



## **EXPLORING THE MENTAL STRUCTURE AND MECHANISM: HOW THE STYLE OF TRUTH-SEEKERS IN MATHEMATICAL PROBLEM-SOLVING?**

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### **Abstract**

The Mathematics students who perform truth-seeking process upon solving mathematical problems were unique. Therefore, the study deems it necessary to know students' mental structure and mechanism so that they can make the right decision by performing truth-seeking. However, no research has delved into the mental structures and mechanisms of Mathematics students, who tend to grapple with truth-seeking processes extensively. This study was explorative qualitative because the aims to describe the types of mental structure and mechanism of Mathematics students upon the truth-seeking process in solving mathematical problems. The research subjects are four Mathematics students at the University of Jember who perform truth-seeking and can communicate fluently when performing think-aloud. Their responses in the answer sheets drove the determination of research subjects' tendency in truth-seeking. Afterward, the results of think-aloud and task-based interview were put under analysis, so as to determine the types of mental structure and mechanism. The research findings have indicated that (1) all mental structures have been constructed by all research subjects and (2) two types of mental mechanism are evident among the subjects, including the process of interiorization coupled with coordination and another process encompassing interiorization, coordination, and reversal.

**Keywords:** Mental Structure and Mechanism, Truth-Seeking, Solving Mathematical Problems

### **Abstrak**

Mahasiswa matematika yang melakukan proses truth-seeking dalam menyelesaikan masalah matematika merupakan mahasiswa yang unik. Oleh karena itu perlu diketahui mekanisme dan struktur mentalnya sehingga mereka mengambil keputusan untuk melakukan *truth-seeking*. Akan tetapi belum ada penelitian yang meneliti mekanisme dan struktur mental mahasiswa matematika yang cenderung melakukan proses truth-seeking. Penelitian ini merupakan penelitian kualitatif eksploratif karena tujuan penelitiannya adalah mendeskripsikan tipe dari mekanisme dan struktur mental mahasiswa matematika selama proses truth-seeking dalam menyelesaikan masalah matematika. Subjek penelitian adalah 4 (empat) mahasiswa matematika di Universitas Jember yang melakukan proses truth-seeking dan mampu berkomunikasi dengan lancar ketika melakukan think-aloud. Penentuan kecenderungan proses truth-seeking subjek penelitian didasarkan pada hasil lembar jawaban dan wawancaranya. Selanjutnya, dilakukan analisis terhadap hasil think-aloud dan wawancara berbasis tugas untuk menentukan tipe mekanisme dan struktur mental. Adapun hasilnya adalah (1) semua struktur mental telah dikonstruksi oleh semua subjek penelitian dan (2) terdapat dua macam proses mekanisme mentalnya yaitu tipe pertama adalah mengalami proses interiorisasi dan koordinasi serta tipe kedua adalah mengalami proses interiorisasi, koordinasi, dan reversal.

**Kata kunci:** Mekanisme dan Struktur Mental, *Truth-Seeking*, Penyelesaian Masalah Matematika

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A one's intellectual or attitudinal upon encountering a problem is a disposition (Lai, 2011). Facione (2000) defines the disposition of critical thinking as a consistent internal motivation to act critically towards certain events or circumstances. In this case, a person with critical thinking disposition is one who always relies on his critical thinking when acting (As'ari, Mahmudi, & Nuerlaelah, 2017). Before

carrying out certain actions, people possessing critical thinking tends to ponder things related to action beforehand.

Some experts have given some indicators portraying the characteristics of people with critical thinking dispositions. Ennis (1985) discuss that there are 13 traits of critical thinking, inter alia, (1) seeking clear statements about a theory or question, (2) seeking or delving into arguments, (3) trying to locate the best information, (4) using available sources (5) seeing situation as a whole, (6) trying to remain relevant to main points, (7) keeping in mind the original and/or basic problems, (8) finding alternatives, (9) being an open thinker, (10) remaining in position and changing position when sufficient evidence and reasons allow so doing, (11) seeking as much precision as possible for justification of material, (12) regularly attempting to work on complex parts as a whole, and (13) being sensitive to feelings, levels of knowledge, degrees of sophistication, and so forth. Facione, Sánchez, Facione, & Gainen (1995) state that there are seven scales in the CCTDI (California Critical Thinking Disposition Inventory) instrument used to define a person's critical thinking disposition, which comprises of truth-seeking, open-mindedness, analyticity, systematicity, self-confidence, inquisitiveness, and maturity. Furthermore, Kokdemir (in Emîr, 2013) says that people with critical thinking dispositions tend to express truth and be open-minded, analytical, systematic, confident, and inquisitive. Referring to these three expert opinions, the characteristics of people mastering the critical thinking disposition, as theorized by Ennis and Kokdemir, can be grouped into seven critical thinking disposition components in the CCTDI instrument.

