Research Report

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RESEARCH REPORT

The Development of a Quadratic Functions Learning Progression and Associated Task Shells

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In this report, we discuss background research on the development of student understanding of quadratic functions and present a provisional learning progression for quadratic functions and quadratic equations. We also describe task shells that are linked to the learning progression. The learning progression and task shells can be used as a starting point to develop tasks that assess student standing with respect to the theory of development. The intention is that this report should find an audience in both researchers and test developers.

Keywords Learning progression; quadratic function; quadratic equations; task shells; assessment

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The study of functions is emphasized in both the algebra curriculum and in the Common Core State Standards for Mathematics (CCSSM; Common Core State Standards Initiative [CCSSI], 2010). The CCSSM includes high school–level standards that are specific to a variety of types of functions: linear, quadratic, polynomial, rational, exponential, trigonometric, radical, and so on. But it also includes a more advanced standard that students should "construct and compare linear, quadratic, and exponential models and solve problems" (CCSSI, 2010, p. 70). Before they can meet this culminating standard by comparing different types of models, students will need focused modeling experiences using each type of function. They will also need practice working with the supporting representations (graphs and equations) for each type; specific expectations in this regard are detailed in the CCSSM.

In this context, learning progressions (LPs) have the potential to be an important tool in assessing student learning and guiding student instruction. An LP is a characterization of how student thinking in an area of study develops, from early conceptions to more complete understanding (Smith, Wiser, Anderson, Krajcik, & Coppola, 2004). An LP is subdivided into levels that constitute qualitative changes in thinking (Deane, Sabatini, & O'Reilly, 2012). As such, it may describe how student thinking evolves with respect to content and/or practice. Based on research in the learning sciences and cognitive psychology, an LP may be used as a basis for assessment design and has the potential to be used as a guide for instruction, pending validation. Once an LP has been validated, a teacher can use it to assess student standing with respect to the progression for the purpose of providing activities that will help the student reach the next level. The word "activities" here may be broadly construed—an activity might be a task, but it might also be a back-and-forth discussion of a misconception typical of a particular level.

As part of the *CBAL®* learning and assessment tool research initiative at Educational Testing Service (ETS; Bennett, 2010; Bennett & Gitomer, 2009), an LP was developed for linear functions (Arieli-Attali, Wylie, & Bauer, 2012; Graf & Arieli-Attali, 2015). The current work can be viewed as a sequel to this work. A quadratic functions LP is not the only possible sequel to a linear functions LP, of course. Although quadratic functions are usually treated before exponential functions in the traditional school curriculum (see, e.g., Blitzer, 2018), some researchers have suggested introducing exponential functions as the sequel to linear functions; linear functions have constant absolute change, whereas exponential functions have constant relative change. As Fife, Graf, Howell, and James (2017) mentioned, there are strong suggestions in the CCSSM that a treatment of exponential functions could follow a treatment of linear functions. For example, in the introduction to the high school section of functions, the standards state: "Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a

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constant percent rate" (CCSSI, 2010, p. 67). There are also connections to be made between linear functions and quadratic functions, however. These are discussed in detail later in the report.

Given this backdrop, exponential functions and quadratic functions are the natural sequels to linear functions. An LP for exponential functions has been developed elsewhere (Fife et al., 2017); in this report, we develop an LP for quadratic functions. To work with quadratic functions, students also need to develop facility with solving quadratic equations (and these procedures are heavily emphasized in the CCSSM as well). For this reason, we decided to address both quadratic functions and quadratic equations in a single comprehensive LP.

Though quadratic functions are an important component of the Algebra II curriculum, not very much research has been devoted to the study of how well students understand quadratic functions in particular. The few papers that have been written suggested that instruction of the topic tends to be highly procedural and focuses solely on the mechanics of graphing and algebraic manipulation. The existing research has also identified numerous conceptual difficulties students have and problems students experience when working with quadratic functions; although the research base is small, consensus is evident with respect to which of the ideas pose difficulty and what kinds of incorrect responses result (Ellis & Grinstead, 2008; Eraslan, 2008; Vaiyavutjamai, 2009; Zaslavsky, 1997). This research potentially paves the way for a focused program of instruction for quadratic functions. The study of quadratic functions is important, not only because it figures prominently in the algebra curriculum and in the standards, but also because, as with many mathematical topics, it has important applications to science and technology (e.g., see Budd & Sangwin, 2004). The capacity to apply quadratic functions with understanding deserves a place on the short list of valued 21st-century skills. To this end, the LP discussed in this report characterizes the levels of sophistication of student thinking with respect to analyzing quadratic functions, solving quadratic equations, and creating quadratic models.

Measuring student competency requires the development of multiple parallel tasks, both so that the construct is sufficiently represented and so that particular tasks are not overexposed. This requirement has resulted in renewed interest in a model-based assessment development approach. According to Mislevy, Steinberg, and Almond (2002):

> A task model provides a framework for describing the situations in which examinees act. In particular, this includes specifications for the stimulus material, conditions, and affordances, or the environment in which the student will say, do, or produce something. It includes rules for determining the values of task-model variables for particular tasks. And it also includes specifications for the "work product," or the form in which what the student says, does, or produces will be captured. Altogether, task-model variables describe features of tasks that encompass task construction, management, and presentation. (p. 112)

Furthermore, instantiating task-model variables with values results in specific tasks, whereas instantiating a proper subset of task-model variables results in a task shell (Mislevy et al., 2002). In accordance with this definition, two task shells for quadratic functions have been developed: the quadratic functions task shell and the quadratic regression task shell. These task shells characterize some of the features of tasks that will elicit evidence of student standing on the LP.

The intention is that this report should find an audience in researchers and test developers. It is important to note that the LP discussed here has not yet been empirically verified—and empirical verification is an important component of the validation of an LP (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009; Corcoran, Mosher, & Rogat, 2009; Deane et al., 2012; Graf & van Rijn, 2016). Nonetheless, before it can be empirically verified, tasks must be developed based on the LP so that it may be used provisionally by test developers. The LP has been used by test developers to develop student-level tasks for CBAL and in the development of items to measure teachers' content knowledge for teaching (CKT; Ball, Thames, & Phelps, 2008). CKT is defined to include teacher knowledge of both students and content; as a result, information about students, including their conceptions and misconceptions and their likely patterns of learning, has been identified as a critical component in designing tests of teachers' CKT (Phelps & Howell, 2016).

The Learning Progression for Quadratic Functions and Quadratic Equations

Correspondence and Covariation Views of Quadratic Functions

One can approach the study of functions from the *correspondence* view or the *covariation* view (Confrey & Smith, 1994, 1995; Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016). The correspondence view is concerned with the dependence of

outputs on inputs, while the covariation view is concerned with representing the nature of change (e.g., linear functions represent constant change). A correspondence approach to quadratic functions leads to the usual algebraic forms for expressing a quadratic function:

Standard form:

$$y = ax^2 + bx + c.$$

Vertex form:

$$y = a(x - h)^2 + k, \text{ where } (h, k) \text{ is the vertex.}$$

Factored form:

$$y = a(x - r)(x - s), \text{ where } r \text{ and } s \text{ are the } x\text{-intercepts.}$$

The relationship between quadratic functions and linear functions is apparent from the standard form of the quadratic function; the quadratic function $y = ax^2 + bx + c$ differs from the linear function $y = bx + c$ in the presence of the quadratic term ax^2.

The covariation approach to functions, with its emphasis on the nature of change, is particularly useful when emphasizing the relationship between exponential functions and linear functions, as discussed previously. But patterns of change can be used to emphasize the relationship between quadratic functions and linear functions as well: Quadratic functions have constant second differences, and linear functions have constant first differences (Berthgold, 2004; Lobato, Hohensee, Rhodehamel, & Diamond, 2012). If f is an arbitrary real-valued real function and the sequence $\{a_n\}$ is defined by $a_n = f(n)$ for $n = 1, 2, 3, \ldots$, then the sequence $b_n = a_{n+1} - a_n$ is called the sequence of *first differences*, and the sequence $c_n = b_{n+1} - b_n$ is called the sequence of *second differences*. For example, if f is the quadratic function $f(x) = \frac{1}{2}x^2 + \frac{1}{2}x$, then the sequence $a_1, a_2, a_3, \ldots$ is the sequence of triangular numbers 1, 3, 6, . . . , the sequence of first differences b_1, $b_2, b_3, \ldots$ is the sequence 2, 3, 4, . . . , and the second differences $c_1, c_2, c_3, \ldots$ are all equal to 1 (see the appendix).

For the general quadratic function $f(x) = ax^2 + bx + c$, we have $a_n = an^2 + bn + c$. A simple calculation then shows that $b_n = 2an + a + b$ and $c_n = 2a$ (see the appendix). Thus, for a general quadratic function, the second differences are constant.

Conversely, Confrey (1992) noted that it is also possible to construct a quadratic function by running the process in reverse, starting with a sequence of constant second differences, together with starting values for the sequence of first differences and the sequence of functional values. With constant second differences $c_n = d$, initial first difference $b_1 = e$, and initial functional value $a_1 = g$, the underlying function is the quadratic function

$$f(x) = \frac{1}{2}dx^2 + \left(e - \frac{3}{2}d\right)x + \left(g - e + d\right),$$

at least when x is an integer; see Table A2 in the appendix. Note that if the second difference d equals 0, the quadratic term is eliminated and the resulting function is linear.

By way of making connections between quadratic functions and other types of functions, Movshovitz-Hadar (1993) also recommended having students construct quadratic functions as the product of two linear functions or as the sum of three monomials. Confrey (1992) outlined three themes in the study of quadratics: rate of change, symmetry, and dimensionality. Rate of change pertains to the property that the second differences of a quadratic function are constant. Symmetry refers to the symmetry of the graph of a quadratic function. Dimensionality refers to expressing a quadratic function as the product of two linear functions.

Conceptual Difficulties in the Study of Quadratic Functions

As with other CBAL mathematics LPs, the model for quadratic functions was designed around the *big ideas*, or concepts and themes that are central to an appreciation of the topic. The LP is based on big ideas and misunderstandings identified in the literature (e.g., Berthgold, 2004; Ellis & Grinstead, 2008; Zaslavsky, 1997) as well as standards included in the CCSSM (CCSSI, 2010).

Numerous misunderstandings pertaining to the study of quadratic functions were observed by Zaslavsky (1997). First, she noted that there are several misunderstandings related to how the graph of a quadratic function is perceived within a window, for example, because of the apparent increase in “steepness,” some students believe the graph of a quadratic function has vertical asymptotes. She also described evidence of confusion surrounding the distinction between

a quadratic function and its corresponding quadratic equation: Given two intercepts of a quadratic function, students would identify the function as $y=(x-r)(x-s)$. Although this function and the target function share the same roots, they need not be the same, because the a parameter (see the factored form, earlier) is still not determined given just these two points. Zaslavsky's interpretation was that students may overgeneralize what may be inferred when working with equivalent equations; that is, $a(x-r)(x-s)=0$ and $(x-r)(x-s)=0$ are equivalent equations because $a \neq 0$ by definition, but this does not imply that $y=a(x-r)(x-s)$ and $y=(x-r)(x-s)$ are the same functions. Vaiyavutjamai (2009) noted that when working with the vertex form $y=a(x-h)^2+k$, a student attended to the sign of the a parameter but not to its value. Zaslavsky (1997) and Vaiyavutjamai (2009) both observed that students do not consider enough points in attempting to specify a quadratic function.

