasoning and criticizing ideas to the PMSMT. Through finding accurate solution for the problems, the PMSMT shared their ideas, criticized them to reach consensus and formed the solution by convincing others. Therefore, it could be stated that problems could enhance the emergence of mathematical argumentations. On the other hand, these opportunities could be provided by the problems on the

HLT. Moreover, the use of the HLT could help the instructor organize the discussions and emergence of argumentations and ideas to develop SMK. To conclude, in order to provide an instructional sequence for the PMSMT to develop their SMK though argumentations, PBL should be used to design this sequence. Also, research in the literature illustrated that discussions including argumentations taking place in problem solving activities facilitated and improved problem solving abilities, scientific thinking by criticizing and justifying claims, knowledge production and conceptual understanding (AbiEl-Mona and Abd-El-Khalick 2011; Duschl and Osborne 2002; Jim´enezAleixandre et al. 2000; Jonassen and Kim 2010; Osborne, Erduran, and Simon 2004; Zembaul-Saul 2005).

It could be stated that the argumentations improved the PMSMT's SMK. For example, initially, as explained in the first mathematical idea, while defining triangles, they produced definitions of triangles without all necessary critical attributes and properties. However, through argumentation, they challenged incorrect and insufficient definitions, determined the missing and unrelated parts of them, and then produced correct definition including critical attributes and properties necessarily. To line with this explanation, research in the literature explain that discussions including argumentations taking place in problem solving activities facilitated and improved problem solving abilities, scientific thinking by criticizing and justifying claims, knowledge production and conceptual understanding (Abi-El-Mona and Abd-El-Khalick 2011; Duschl and Osborne 2002; Jim´enez-Aleixandre et al. 2000; Jonassen and Kim 2010; Osborne, Erduran, and Simon 2004; Zembaul-Saul 2005). In addition, argumentations facilitate doing mathematics and discussing claims in a social environment in which the learners communicate and make reasoning to form the discourse, knowledge, and classroom culture (Abi-El-Mona and Abd-El-Khalick 2011). Therefore, it can be stated that the argumentations are useful to encourage the development of SMK of learners. In other words, when the PMSMT's learning of triangles through mathematical ideas was considered, it was observed that the discussion periods including argumentations and enhancing the formation of the arguments could improve their SMK.

To conclude, the findings about six-week instructional sequence including argumentations helping the PMSMT develop their SMK about triangles were documented in the current study. It was observed that this sequence improved the PMSMT's SMK about the mathematical concept of triangles. Hence, it can be stated that argumentations can be useful to help teacher candidates attain the necessary knowledge and understanding in teacher education programs. Also, this study can provide useful feedback for the current and future studies about preservice teachers and development of SMK about mathematical concepts. They provided different perspectives for a case or problem. Hence, they formed the idea accurately. This finding is parallel to other research since argumentations as a kind of discourse can be useful for teachers. The instructors can form an environment including multiple ways of constructing mathematics and solving mathematical problems for the students challenging, judging and justifying their ideas (Andrews 1997; Owen 1995).

The discussion period including argumentations improved their geometric thinking and knowledge of triangles in the study. The previous research validated this finding as in the study of Olkun and Toluk (2004) who found that in-class discussions improved the learners' geometric thinking. Also, research in the literature illustrated that discussions including argumentations taking place in problem solving activities facilitated and improved problem solving abilities, scientific thinking by criticizing and justifying claims, knowledge production and conceptual understanding (Abi-El-Mona & Abd-El-Khalick, 2011; Duschl & Osborne, 2002; Jim´enez-Aleixandre et al., 2000; Jonassen & Kim, 2010; Osborne, Erduran, & Simon, 2004; Zembaul-Saul, 2005).

Note

This article has been produced from the author's doctoral dissertation entitled "*Developing mathematical practices in a social context: A hypothetical learning trajectory to support preservice middle school mathematics teachers' learning of triangles*".

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