The disposition of students' critical thinking can influence the problem-solving process (Biber, Tuna, & Incikabi, 2013; Karagöl & Bekmezci, 2015, 2015; Özyurt, 2015; Tumkaya, Aybek, & Aldag, 2009). Therefore, in problem-solving, critical thinking disposition needs to develop (Kim & Choi, 2014), since problem-solving is a dimension of critical thinking disposition. As such, when a person's critical thinking disposition escalates, problem-solving skills will also improve (Kanbay & Okanlı, 2017). Therefore, when students master critical thinking disposition, they do not immediately solve a problem at hand, but they will first check the truth behind the problem and classify things associated with the problem (Kurniati & Zayyadi, 2018). Conversely, if students do not master critical thinking disposition, then when solving a particular mathematical problem, they will not check the entire set of questions and the truth behind the information embedded in the problem (As'ari, et al. 2017). This will lead to errors in solving mathematical problems. Therefore, the disposition of critical thinking holds very pivotal roles for every student, especially Mathematics students, when dealing with solving mathematical problems.

Truth-seeking is a critical component of critical thinking disposition, and it is imperative that students master critical thinking disposition (Facione, et al. 1995). This is because truth-seeking denotes the tendency to always search for truth when encountering a problem. Therefore, by performing truth-seeking, Mathematics student will specifically check on the given mathematical problem. The checking focused on the entirety of speech in question, the truth behind the information

in the problem, the use of mathematical symbols, and the applications of logic and logical argumentation in a mathematical problem. This edifice of problem-solving tasks allows Mathematics students to come up with exemplary problem-solving. Person capable of performing the truth-seeking process is characterized by (1) always aiming at the best understanding of a particular situation, (2) strongly emphasizing evidence and reasoning, even on matters already acknowledged, (3) questioning the established beliefs of a person, and (4) always taking important details into concern (Assessment, 2017).

The ability is influence the disposition of Mathematics students to construct the mathematical knowledge existing in their mind (Cansoy & Turkoglu, 2017; E. D. Jacobson, 2017; E. Jacobson & Kilpatrick, 2015). However, no research delves into the relationship between critical thinking dispositions, especially the truth-seeking of Mathematics students with the ability to construct their knowledge when solving problems based on their mental mechanisms and structures. Some researchers conduct studies focusing on the mental mechanism and structure of students in solving mathematical problems or proving a theorem without activating the disposition of critical thinking, especially the truth-seeking component (Brijlall & Maharaj, 2015; Syamsuri, Purwanto, Subanji, & Irawati, 2017). In general, Mathematics student-teachers do not perform the process of encapsulation to object conception upon solving the problem of an infinite set (Brijlall & Maharaj, 2015). That statement is in line with the notion claiming that students fail to construct formal mathematical evidence because the process of thinking in its mental mechanism has yet to manifest encapsulation, de-encapsulation, and generalization (Syamsuri, et al. 2017).

The present study deems it necessary to delve into the mental mechanism and structure of Mathematics students during the truth-seeking process in solving mathematical problems. The truth-seeking process is worth exploring because there is a possibility of revealing different mental mechanisms and structures among mathematics student. The study seeks to gain glaring understanding on the mental structure and mechanism of Mathematics students who master truth-seeking skills in solving Mathematics problem. This investigation further seeks to shed lights on determining appropriate instructional approach or model, which can foster truth-seeking in every Mathematics instruction. What is more, fine-cut understanding on the process of decision-making related to the process of mental structure and mechanism in the students of mathematics can be brought to the surface, helping teachers to encourage the students' truth-seeking upon working on mathematical problems.

In this study, the phases of mental structure and mechanism made operative refer to the theory of APOS, projected to understand the mechanism of reflective abstraction, as introduced by Piaget (Dubinsky, 2002). According to this theory, there are five types of reflective abstractions or mental mechanisms, namely interiorization, coordination, reversal, encapsulation, and generalization, leading to the construction of mental structures namely Action, Process, Object and Scheme (Arnon, et al. 2014; Monica, et al. 2012). Figure 1 displays the detailed mental structure and mechanism of

Mathematics students during the process of truth-seeking in solving mathematical problems. In the present study, each of the traits characterizing the capability of the truth-seeking process is elaborate the translation of the mental structure and mechanism of Mathematics students to construct mathematical knowledge based on APOS theory in solving mathematical problems.