There is also evidence that although students may have learned procedural rules in association with one or more of the algebraic forms given earlier, they may not understand their mathematical meaning. The roles of the parameters (especially the a and b parameters in the standard form) are not obvious, and students may have difficulty distinguishing among the different forms. For example, Eraslan (2008) described how when given the function $y=x^2+2x-3$, a student graphed the parabola so that it "opened up" but located the vertex at the point (2, −3). When asked to express the graph of $y=(x+1)^2+4$ as an equation, the same student wrote $y=-x^2-1x+4$. Both of these errors are consistent with conflating the standard and vertex forms. According to Ellis and Grinstead (2008), who focused on students working with the standard form, although the a parameter is interpreted as influencing the shape of the parabola, students also thought that varying the a parameter would not change the location of the vertex. This finding may also be explained by confusion with the vertex form: Because the a parameter plays a somewhat different role in the standard and vertex forms, changing the a parameter in the vertex form does not change the location of the vertex. It is clear why confusions among these forms arise: Although they have different mathematical interpretations, they also have many similar surface characteristics. If students' schemas for working with quadratic functions are based on memorized rules (e.g., "If a is negative, the graph 'opens down'") without mathematical meaning, then it might be expected that these confusions will occur.

Students may not recognize special cases of the quadratic function, for example, believing that the c parameter does not exist when it is equal to zero (Zaslavsky, 1997). Graph translation activities are often explored in the context of studying quadratic functions. Vaiyavutjamai (2009) found that while students can sometimes successfully apply rules to translate the graphs of quadratic functions given in the vertex form (e.g., if a constant is added outside the parentheses, shift the graph "up"; if a constant is added inside the parentheses, shift the graph "left"). However, a student who successfully implemented such strategies indicated in an interview that he did not understand why the rules worked—he applied them as they had been taught.

Both Zaslavsky (1997) and Ellis and Grinstead (2008) noted interference with concepts learned about linear functions. For example, students sometimes attempt to calculate the slope of a parabola. When working with the standard form, students tend to interpret the a parameter as the slope of the parabola (Ellis and Grinstead noted that most of their interview participants gave this interpretation at some point). Zaslavsky (1997) described how one student attempted to find the a parameter by calculating $(y_2-y_1)/(x_2-x_1)$ for two points on the graph of a quadratic function.

In general, students have difficulty with interpreting the parameters of quadratic functions. When working with standard form, interpreting the c parameter (the y-intercept) appears to be more straightforward for most students. Students seem to have a partial understanding of the a parameter, though the differing roles of the a parameter in the standard and the vertex forms may be a point of confusion. When working with the vertex form, most students can readily identify the vertex, but many still have difficulty with transforming the graph, even when using this form. The b parameter in the standard form is difficult to understand and difficult to teach, but exploring its role may be worthwhile for helping to make the link between linear and quadratic functions.

Assumptions of the Progression

It is assumed that students at Level 1 have reached at least Level 3 of the linear functions LP (Arieli-Attali et al., 2012; Graf & Arieli-Attali, 2015) but may have had no specific instruction on quadratic functions beyond that. The big ideas identified from the research literature and the CCSSM were instantiated as progress variables, or learning goals, within the progression (see Corcoran et al., 2009). Identifying the progress variables involved several iterations of reflecting on the standards and the literature, followed by discussion. For example, solving quadratic equations was not initially one of the progress variables in the progression, but working with quadratic functions so often leads to solving quadratic equations

that we felt that methods for solving quadratic equations needed to be addressed to completely cover the construct. The progress variables are listed subsequently, together with brief descriptions of what each one entails.

Compare Quadratic Functions to Other Functions

At the earlier levels, this involves comparing the characteristics of quadratic functions to functions that are obviously different by visual inspection (e.g., linear functions). At the later levels, students will be able to compare quadratic functions to linear and exponential functions with respect to their rates of change.

Model Situations Using Quadratic Functions

Consistent with the CBAL mathematics competency model (Graf, Harris, Marquez, Fife, & Redman, 2009, 2010), this skill encompasses using quadratic functions to model real-world situations and applying those models to the real world, for example, by using them to make predictions (see Lesh & Lamon, 1992). Tasks involving this skill may also require students to *interpret* a given quadratic function with respect to a real-world situation. At the earlier levels, it is expected that students will require significant scaffolding support to use quadratic functions to model situations. Later in development, students may be able to choose a quadratic model on the basis of fit, though they are unlikely to justify the selection on theoretical grounds. Finally, at the most advanced levels, it is expected that students will have internalized schemas that allow them to recognize situations that may be modeled by a quadratic function.

Graph Quadratic Functions and Characterize Properties of Quadratic Functions Based on Their Graphs

At the early levels, it is expected that students can graph quadratic functions by constructing a table of values and plotting points. At the most advanced levels, it is expected that students will be facile with curve sketching and have moved well beyond plotting from points. Students at the more advanced levels also understand when they have sufficient information to construct a graph.

Solve Quadratic Equations Using a Variety of Methods

Students at the earlier levels can solve quadratic equations by inspection or using factoring techniques (where possible); at the intermediate levels, they become facile with using additional methods, such as applying the quadratic formula and completing the square. An understanding of the connections among these approaches develops gradually, with students at more advanced levels starting to understand why some solution approaches “work” when others do not. Students also gradually develop an understanding of the relationship between the graph of a quadratic function and the solution of the corresponding quadratic equation. Understanding the fundamental theorem of algebra, which is an advanced CCSSM standard, would not be expected (with respect to quadratic functions) until the most advanced level of the progression.

Use Algebraic Forms

Express quadratic functions in algebraic form, use an algebraic form to evaluate, and translate among the different algebraic forms. In particular, the three forms of interest are the standard form, the vertex form, and the factored form. Each form highlights different features of a quadratic function. At the earlier levels, it is expected that students can evaluate the function if given the function in different forms; later they learn to translate among the forms, and finally, they recognize the advantages of different forms for different applications.

Compare Quadratic Functions to Each Other Using Different Representational Forms

The representational forms of interest include tables, graphs, and equations. At the earlier levels, it is expected that students may have difficulty comparing functions expressed in different representational forms (e.g., see the description in Level 3, which is consistent with common patterns of errors observed by Eraslan, 2008). At the later levels, students become more facile with comparing functions expressed in different representational forms, and at the most advanced levels, they may not even have to convert them into a common form because they have internalized higher level schemas involving the characteristics of functions.

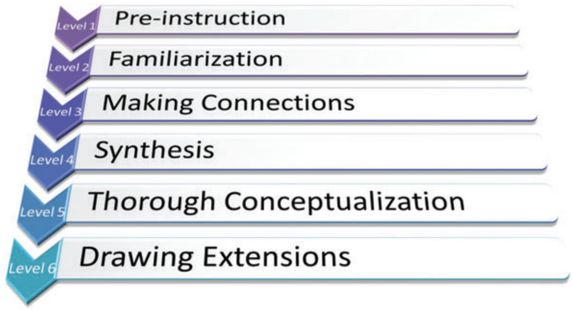


Figure 1 The six levels of the quadratic functions and quadratic equations learning progression.

Understand the Role of Parameters and the Relationship Between Parameters and Families of Quadratic Functions

It is challenging for students to develop an understanding of the role of parameters, in part because while an a parameter is used in both the standard and the vertex forms of a quadratic equation, it plays a different role in each. It is likely that understanding the role of the c parameter in standard form will develop first, with an understanding of the role of the b parameter developing last.

The Learning Progression

With respect to each of the progress variables, we assume that students develop across six levels (preinstruction through drawing extensions, as shown in Figure 1). The LP is represented as a set of seven tables, one table corresponding to each progress variable (see Tables 1–7). Each table provides a description of what is expected at each level of the corresponding progress variable. The levels are not always distinct on every progress variable (i.e., there may be one or more progress variables for which adjacent levels are the same). Also, we do not make the strong claim that levels are necessarily aligned across progress variables (e.g., a student at Level 3 on solve a quadratic equation using a variety of methods may not be at Level 3 on model situations using quadratic functions.

It is important to note that although we present these six levels as an organizing framework, identifying names for them was a bottom-up process. The descriptions of the levels, which are content specific, are based on the CCSSM and the findings from the literature discussed earlier. Only after we developed the content-based descriptions did we come up with the more generic short names for the levels shown in Figure 1. Each name captures the essence of the corresponding level, in our judgment. It was also our intent that the levels should be generic enough that they might be used for another LP in a different area of mathematics.

Nevertheless, the six levels do have some similarities to other overarching frameworks, namely, the original version of Bloom's (1956) taxonomy and the structure of the observed learning outcome (SOLO) taxonomy (Biggs & Collis, 1982). The main categories of Bloom's taxonomy include knowledge, comprehension, application, analysis, synthesis, and evaluation. We also have a synthesis level (Level 4), and it is defined as Bloom defined it: "The putting together of elements and parts so as to form a whole. This involves the process of working with pieces, parts, elements, etc., and arranging and combining them in such a way as to constitute a pattern or structure not clearly there before" (p. 206).

The SOLO taxonomy has five levels: prestructural, unistructural, multistructual, relational, and extended abstract (Biggs & Collis, 1982). At our preinstruction level, one might expect the kinds of performances described at the prestructural level to take place, though not necessarily. The prestructural level is characterized in terms of the student's nature of thinking (in that the student tends to be inconsistent and to draw on irrelevant information). The preinstructional level is characterized more with respect to the student's knowledge of the content: These are the kinds of performances one might expect before the student has been exposed to the material. Familiarization might be most similar to the SOLO unistructural and multistructural levels, expect that we make no claims about how many aspects of the system the student attends to, but we do claim that a student at this level tends to carry out rote procedures, generally without adaptation. The making connections, synthesis, and thorough conceptualization levels are most similar to the SOLO relational level. We have broken this into three levels because, based on our reading of the literature and our observations of student work, we believe the process of establishing relationships is a relatively slow one, carried out across more than one level.

Table 1 Quadratic Functions Learning Progression: Compare Quadratic Functions to Other Functions

Level	Description
Level 6: Drawing Extensions	Identical to Level 5
Level 5: Thorough Conceptualization	Students at this level conceptualize functions with respect to their rates of change and for this reason can distinguish between functions that when graphed appear very similar. They can compare rates of change for linear, quadratic, and exponential functions using first and second differences and can rule out the possibility that a function is quadratic if its second differences are not constant. They know that a quadratic function has different rates of change over its domain—positive, negative, and zero—while a linear function has a constant rate of change over its domain. Students understand and can explain that the minimum or maximum of the parabola is the point where the function is going from increasing to decreasing, or vice versa.
Level 4: Synthesis	Students at this level do not yet understand the nature of the rate of change for a quadratic function, though they certainly recognize that it is not constant. Like students at Level 3, they can communicate features that are common and different between quadratic functions and other functions (e.g., they realize that a quadratic function does not have vertical asymptotes).
Level 3: Making Connections	Students at this level can communicate features that are common and different between quadratic functions and other functions (e.g., they realize that a quadratic function does not have vertical asymptotes). Because they have not reached the level where they understand a quadratic function with respect to rates of change, comparisons to other functions in this respect are incomplete. For example, at this level, students could not yet articulate that the rate of change for a quadratic is not constant, though they can certainly observe that the graphs of linear and quadratic functions look different.
Level 2: Familiarization	Students at this level can identify characteristics of a quadratic function, e.g., symmetry, vertex as maximum or minimum, and intercepts (if any), though not necessarily using mathematical terminology. Because of this, they are able to distinguish quadratic functions from functions that do not have these characteristics. Students at this level do not necessarily understand end behavior (some may believe, e.g., that a quadratic function has vertical asymptotes)—also see the "Graph" progress variable.
Level 1: Preinstruction	Students at this level can recognize when looking at a graph (or perhaps a table) that a quadratic function is not linear, but they may not be able to distinguish quadratic functions from other nonlinear functions (e.g., if working in quadrant 1, they may identify an exponential function as quadratic).

Finally, the drawing extensions level is very similar to the SOLO extended abstract level—it is at this level that the student generalizes beyond the system. In drawing extensions, we also intend to suggest that the student generalizes to other mathematical content. Wilmot, Schoenfeld, Wilson, Champney, and Zahner (2011) used a slightly adapted version of the SOLO taxonomy as a framework for a LP for functions; they included a sixth level, prealgebraic, just below prestructural.

There are also frameworks of development that, while still general, are more specific to mathematics: These include Sfard's (1991) model of concept development and APOS theory (Dubinsky & Harel, 1992; Dubinsky & McDonald, 2001; Dubinsky & Wilson, 2013). Sfard's (1991) model includes three steps: *interiorization*, *condensation*, and *reification*. The interiorization step is most similar to our familiarization level; it is during interiorization that a student focuses on operational processes. During condensation, the student chunks processes and focuses less on discrete steps. Our progression does not really have an equivalent to the *condensation* step. Finally, during *reification*, the student conceives of the concept as an object that can be operated on in its own right. This step is most similar to Levels 5 and 6 of our progression. The stages of APOS theory are action, process, object, and schema (Dubinsky & McDonald, 2001). At the action level, the student can carry out procedures step by step and must have access to external stimuli. At the process level, the student can imagine carrying out procedures without actually enacting them and without access to external stimuli, such as manipulatives. At the object level, the student sees a concept as an object and can act on it (this is like Sfard's reification step). Finally, at the schema level, the student sees how actions, processes, and object views are integrated as part of the same concept. Level 2 of our progression (familiarization) is most similar to the action phase of APOS. We do not make

Table 2 Quadratic Functions Learning Progression: Model Situations Using Quadratic Functions

Level	Description
Level 6: Drawing Extensions	Identical to Level 5
Level 5: Thorough Conceptualization	As at Level 4, students can select a suitable function (given a choice among a linear, quadratic, and exponential function) to model a set of data. Level 5 students have internalized schemas that allow them to select the appropriate model by observing patterns of change. For example, a student at this level recognizes that if the terms of a sequence are linear, then the cumulative sum of the terms can be modeled by a quadratic function. Given a scatterplot that represents data in two quantitative variables, students can use a calculator or computer to visually compare fits among linear, quadratic, and exponential models. In contrast to students at Level 4, students at Level 5 can fully justify the selection of a model on theoretical grounds, as well as compare fits among different models, and can explain variable rate in terms of the situation they are modeling. They can connect each part of the graph to the situation being modeled.
Level 4: Synthesis	Students at this level can select a suitable function (given a choice among a linear, quadratic, and exponential function) to model a set of data. Given a scatterplot that represents data in two quantitative variables, students can use a calculator or computer to visually compare fits among linear, quadratic, and exponential models. In contrast to students at Level 3, students at Level 4 do not select a function solely on the basis of fit but are also able to partially justify the selection on theoretical grounds by relating the mathematical model to the real-world situation it represents. Students at this level can start to use quadratic functions to model novel situations.
Level 3: Making Connections	Students at this level can use quadratic functions to model real-world situations (such as projectile motion), provided they know at the outset that a quadratic function will be used to fit a set of data. If given a scatterplot that represents data in two quantitative variables, students can use a calculator or computer to compare the fitted curve from a quadratic regression to a fitted line from a linear regression and make the more suitable choice if it is clear to the eye which model provides the better fit. At this stage, they may still struggle with interpreting the relationship between the two variables. Note that it is not assumed here that the student is familiar with methods for finding lines or curves of best fit nor with methods for comparing fits between models.
Level 2: Familiarization	With scaffolding support, students at this level might begin modeling some more straightforward situations (e.g., vertical displacement with respect to time when a body falls from rest).
Level 1: Preinstruction	Students at this level are not expected to model using quadratic functions, at least not without a great deal of scaffolding support.

claims in our progression with respect to whether the student requires access to external stimuli. We do not really have a level similar to process; the object level is most similar to Levels 5 and 6 of our progression. The schema level is most similar to Level 6 of our progression, however, schema is focused on integrating alternative conceptions, while Level 6 of our progression is focused on extending a concept to new contexts and new domains.

We expect that most student performances will fall between Levels 2 and 5 of our progression. Levels 1 and 6 are provided primarily as anchors—Level 1 represents the kind of thinking expected before a student has studied quadratic functions, and Level 6 represents expert-level thinking, which might only be expected after studying mathematics courses well beyond Algebra II.

Task Shells

As discussed earlier, a model-based approach to task development is desirable because the development of parallel tasks mitigates the risk of overexposure and also helps ensure that student understanding is assessed across contexts. In this section, we motivate the decision to develop two task shells: the quadratic functions task shell and the quadratic regression task shell. Each shell corresponds to a different category of tasks. The roots of this decision can in part be traced

Table 3 Quadratic Functions Learning Progression: Graph Quadratic Functions

Level	Description
Level 6: Drawing Extensions	Identical to Level 5
Level 5: Thorough Conceptualization	As at Level 4, students at Level 5 are accomplished at curve sketching and know that three unique points determine a quadratic function.
Level 4: Synthesis	Students at this level know if asked that three unique points determine a quadratic function, though they may still make slips in which they attempt to identify a function without sufficient information. Apart from occasional slips, they are accomplished at curve sketching, and in general, they recognize that a family of quadratic functions passes through a given vertex or a pair of distinct points.
Level 3: Making Connections	Students at this level understand how the solutions of a quadratic equation with real roots are related to the corresponding graph and are starting to sketch graphs of quadratic functions, though they are still expected to have some difficulty with this because they do not yet understand necessary and sufficient criteria for creating the sketch (i.e., that three distinct points determine a quadratic function). For example, they may attempt to make a sketch after only finding the roots or only after finding the vertex.
Level 2: Familiarization	When given the symbolic representation, students can use it to construct a table of values and plot the points. They may be able to find the vertex given the vertex form or the intercepts given the factored form. Because they are familiar with the characteristics of the function, they are likely to notice and correct obvious plotting errors (e.g., departures from symmetry, too many extrema). While they may be able to solve some quadratic equations, they do not yet understand the connection between finding the roots of a quadratic function and solving the corresponding quadratic equation. When identifying graphs, students at this level may confuse graphs of quadratic functions with graphs of parabolas that are not functions. Students at this level may also, for example, confuse graphs of quadratic functions with graphs of hyperbolas, particularly if they cannot see the whole graph.
Level 1: Preinstruction	With scaffolding support, students can graph a quadratic function by plotting points when given an input/output table (or perhaps by constructing it). They will not notice plotting errors (such as those that result in departures from symmetry), nor can they sketch the graph because they are not yet familiar with the characteristics of a quadratic function. While they may be able to solve some quadratic equations, they do not yet understand the connection between finding the roots of a quadratic function and solving the corresponding quadratic equation. For example, for a quadratic function with two real roots, students would not yet recognize that the solutions to the corresponding quadratic equation indicate where the quadratic function crosses the x-axis.

to the CCSSM. Although the Common Core standards for algebra do not differentiate between Algebra I and Algebra II, there are several suggested course outlines in the appendix to the standards. The outline for a traditional Algebra I/Geometry/Algebra II sequence shows how the standards on quadratic functions are to be combined with the other CCSSM standards and divided between the first and second algebra courses. In particular, the Algebra I course includes the topics (a) solving quadratic equations by factoring, completing the square, using the quadratic formula, and graphing; (b) understanding quadratic functions as examples of functions; (c) understanding that solutions of quadratic equations are zeros of the associated quadratic function; and (d) using quadratic functions to model real-life situations. The Algebra II course includes the more theoretical topics of complex roots, the fundamental theorem of algebra, and the relationship between zeros of a polynomial and factors of the polynomial.

Prior to the units on quadratic functions, the Algebra I outline includes a unit on descriptive statistics, which includes plotting data points and using linear regression to model the data with a linear function. Although quadratic regression is never explicitly mentioned, either in the standards or in the appendix, students are nonetheless expected to build quadratic functions that model real data, and these activities are not limited to those situations in which it is reasonable to assume that a quadratic function describes the data precisely.

Table 4 Quadratic Functions Learning Progression: Solve Quadratic Equations Using a Variety of Methods

Level	Description
Level 6: Drawing Extensions	As at Level 5, students are accomplished with common methods that may be applied to the problem of solving quadratic equations with real coefficients and complex solutions, including factoring over the integers, using the quadratic formula, finding roots graphically, and completing the square. They understand that if a quadratic cannot be factored over the integers, then the solutions may still be found by completing the square or using the quadratic formula. Not only can they complete the square in a quadratic expression but they can use the result to reason about the maximum or minimum of the corresponding function without creating a sketch. They can recognize both graphically and algebraically when a quadratic function will have non-real roots. Additionally, students at Level 6 can use the quadratic formula to find all complex roots and interpret the fundamental theorem of algebra with respect to quadratic functions. They recognize that roots will be either real or complex.
Level 5: Thorough Conceptualization	Students at Level 5 are accomplished with common methods that may be applied to the problem of solving quadratic equations with real coefficients and complex solutions, including factoring over the integers, using the quadratic formula, finding roots graphically, and completing the square. They understand that if a quadratic cannot be factored over the integers, then the solutions may still be found by completing the square or using the quadratic formula. At Level 5, not only can students complete the square in a quadratic expression but they can use the result to reason about the maximum or minimum of the corresponding function without creating a sketch. Like students at Level 4, they can recognize both graphically and algebraically when a quadratic function will have non-real roots, use the quadratic formula to find all complex roots, and interpret the fundamental theorem of algebra with respect to quadratic functions.
Level 4: Synthesis	Students are accomplished with common methods for solving quadratic equations with rational coefficients and real solutions, such as factoring over the integers, using the quadratic formula and with completing the square. Procedural errors (apart from arithmetic errors) are rare at this level. Although they are accomplished with these methods, they may still perceive them as equivalent, which can result in confusion if they can find solutions using one method but not with another. They understand the correspondence between the graph of a function and the solution of the corresponding quadratic equation. They recognize that a quadratic equation has at most two real solutions and can relate the number and value of the solutions with respect to the graph (i.e., they can determine, both algebraically and graphically, whether the function has real roots and if so whether it has a double root). Though again, at this level, students may still see it as a contradiction if the graph of a function suggests that it has real roots that cannot be identified by factoring over the integers.
Level 3: Making Connections	Students are procedurally competent with common methods for solving quadratic equations with rational coefficients and with rational roots, such as factoring over the integers, using the quadratic formula and with finding roots graphically. They may need support to recognize quadratic equations if they are not expressed in one of the expected forms (e.g., if the x^2 term is not first). They still struggle with factoring more difficult equations (e.g., those with a leading coefficient other than 1 or with noninteger coefficients). Students at this level have been introduced to the zero product property and may be able to provide some explanation for it when asked but are still prone to overgeneralization; e.g., when asked to solve $(x-1)(x-1)=3$, the student may write "$x-1=3$ or $x-1=3$." Or, occasionally, the student may "lose a root" by dividing through by one of the factors (also an indication that the zero product property is not completely understood). They are starting to understand the correspondence between the graph of a function and the solution of the corresponding quadratic equation, but errors are expected in this regard. For example, given a quadratic function with real roots expressed in symbolic form, a student may solve the corresponding quadratic equation by factoring and assert that the function must be $y=(x-r)(x-s)$, not realizing that the roots alone are insufficient to specify the function. Although they can use the quadratic formula and complete the square (though this may still pose some difficulty), they do not yet understand that the quadratic formula results from completing the square on the generalized form of the equation.

Table 4 Continued

Level	Description
Level 2: Familiarization	Students at this level can solve simple quadratic equations by inspection; some students can also solve by using factoring methods (e.g., for quadratic equations with integer roots for which the coefficient of the squared term is 1 and for which the constant term has a small number of factors). The students may have more or less experience with other methods, depending on the sequence of instruction, though it is generally expected that completing the square is most difficult. Students at this level may have been introduced to the zero product property. Still, for those students who can solve by factoring methods, the connection to zero product is tenuous and procedural; for example, students may know that they can set each factor equal to zero but not understand why.
Level 1: Preinstruction	Students at this level may be able to solve very simple quadratic equations (e.g., $x^2 = 81$) by inspection. They may be more or less familiar with different methods for solving quadratic equations, depending on how instruction has been sequenced.

Table 5 Quadratic Functions Learning Progression: Use Algebraic Forms

Level	Description
Level 6: Drawing Extensions	As at Levels 4 and 5, students at this level are facile with the three algebraic forms of the quadratic function (standard, vertex, and factored over the real numbers) and can translate among the forms. At Level 6, students recognize a quadratic function as a special case of a polynomial expressed in the form $$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0,$$ where $a_n \neq 0$ and $n = 2$.
Level 5: Thorough Conceptualization	As at Level 4, students at this level are facile with the three algebraic forms of the quadratic function (standard, vertex, and factored over the real numbers) and can translate among the forms (they can, e.g., complete the square to translate the standard form to vertex form). Their understanding is not merely procedural: They can explain the advantages of different forms in different situations.
Level 4: Synthesis	Students at this level are facile with the three algebraic forms of the quadratic function (standard, vertex, and factored over the real numbers) and can translate among the forms (they can, e.g., complete the square to translate the standard form to vertex form). Their understanding is not merely procedural: They can explain the advantages of different forms in different situations.
Level 3: Making Connections	Students at this level use algebraic forms to evaluate; they are also attempting to express quadratic functions in algebraic form (using a graph or table of values), though errors are still expected because understanding of the necessary and sufficient conditions to define the quadratic function is not complete. Students at this level are starting to work with each of the three common symbolic forms (standard, vertex, and factored over the real numbers) and can follow procedures to translate a quadratic function from vertex form or factored form to standard form and from standard form to factored form. Because they can complete the square, they may be able to translate from standard form to vertex form, though this may still pose difficulty. Although they may be able to follow procedures for translating forms, understanding of the relationships among them is incomplete.
Level 2: Familiarization	Students at this level can use algebraic forms for evaluation, e.g., when given the symbolic representation, they can use it to construct a table of values or test whether a point belongs to a function (though the latter may be more difficult).
Level 1: Preinstruction	Students at this level are not expected to express quadratic functions in algebraic form or to translate among the forms; while they can probably use a symbolic representation of a form to evaluate $f(x)$, the meaning of $f(x)$ with respect to the corresponding graph is not completely understood.

Table 6 Quadratic Functions Learning Progression: Compare Using Different Representational Forms

Level	Description
Level 6: Drawing Extensions	As at Levels 4 and 5, students at this level are facile with the three algebraic forms of the quadratic function (standard, vertex, and factored over the real numbers) and can translate among the forms. At Level 6, students recognize a quadratic function as a special case of a polynomial expressed in the form $$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$ where $a_n \neq 0$ and $n = 2$.
Level 5: Thorough Conceptualization	As at Level 4, Students at this level are accomplished at comparing quadratic functions whether expressed in symbolic, graphical, or tabular form. Additionally, at Level 5, students can focus on comparing aspects of change for functions expressed in different forms. For example, in comparing the graph of an exponential function to a table for a quadratic function, a Level 4 student might graph the values from the table and note that the two graphs are not of the same type. A Level 5 student might be able to analyze the pattern of change evident in the table and to conclude that the two functions are different types without creating a graph. In general, Level 5 students may be less likely to convert different functions to a common representation before comparing them, because they have internalized their corresponding features.
Level 4: Synthesis	Students at this level are accomplished at comparing quadratic functions whether expressed in symbolic, graphical, or tabular form.
Level 3: Making Connections	At this level, students attempt to compare quadratic functions expressed in different forms (e.g., symbolic, graphical, in a table of values), but errors are expected. Understanding of necessary and sufficient criteria to define a quadratic function is incomplete, resulting in difficulties in making such comparisons. Also, students may still attend to surface characteristics of graphs and equations and attempt to relate them. For example, given the equation $y = x^2 - 4x + 3$, the student may identify the vertex as $(-4, 3)$, even if he or she has learned the procedures to translate from standard to vertex form. Asked to translate the graph of $y = (x - 3)^2 - 2$ to symbolic form, a student may write $y = x^2 - 3x - 2$.
Level 2: Familiarization	Students at this level are not expected to compare quadratic functions using different representational forms.
Level 1: Preinstruction	Students at this level are not expected to compare quadratic functions using different representational forms.

It is important to note, however, that tasks in which students use quadratic regression or other means to fit a quadratic function to a data set have a different flavor from tasks in which a quadratic function is given. The most important distinction is just that: On one hand are tasks in which the student is presented with a specific quadratic function (either a pure function or a function that models a real-life situation), while on the other hand are tasks in which the student is presented with a set of data and is asked to find the quadratic function that best models the data, using the quadratic regression feature of a graphing calculator or other technology.

Another difference between the two types of tasks—tasks in which the quadratic function is given versus tasks in which a quadratic regression is fitted to a data set—is that the first one offers the task designer several options for choosing a path through the different possible representations of a quadratic function—symbolic, numeric, and graphic—with these different paths representing different cognitive skills and different levels in the LP, while the mathematical requirements inherent in regression analysis prescribe a fixed path; this point is discussed at length later.

Finally, a third difference between the two types of tasks results from the possibly differing nature of the quadratic functions presented. Tasks of the first type can present a quadratic function in either a pure context or as a model of a real-life situation. In either case, the function is meaningful for all real-number arguments, or at least for all real numbers for which the value of the quadratic function is positive or otherwise satisfies a constraint imposed by the real-life situation and not by the model itself. Tasks using quadratic regression, however, present quadratic functions that are derived from a limited set of data and may only be applicable for the limited range of the data. Although the quadratic function can be used to interpolate values within the range of data, it may not be meaningful to extrapolate values outside the range of data. Thus, whereas tasks of the first type can include questions concerning the global nature of the quadratic function, some

Table 7 Quadratic Functions Learning Progression: Understand the Role of Parameters

Level	Description
Level 6: Drawing Extensions	Identical to Level 5
Level 5: Thorough Conceptualization	Students at this level understand the role of parameters in all three forms and can explain the role of the *b* parameter in the standard form following an exploration with an animated graph. Students at Level 5 can express parameters in one algebraic form in terms of parameters from another algebraic form.
Level 4: Synthesis	Students at this level understand the role of all parameters in all three forms, except for the role of the *b* parameter in the standard form; however, they are ready to learn about its role via exploration with animated graphs.
Level 3: Making Connections	Students at this level understand the role of the *c* parameter in the standard form. They also know that the sign of the *a* parameter determines whether the graph opens upward or downward and have a general sense that increasing the magnitude of the *a* parameter "narrows" the graph. They still have confusion about the *a* parameter, however. Students may not make a distinction between the role of the *a* parameter when working with the vertex form vs. the standard form. For example, when working with the standard form, students may predict that changing the *a* parameter will not change the vertex (possibly due to confusion with the role of the *a* parameter in the vertex form or overgeneralization from examples in the standard form where the *b* parameter is equal to zero and changing the *a* parameter does not change the vertex). Some students may attempt to find the value of the *a* parameter by computing the linear slope between two points on a quadratic function. The role of the *b* parameter is not understood. Students at this level understand the connection between the vertex form and the coordinates of the vertex.
Level 2: Familiarization	Students at this level partially understand the role of the *c* parameter in the standard form, though they may assert that it does not exist when $c=0$. They may have learned that the sign of the *a* parameter (when given the standard, vertex, or factored form) indicates whether the graph opens upward or downward but are unlikely to understand the role of its magnitude. The role of the *b* parameter in the standard form is not understood. Students may understand that *h* and *k* in the vertex form are related to the vertex but may make sign errors in trying to identify its coordinates.
Level 1: Preinstruction	Students at this level are not expected to understand the role of parameters.

tasks of the second type may not be able to ask such questions. For these reasons, two task shells for quadratic functions are presented: the quadratic functions task shell and the quadratic regression task shell.

History and General Description of the Quadratic Functions Task Shell

The quadratic functions task shell has antecedents in two CBAL tasks: Put the Brakes On and Moving Sidewalks — 1 Rider. Put the Brakes On starts with a graph, Braking Distance, that shows the total distance a car moves after the brakes are applied (see Figure 2). Although the task does not require it, the first 5 s of the Braking Distance graph can be modeled by a quadratic function (after which it can be modeled by a constant function). Students are asked questions that require reading the graph (including an explanation for how the graph was used to answer the questions). Next, students are shown the corresponding speed versus time graph, Speed During Braking, over the same time interval (see Figure 3).

The first 5 seconds of the Speed During Braking graph from the Put the Brakes On task can be modeled by a decreasing linear function. Students are asked to answer questions that require reading the graph, providing a written rule characterizing the graph, reasoning about the slope, and providing an equation using *t* and *s* (time and speed) to find the speed in the interval from $t=0$ to $t=5$. Note that the fact that the Speed During Braking graph is linear implies that the function represented in the Braking Distance graph has constant second differences and hence is a quadratic function.

The task then returns to the Braking Distance graph, in which students are asked questions to help them observe its nonlinear properties. (For example, they are asked to find first differences.) Finally, students are asked to relate the two graphs (Braking Distance and Speed During Braking). Although quadratic functions are not explicitly mentioned and students do not need to provide an equation for the quadratic function, they are beginning to observe its nonlinear

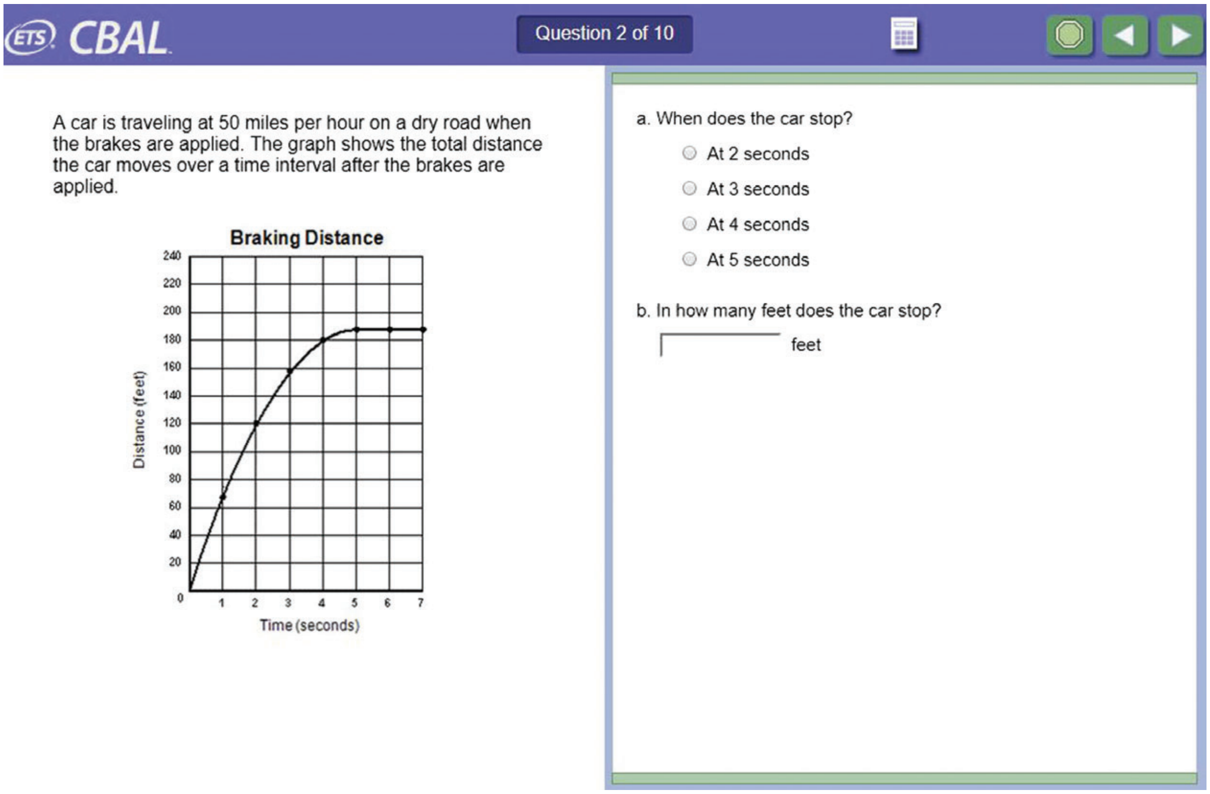


Figure 2 Braking Distance graph from Put the Brakes On task.

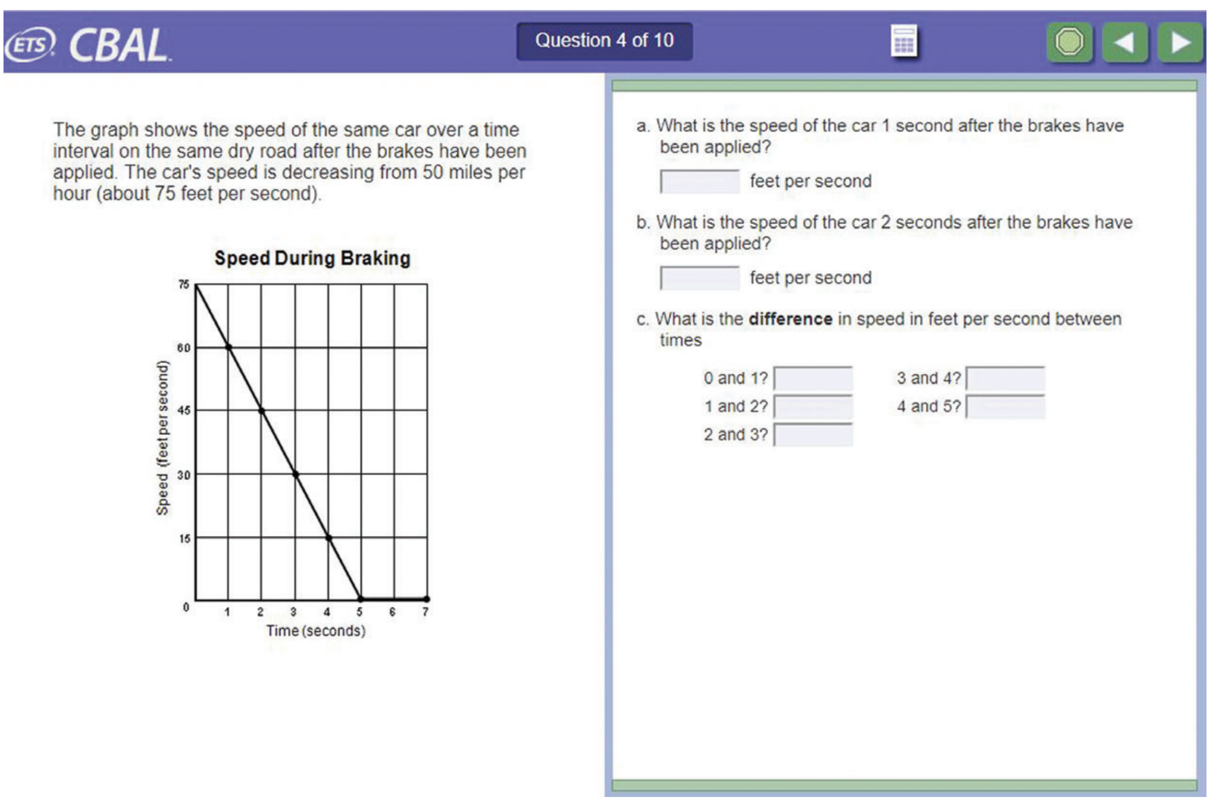


Figure 3 Speed During Braking graph from Put the Brakes On task.

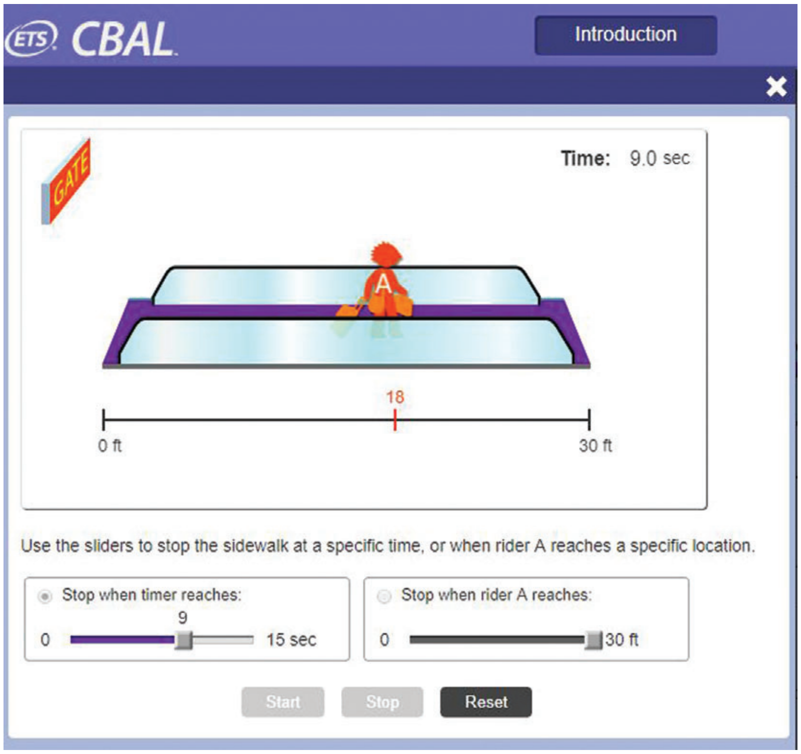


Figure 4 Simulation from the Moving Sidewalks—1 Rider task.

properties in an informal way and are being asked to consider the relationship between a quadratic function (Braking Distance) and a linear function (Speed During Braking).

Put the Brakes On has the potential to be extended to the high school level by asking students to explicitly model the quadratic function. Instead of starting with the graph, it could start with a table of values (total distance vs. time) or even a simulation that shows a car braking and slowing to a stop. Time and distance would be indicated. Students could be asked to plot the values from the table on a graph and to fit a quadratic function to the graph. Depending on the types of skills we want to assess, we could ask them to do this algebraically or by using software. (a graphing calculator can do this, as can Excel; curve-fitting functionality could be incorporated into the task itself).

In the existing Moving Sidewalks—1 Rider task, a rider moves away from a gate along a sidewalk at a constant speed. The student is asked to model the rider's distance from the gate using tables, graphs, and equations. The situation can be modeled by a linear function, and the student has access to an interactive simulation that can help draw this conclusion. On any run of the simulation, the student can indicate whether the rider should stop at a particular distance from the gate or at a particular time on the clock, and then both are displayed (see Figure 4).

This is a simplified situation. Some high-speed moving sidewalks have an acceleration phase at the beginning and a deceleration phase at the end. The Pearson International Airport in Toronto has such a moving sidewalk. If it can be assumed that the acceleration and deceleration are constant, then the position versus time graphs during the acceleration and deceleration phases can be modeled using quadratic functions. (We might call these tasks Accelerating Sidewalk and Decelerating Sidewalk.) The existing Moving Sidewalks—1 Rider simulation would need to be adapted to accommodate changes in speed. Note that the same general engine (one that allows for acceleration/deceleration at a constant rate) could be used for the quadratic versions of Braking Distance, Moving Sidewalks—1 Rider, and other tasks involving falling bodies or projectiles.

Mathematical Characterization of the Quadratic Functions Task Shell

The tasks covered by this shell could be pure mathematics problems or they could be tasks that use quadratic functions to model authentic and engaging real-life situations. Examples of the latter include the law of falling bodies, braking

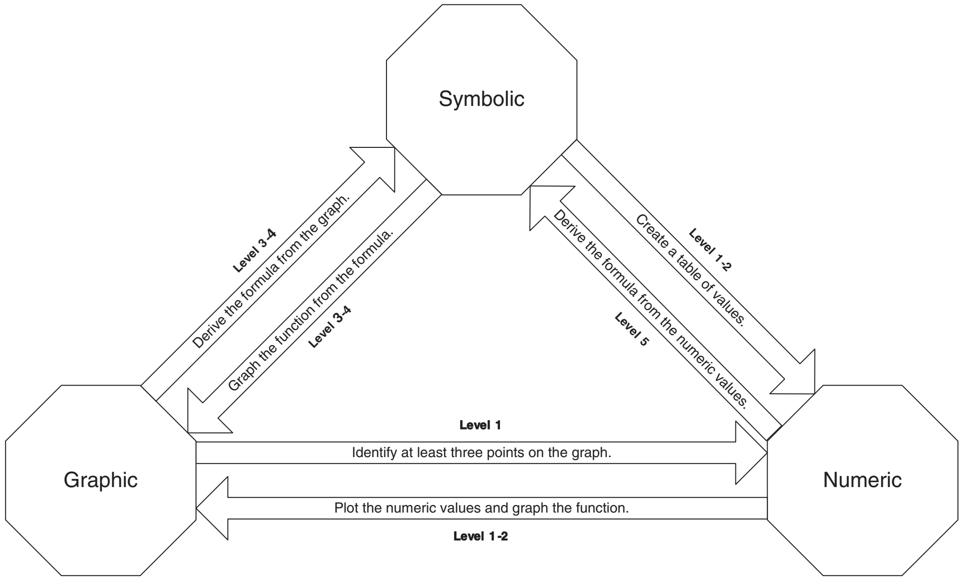


Figure 5 Moving between the three representations of a quadratic function.

distance, profit/loss in a competitive market, and heights that snowboarders can reach doing tricks in a half-pipe. The examinee is presented with either a specific pure quadratic function or a specific quadratic model of a real-life situation and is asked to find and interpret critical features of the function. The function can be presented symbolically (e.g., $f(x) = ax^2 + bx + c$ for specific real numbers a, b, and c). For example, the quadratic function for Braking Distance can be expressed symbolically as $f(x) = -a(x - x_0)^2 + b(x - x_0)$, or $f(x) = -ax^2 + bx$ when $x_0 = 0$, where a is half the acceleration constant and b is the initial velocity. The car stops moving at time x_1, so x_1 is chosen so that $-2a(x_1 - x_0) + b = 0$; that is,

$$x_1 = \frac{b}{2a} + x_0.$$

The Accelerating Sidewalk version of Moving Sidewalks—1 Rider can be modeled by the quadratic function $f(x) = a(x - x_0)^2$, where a is positive.

Alternatively, the function can be presented numerically (i.e., with a table of values) or graphically. The examinee can then be asked to derive the other two representations. For each of the three possible initial presentations of the function, there are two possible paths to deriving the other two representations, as illustrated in Figure 5 and Table 8. These various paths represent different cognitive skills and different levels in the compare using different representational forms progress variable of the LP, as indicated in the diagram. Some are quite easy (such as plotting points from a table) or can be easily accomplished using technology (e.g., producing the graph from the formula); others, like deriving the formula algebraically either from the graph or from three points, are quite difficult and may represent tasks that belong to the most advanced levels of the LP.

When these tasks are presented in a formative setting, examinees can check their work with a graphing calculator if the calculator was not used in the initial response to the tasks. For pure problems, the examinee should choose the viewing window that shows all the interesting behaviors of the parabola (especially the vertex of the parabola and any real roots); for real problems, the examinee should choose a viewing window that applies to the real-life situation, although later, the examinee may be asked to extend the graph to include the vertex and the roots, if any.

After obtaining these alternate representations of the function, the test taker can be asked questions like "What is $f(x)$ when $x = \_$?" and "What is x when $f(x) = \_$?" For real-life models, these questions would be asked in terms of the real-life situation, for example, How high is the snowboarder after 3 seconds? When will the snowboarder first reach a height of 15 feet?

To answer the first type of question, examinees should be required to evaluate and find the answer from a graph. Answering the second question may involve solving a quadratic equation; how this is done depends on the complexity of the equation. It may be possible to factor the equation, but there will be tasks for which completing the square is necessary.

Table 8 Moving Between the Three Representations of a Quadratic Function

The examinee is presented with this representation	The examinee is given this task	Then the examinee is given this task
Symbolic ($y=ax^2+bx+c$)	Create a table of values. This can be done by hand or using technology.	Plot the values in the table, and then sketch the graph of the function by hand.
	Graph the function. This can be done algebraically or using technology. To determine the graph algebraically, derive the vertex form of the equation ($y=a(x-h)^2+k$). The vertex of the parabola is at the point (h, k), and the sign of a determines whether the parabola opens upward or downward. Also, the c parameter in the standard form of the equation determines the y-intercept.	Identify some points on the graph.
Numeric (table of values)	Plot the values in the table and then sketch the graph of the function by hand.	Use the graph to derive the formula for the function. From the graph, determine the vertex (h, k) and the y-intercept c. The equation is $y=a(x-h)^2+k$, where $a=(c-k)/h^2$.
	Use three values from the table to determine the formula for the function. This can be done algebraically (solving a system of three equations in the three unknowns a, b, and c) or using technology, using the quadratic regression feature, which in this case will yield the exact quadratic that fits the three points.	Graph the function. This can be done algebraically or using technology. To determine the graph algebraically, derive the vertex form of the equation ($y=a(x-h)^2+k$). The vertex of the parabola is at the point (h, k) and the sign of a determines whether the parabola opens up or down. Also, the c parameter in the standard form of the equation determines the y-intercept.
Graphic	Identify at least three points on the graph.	Use the three points to determine the formula for the function. This can be done algebraically (solving a system of three equations in the three unknowns a, b, and c), or using technology, using the quadratic regression feature, which in this case will yield the exact quadratic that fits the three points.
	Use the graph to derive the formula for the function. From the graph, determine the vertex (h, k) and the y-intercept c. The equation is $y=a(x-h)^2+k$, where $a=(c-k)/h^2$.	Create a table of values. This can be done by hand or using technology.

In particular, real-life models should not be rigged to avoid equations that require completing the square. Examinees should also solve the equations graphically on a graphing calculator.

Examinees can be asked to identify the roots of the functions and the vertex of the parabola, indicating if the vertex is a maximum or a minimum of the function, extending the graph if necessary. They should also identify the axis of symmetry. For real-life problems, they should be asked these questions in terms of the real-life situation.

Finally, examinees should look at the rate of change between various pairs of points x and $x+\Delta x$ for fixed Δx. These values can be displayed in a table, and examinees should be asked to describe the rate of change $f(x+\Delta x)-f(x)$ as a function of x. They can be asked to compare this situation with the situation when $f(x)$ is linear or exponential.

Piecewise-Defined Functions

It is also possible to consider real-life situations that are modeled by piecewise-defined functions that combine quadratic and linear functions. The accelerating and decelerating version of Moving Sidewalks—1 Rider is such an example. In this example, there are four time points: x_0, the start of the rider's acceleration period (most likely $x_0 = 0$); x_1, when the rider transitions from the acceleration zone to the constant velocity zone; x_2, when the rider transitions from the constant velocity zone to the deceleration zone; and x_3, when the rider stops.

The function for distance with respect to time for an accelerating sidewalk (when the rider is in the acceleration zone, i.e., when $x_0 \le x \le x_1$) is quadratic and can be expressed in the form $y = a(x - x_0)^2$, where a is positive.

When the rider is in the constant velocity zone (i.e., when $x_1 \le x \le x_2$), the function for distance with respect to time is linear, with slope m equal to the derivative of the previous quadratic function when $x = x_1$; that is, $m = 2a(x_1 - x_0)$. It follows that this function is $y = 2a(x_1 - x_0)(x - x_1) + y_1$, where y_1 is the value of the previous quadratic function at $x = x_1$. Therefore $y = 2a(x_1 - x_0)(x - x_1) + a(x_1 - x_0)^2$.

When the rider is in the deceleration zone (when $x_2 \le x \le x_3$), the function is quadratic. Because we are assuming that the constant deceleration is the negative of the constant acceleration in the acceleration zone, the function for distance with respect to time in the deceleration phase can be written in the form $y = -a(x - x_3)^2 + y_3$. By symmetry, $x_3 - x_2 = x_1 - x_0$ and $y_3 - y_2 = y_1 - y_0$. It follows that

$$\begin{aligned} y &= -a\left(x - \left(x_2 + x_1 - x_0\right)\right)^2 + 2a\left(x_1 - x_0\right)\left(x_2 - x_1\right) + 2a\left(x_1 - x_0\right)^2 \\ &= -ax^2 + 2a\left(x_2 + x_1 - x_0\right)x - a\left(x_2^2 + x_1^2 - x_0^2\right). \end{aligned}$$

Note that the coefficient of the x^2 term must be negative because a is positive by definition, the coefficient of the linear term must be positive because a is positive and $x_2 + x_1 - x_0$ is positive, and the coefficient of the constant term must be negative because a is positive and $x_2^2 + x_1^2 - x_0^2$ is positive.

The Task Shell, the CBAL Competency Model, and the Common Core Standards

Table 9 shows how the items in this task shell assess the relevant CBAL competencies and Common Core standards.

Example of a Task Based on the Quadratic Functions Task Shell

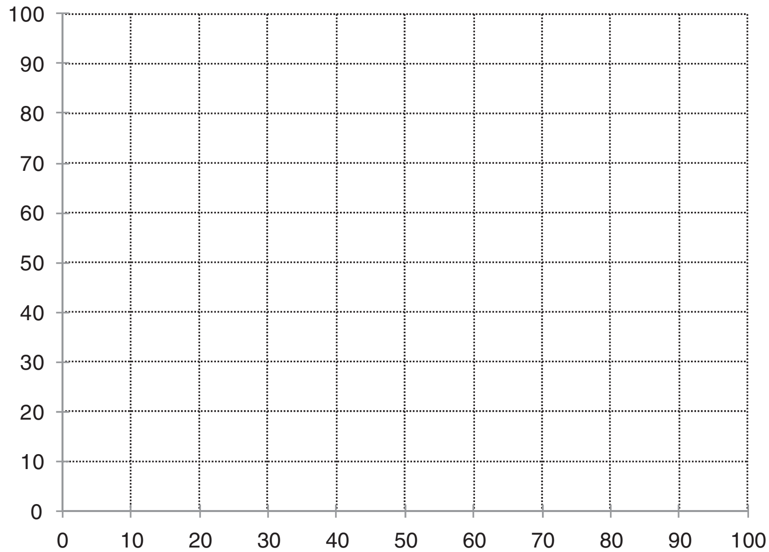
To illustrate how a task can be developed from the quadratic functions task shell, following is an example, based on Fey and Heid (1995):

> For a car with a mass of 1,000 kilograms, traveling on a dry asphalt road over a flat surface, the rule $f(x) = 0.005x^2 + 0.14x$ can be used to predict stopping distance $f(x)$ in meters, where x is the speed in kilometers per hour.

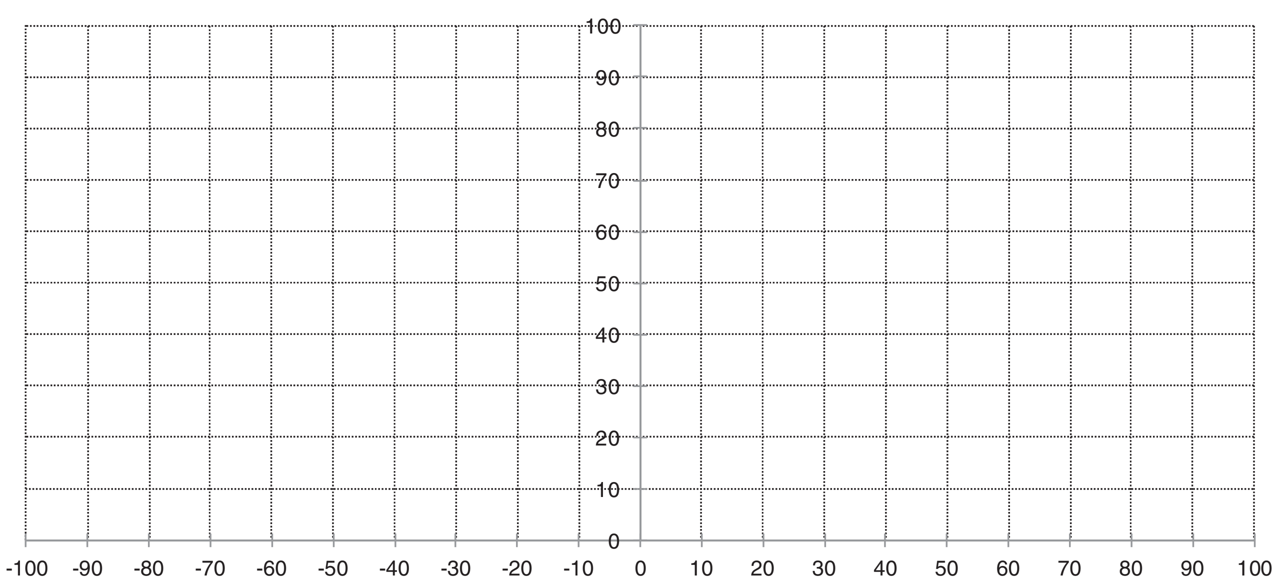
1. Complete the following table. Some values have been filled in for you:

Speed (in kilometers per hour)	**Stopping distance (in meters)**
10	1.9
20	4.8
30	
40	13.6
50	19.5
60	
70	34.3
80	43.2
90	
100	64

2 On the grid below, plot the graph of $y = f(x)$.



3. What is the stopping distance when the car is traveling at a speed of 80 kilometers per hour?
4. Approximately how fast can the car travel and be able to stop in 20 meters?
5. You want to know how fast the car can travel and be able to stop in 100 meters. Write an equation of the form $ax^2 + bx + c = 0$ that you can solve to answer your question.
6. If x is the top speed the car can travel and stop in 100 meters, then x satisfies the $0.005x^2 + 0.14x - 100 = 0$. To solve this equation, first divide both sides by 0.005.
7. You should now have the equation $x^2 + 28x - 20,000 = 0$.
    a. By completing the square, find the solutions to this equation. Round your answers to the nearest whole number.
    b. How many solutions are there?
    c. Which solution represents the top speed at which the car can stop in 100 meters?
8. Extend the graph of $y = 0.005x^2 + 0.14x$ so that $-100 \le x \le 100$.



a. What is the vertex of the parabola?
b. Is the vertex a maximum or a minimum?
c. What feature of the equation of the parabola tells you whether the vertex will be a maximum or a minimum?
d. What is the axis of symmetry of the parabola?

9. When the speed of the car increases by 10 kilometers per hour, how much does the stopping distance increase? Complete the values in the following table:

x = speed	y = stopping distance	increase in y
10	1.9	
20	4.8	$4.8 - 1.9 = 2.9$
30	8.7	$8.7 - 4.8 = 3.9$
40	13.6	
50	19.5	
60	26.4	
70	34.3	
80	43.2	
90	53.1	
100	64.0	

10. As x increases, how would you describe the increase of the increase in y? That is, if you let Δy be the increase in y in the third column of the table in Question 8, which of the following best describes Δy as a function of x?
a. Δy is a linear function of x.
b. Δy is a quadratic function of x.
c. Δy is an exponential function of x.
d. Δy is a function of x that is not linear, quadratic, or exponential.

General Description of the Quadratic Regression Task Shell

This task shell has many points of intersection with the first task shell; in both, the student is asked to analyze a quadratic function, obtain various representations of the function, and find and interpret critical features of the function. But as stated earlier, there are several differences between the two shells. The most important distinction is that in tasks based on the quadratic functions task shell, the student is presented with a specific quadratic function (either a pure function or a function that models a real-life situation), while in tasks based on the quadratic regression task shell, the student is presented with a set of data and is asked to find the quadratic function that best models the data, using the quadratic regression feature of a graphing calculator or other technology.

Another difference between the shells is that the first one offers the task designer several options for choosing a path through the different possible representations of a quadratic function—symbolic, numeric, and graphic—with these different paths representing different cognitive skills and different levels in the LP, while the mathematical requirements inherent in regression analysis prescribe a fixed path; see Figure 6 and compare with the quadratic functions task shell (see Figure 5 and Table 8).

A third difference between the two task shells results from the possibly differing nature of the quadratic functions presented in instances of the shells. As stated earlier, tasks generated from the quadratic functions task shell can present a quadratic function in either a pure context or as a model of a real-life situation. In either case, the function is meaningful for all real-number arguments, or at least for all real numbers for which the value of the quadratic function is positive or otherwise satisfies a constraint imposed by the real-life situation and not by the model itself. Tasks generated from the quadratic regression task shell, however, present quadratic functions that are derived from a limited set of data and may only be applicable for the limited range of the data. Although the quadratic function can be used to interpolate values within the range of data, it may not be meaningful to extrapolate values outside the range of data. Thus, although tasks generated from the quadratic functions task shell can include questions concerning the global nature of the quadratic function, some tasks generated from the quadratic regression task shell may not be able to ask such questions. In fact, it might be appropriate to include questions that assess the student's understanding of the inappropriateness of extrapolation.

Table 9 CBAL Competencies and CCSSM Standards Assessed, by Question Type

Question type	CBAL competency	Common Core standard
Student answers several questions about the simulation, e.g., after so much time has elapsed, what is the displacement, how much displacement at such-and such time?	Interpret Representational Devices	
Student creates/completes a table with columns for time and displacement.	Create/Translate Representational Devices	
Student creates a graph based on a table.	Create/Translate Representational Devices	
Student is asked to describe features of the graph (where the function is increasing, decreasing, etc.).	Interpret Representational Devices Use, Graph, and Reason about Linear Functions and Some Nonlinear Functions	"For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given in a verbal description of the relationship." (CCSSI, 2010, Standard F-IF 4, p. 69)
Student is asked to identify the subdomain for which the graph is nonlinear.	Interpret Representational Devices Use, Graph, and Reason about Linear Functions and Some Nonlinear Functions	"Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes." (CCSSI, 2010, Standard F-IF 5, p. 69) "Construct and compare linear, quadratic, and exponential models and solve problems." (CCSSI, 2010, Standard F-LE, p. 70)
Student is told that the nonlinear piece of graph can be modeled by a quadratic function. Student is asked to fit a function and find an equation for it (manually or using technology).	Abstract to Models Translate Representational Devices Use and Understand Algebraic Notation	"Construct and compare linear, quadratic, and exponential models and solve problems." (CCSSI, 2010, Standard F-LE, p. 70)
Student is told that the nonlinear piece of graph can be modeled by a quadratic function. Student is asked to use values from the table to find an algebraic equation for the function.	Abstract to Models Translate Representational Devices Use and Understand Algebraic Notation Generate an Equation	"Construct and compare linear, quadratic, and exponential models and solve problems." (CCSSI, 2010, Standard F-LE, p. 70)
Student is asked to express the equation in vertex form.	Translate Representational Devices Use and Understand Algebraic Notation Generate an Equation	"Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function." (CCSSI, 2010, Standard F-IF 8, p. 69)
Student is asked to find zeros of the function (algebraically) and compare this to what is evident from the graph (and explain any differences).	Interpret Representational Devices Use and Understand Algebraic Notation Generate an Equation Solve Equations and Inequalities and Simplify Equations	"Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation.." (from CCSSI, 2010, Standard A-REI 4b, p. 65) "Factor a quadratic expression to reveal the zeros of the function it defines." (CCSSI, 2010, Standard A-SSE 3a, p. 64) "Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions*." (CCSSI, 2010, Standard A-CED 1, p. 65) "Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context." (CCSSI, 2010, Standard F-IF 8a, p. 69)

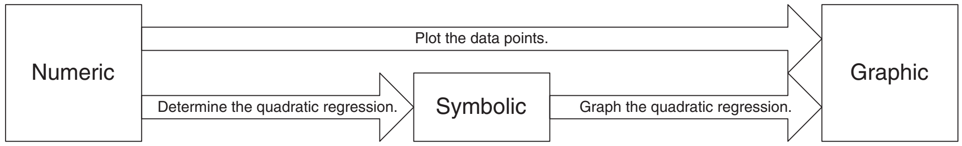


Figure 6 Moving between the representations of a quadratic function.

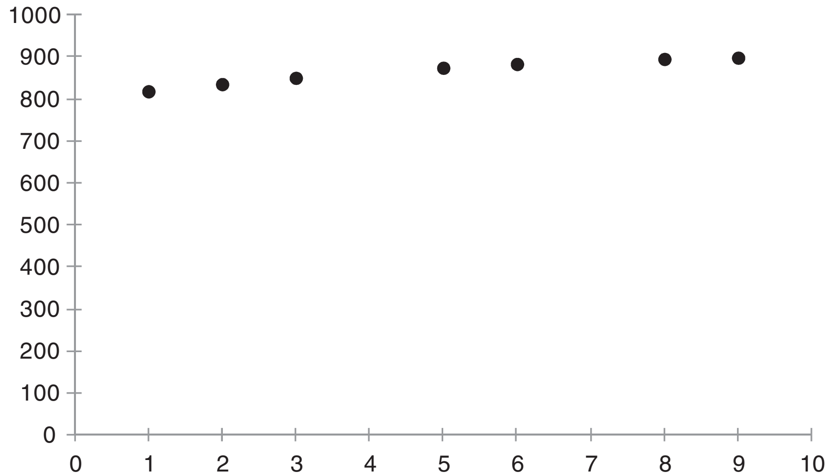


Figure 7 Near-linear data.

Because of its use of regression analysis to generate the function that models the real-life situation, the quadratic regression task shell is the descendent of CBAL tasks that ask the student to construct linear models that best fit data; examples of such tasks include Dams and Droughts and Heights and Growth. These tasks do not use technology to calculate the linear models, however. Instead, they ask the student to draw (in a graph editor) the graph of a line that provides a good model of the data. Later, they give the student the equation of the line of best fit. For quadratic functions, the graphical approach would involve more complex manipulation of the graphic images—it would be necessary for the student to move three points around instead of just two. Because we may want the students to use technology to analyze the quadratic models, it seems reasonable for the students to use technology to generate the models.

Mathematical Characterization of the Quadratic Regression Task Shell

In the tasks covered by this shell, the student is given a set of data points that can be modeled by a quadratic function. The student is asked to plot the points in a graphing editor and to guess if the data would be best modeled by a linear function, a quadratic function, or an exponential function. It may be obvious that a quadratic function is best, in which case, the student would be expected to guess correctly and would then be asked to use technology to determine the quadratic function that best models the data. But if the data are close to being linear, it may not be visually obvious which function is best, and students may need to examine all three models (linear, quadratic, and exponential) and determine the best. For example, the data in Figure 7 nearly fall in a line and are approximated well by a linear function. But in fact the data fall exactly on the quadratic function $y = -x^2 + 20x + 800$. Of course, for these data points, a linear model would be easier to work with than a quadratic model and may be a better choice. But the point we are making here is that it may not be obvious by looking at the data what the best model is, and students may need to generate several models and then choose, based on goodness of fit and other considerations.

The student could then be asked to use technology to graph the quadratic function. The graph itself will not be scored. The student can enter the formula for the quadratic function in a graphing calculator and let the calculator generate the graph; or the student can plot points in Excel or another spreadsheet program, using the quadratic function to calculate

the y-values; or the student can use the trend functionality in Excel to generate the graph. Subsequent questions will be answered (and scored) based on the graph the student generates.

Alternatively, the correct quadratic function and its graph can be given to the student and used to answer subsequent questions. This eliminates the need for conditional scoring. Conditional scoring itself is not difficult; the problems arise when the student's errors on previous questions affect the difficulty of subsequent questions.

The test taker can then be asked questions that ask him or her to analyze the quadratic function by determining its roots, its maximum or minimum, its regions of increase and decrease, and so on. These questions are similar to those asked in tasks generated from the task shell for quadratic functions but would be limited to those that are relevant to the real-life situation.

Example of a Task Based on the Quadratic Regression Task Shell

Finally, we present a task developed from the quadratic regression task shell, based on data from Ron Lewis-Smith (as cited in Brown & Rothery, 1993, p. 127):

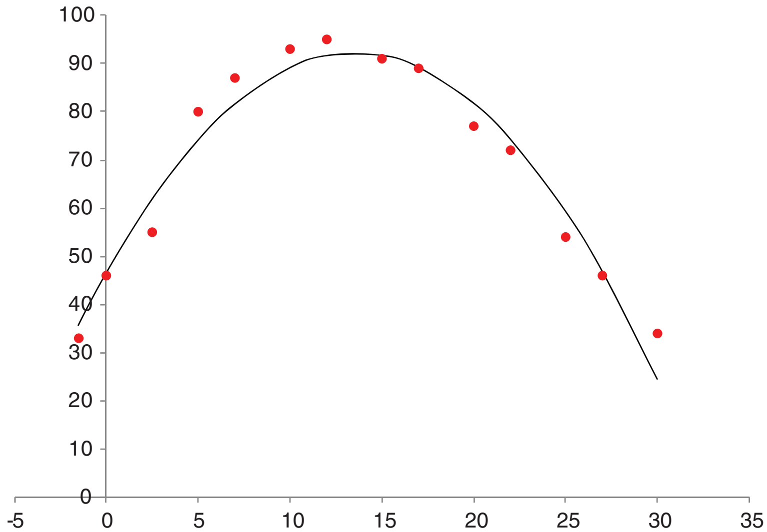
> Photosynthesis is the process by which plants convert light energy to chemical energy. The efficiency of the photosynthesis is the percentage of the light energy that is converted to chemical energy. Photosynthetic efficiency can depend on several factors, including the ambient temperature. The table below shows the photosynthetic efficiency of a certain Antarctic species of grass at various temperatures. The objective is to determine the temperature at which photosynthesis for this species of grass is most efficient.

Efficiency of Photosynthesis at Various Temperatures

Temperature (°C)	Efficiency (%)
−1.5	33
0	46
2.5	55
5	80
7	87
10	93
12	95
15	91
17	89
20	77
22	72
25	54
27	46
30	34

1. Plot the data in the table on the xy-coordinate plane, where x is the temperature in degrees Celsius and y is the efficiency.
2. You would like to model these data with a function $y = f(x)$. Which of the following types of functions do you think would make the best model?
   a. A linear function
   b. A quadratic function
   c. An exponential function
3. Use your graphing calculator to find the quadratic function $y = ax^2 + bx + c$ that best fits the data. Indicate below the values of a, b, and c. Round your answers to two decimal places:
   $a =$ ___________
   $b =$ ___________
   $c =$ ___________

In the figure below, the points in the table have been plotted along with the quadratic function $y = -0.25x^2 + 6.77x + 46.37$. This is the quadratic function that best fits the data.



Efficiency of photosynthesis at various temperatures.

Efficiency of photosynthesis at various temperatures.

4. According to this model, at what temperature is photosynthesis most efficient in Antarctic grass? Use your graphing calculator to find the value of x at which this quadratic function has a maximum. Write your answer below; round your answer to two decimal places. Photosynthesis is most efficient when the temperature is __________°C.
5. According to this model, what is the efficiency of photosynthesis at the temperature at which it is most efficient? The efficiency of photosynthesis at the temperature at which it is most efficient is __________%.
6. Use completing the square to write the equation $y = -0.25x^2 + 6.77x + 46.37$ in the vertex form $y = a(x - h)^2 + k$. The value of the coefficient a is −0.25; what are the values of h and k?
   $h =$ __________
   $k =$ __________
7. What is the relationship between h and your answers to previous questions?
8. What is the relationship between k and your answers to previous questions?

Summary and Next Steps

On the basis of a review of the existing research and the CCSSM standards, we identified seven progress variables on which to base an LP for quadratic functions and quadratic equations:

1. Compare quadratic functions to other functions.
2. Model situations using quadratic functions.
3. Graph quadratic functions and characterize properties of quadratic functions based on their graphs.
4. Solve quadratic equations using a variety of methods.
5. Use algebraic forms.
6. Compare quadratic functions to each other using different representational forms.
7. Understand the role of parameters and the relationship between parameters and families of quadratic functions.

We crossed these seven progress variables with six levels of learning:

Level 1: Preinstruction
Level 2: Familiarization
Level 3: Making connections
Level 4: Synthesis

Level 5: Thorough conceptualization

Level 6: Drawing extensions

The task shells for quadratic functions grew naturally out of the LP, the CCSSM, and the CBAL mathematics competency model. For various reasons, it seemed appropriate to write two separate task shells. Although there is significant overlap between the two shells, the salient feature in which they differ is that in one of the models, the quadratic function is assumed, whereas in the other shell, the student must construct the model from data using a quadratic regression.

A theme throughout the CCSSM is the idea of looking at linear, quadratic, and exponential models to see which is more appropriate in particular situations. Although our CBAL task shells and tasks do not assume familiarity with exponential functions and models, we do ask students to assess linear and quadratic models and determine which is better for various data sets. For the most part, these determinations are made empirically, by observing which model seems to better fit the data. As their understanding develops, however, students can make judgments based on theoretical grounds. Though purely empirical approaches have their place, advanced students will have internalized schemas that include some sort of explanatory mechanism.

Finally, advanced students will have a complete understanding of the relationship between the roots of a quadratic equation and the factors of the associated quadratic function, even when the roots are complex. One of the advanced standards in the CCSSM states that students should know the fundamental theorem of Algebra but does not explain what it means to "know" the theorem. We assume it means that advanced students understand the relationship between roots and factors; understand that knowing the roots of a polynomial enables knowing the factors of the polynomial; and understand that, up to multiplicity, a polynomial has as many roots (and therefore factors) as its degree.

As a next step, the task shells might be augmented to include generalized rubrics that specify how different student responses correspond to levels of understanding from the LP.

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Appendix

Recall that if f is an arbitrary real-valued real function, the sequence $\{a_n\}$ is defined by $a_n = f(n)$, the sequence $\{b_n\}$ is defined by $b_n = a_{n+1} - a_n$, and the sequence $\{c_n\}$ is defined by $c_n = b_{n+1} - b_n$. The sequence $\{b_n\}$ is called the sequence of *first differences*, and the sequence $\{c_n\}$ is called the sequence of *second differences*. For example, if f is the quadratic function $f(x) = \frac{1}{2}x^2 + \frac{1}{2}x$, then the sequence $a_1,\ a_2,\ a_3,\ \dots$ is the sequence of triangular numbers 1, 3, 6, . . . , the sequence of first differences $b_1,\ b_2,\ b_3,\ \dots$ is the sequence 2, 3, 4,. .., and the second differences $c_1,\ c_2,\ c_3,\ \dots$ are all equal to 1. See Table A1. More generally,

$$a_n = \frac{1}{2}n^2 + \frac{1}{2}n,$$

$$b_n = a_{n+1} - a_n = \left(\frac{1}{2}(n+1)^2 + \frac{1}{2}(n+1)\right) - \left(\frac{1}{2}n^2 + \frac{1}{2}n\right) = n + 1,$$

$$c_n = b_{n+1} - b_n = (n+2) - (n+1) = 1.$$

Table A1 Sequences of First and Second Differences for the Triangular Numbers

n	$a_n = f(n)$	$b_n = a_{n+1} - a_n$	$c_n = b_{n+1} - b_n$
1	$\frac{1}{2} \times 1^2 + \frac{1}{2} \times 1 = 1$	$3 - 1 = 2$	$3 - 2 = 1$
2	$\frac{1}{2} \times 2^2 + \frac{1}{2} \times 2 = 3$	$6 - 3 = 3$	$4 - 3 = 1$
3	$\frac{1}{2} \times 3^2 + \frac{1}{2} \times 3 = 6$	$10 - 6 = 4$	$5 - 4 = 1$
4	$\frac{1}{2} \times 4^2 + \frac{1}{2} \times 4 = 10$	$15 - 10 = 5$	$\vdots$
5	$\frac{1}{2} \times 5^2 + \frac{1}{2} \times 5 = 15$	$\vdots$	
$\vdots$	$\vdots$		

Table A2 Generating a Quadratic Function From Constant Second Differences

n	$c_n = d$	$b_n = b_{n-1} + c_{n-1}$	$a_n = a_{n-1} + b_{n-1}$
1	d	e	g
2	d	$e + d$	$g + e$
3	d	$e + 2d$	$(g + e) + (e + d) = g + 2e + d$
4	d	$e + 3d$	$(g + 2e + d) + (e + 2d) = g + 3e + 3d$
5	d	$e + 4d$	$(g + 3e + 3d) + (e + 3d) = g + 4e + 6d$
6	d	$e + 5d$	$(g + 4e + 6d) + (e + 4d) = g + 5e + 10d$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	d	$e + (n-1)d$	$g + (n-1)e + \left(\sum_{i=1}^{n-2} i\right) d$ $= g + (n-1)e + \frac{1}{2}(n-1)(n-2)d$ $= \frac{1}{2}dn^2 + \left(e - \frac{3}{2}d\right)n + \left(g - e + d\right)$

For the general quadratic function $f(x) = ax^2 + bx + c$, we have $a_n = an^2 + bn + c$. A simple calculation then shows that

$$\begin{aligned} b_n &= a_{n+1} - a_n \\ &= \left(a(n+1)^2 + b(n+1) + c\right) - \left(an^2 + bn + c\right) \\ &= \left(an^2 + 2an + a + bn + b + c\right) - \left(an^2 + bn + c\right) \\ &= 2an + a + b, \end{aligned}$$

and therefore

$$
\begin{aligned}
c_n &= b_{n+1} - b_n \\
&= (2a(n+1) + a + b) - (2an + a + b) \\
&= (2an + 2a + a + b) - (2an + a + b) \\
&= 2a.
\end{aligned}
$$

Conversely, if the second differences are constant, say, $c_n = d$, and $b_1 = e$ and $a_1 = g$, then $f(n) = \frac{1}{2}dn^2 + \left(e - \frac{3}{2}d\right)n + \left(g - e + d\right)$; see Table A2.

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