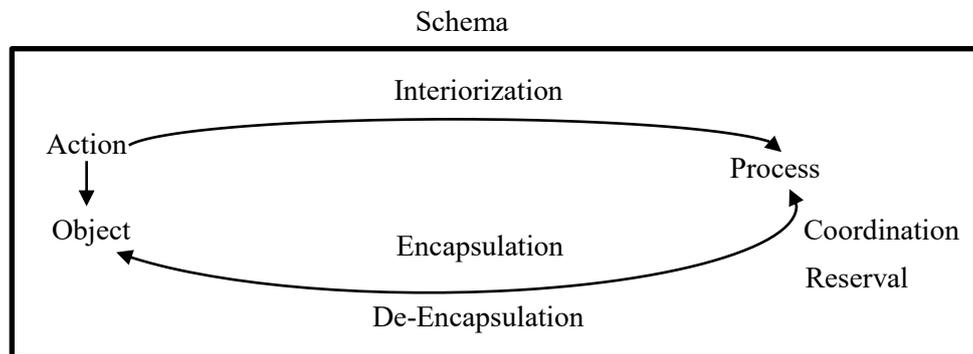


Figure 1. The Mental Mechanism and Structure Based on APOS Theory

## METHOD

This research was explorative qualitative in nature, as it was aimed at revealing the types of mental structure and mechanism of Mathematics students in the truth-seeking process upon solving mathematical problems. Mathematical problems used for testing purposes consisted of two items, comprising of one with contradiction and another one erroneous completion. Both questions were designed to find out whether or not the Mathematics student mastered truth-seeking and to scrutinize their mental structure and mechanism. The questions operative for these aims are as follows.

1. Question number 1. The question with contradictory information.

If  $x + 3y + 7z = 50$  with  $x = 2k + 1, y = 2l + 1, z = 2m + 1$ , and  $k, l, m \in N$ , determine the value of  $x, y$ , and  $z$ .

2. Question number 2. The question with erroneous problem-solving phases.

If  $x^2 = 4$  and  $x = 2$  and the consequence is  $x^2 - 4 = x - 2$ . Due to  $x^2 - 4 = (x - 2)(x + 2)$  so  $x + 2 = 1$ . If  $x = 2$ , meaning  $4 = 1$ , is this statement correct or incorrect? If it is incorrect, then where does the error lie in the problem completion? Explain your answer. If it is correct, explain why it is correct.

In the first question, before working on it, a truth-seeking conversant student will check the truth of the given problem by showing that the sum of the three odd numbers will produce an even number. As a corollary, the student will conclude that the question is incorrect, and it is thus impossible to determine the values of  $x, y$ , and  $z$ . The action indicator on question 1 is giving some correct examples to assert that no values correspond to  $x, y$ , and  $z$  and satisfy the equations in the question. The process indicator is that it can make the equation model of the sum of three odd numbers equal to an odd number. The object indicator for question number 1 is being able to make

another representation of the mathematical equation model confirming that  $x$ ,  $y$ , and  $z$  are odd and odd numbers, and are an even number. Furthermore, the students' scheme indicator for item number 1 is constructing a good and correct scheme between the concept of natural numbers, the concept of odd and even numbers, as well as operations on the original numbers in the proposition.

In the second question, a truth-seeking competent student will spot the error during the problem-solving process. The error lies in the statement if  $x^2 = 4$  and  $x = 2$  then result in  $x^2 - 4 = x - 2$ . The statement  $x^2 = 4$  and  $x = 2$  is true, while the statement  $x^2 - 4 = x - 2$  is incorrect. Therefore, based on the rules of mathematical logic, if the first statement is correct and the result of the second statement is incorrect, then the conclusion is consequently incorrect. The action indicator on problem 2 is giving some correct examples to confirm that the statement if  $x^2 = 4$  and  $x = 2$  then  $x^2 - 4 = x - 2$  is incorrect. The process indicator is to make a model of the quadratic equation to express the factorization of  $x^2 = 4$ . The object indicator for question 2 is being able to make another representation by stating that  $x^2 - 4 = (x + 2)(x - 2)$  and  $x = -2$  or  $x = 2$ . The students' scheme indicator for item 2 is having a good and correct scheme among the principle of quadratic equation, the principle of equality of two integers, and the concept of a solution of quadratic equations in a proposition.

Subjects in this study were 4 (four) 6<sup>th</sup>-semester students in the Department of Mathematics Education at the Faculty of Teacher Training and Education of Universitas Jember during in the 2017/2018 Academic Year. The four subjects were selected because they met the following conditions: (1) fulfilling the four truth-seeking indicators in completing two given questions, (2) mastering excellent fluent communication skills when doing think-aloud based on the recording, and (3) willing to be the research subject.

After determining the research subjects, the researchers analyzed the results of recorded think-aloud of each subject. The analysis focused on the tendency evident in their mental structure and mechanism upon solving mathematical problems. Next, the researchers conducted an unstructured interview to confirm the process of mental structures and mechanisms during truth-seeking performed by the subject when solving mathematical problems. The interviews investigated the process of solving mathematical problems. The final stages encompassed analyzing answer sheets, recording think-aloud, direct observation record of truth-seeking, and interviewing the result to portray the Mathematics students' tendency in mental structure and mechanism during the very truth-seeking process, concerning APOS theory.

## RESULTS AND DISCUSSION

### *The Case of Question Number 1*

The first question given to the research subjects was described as follow.

If  $x + 3y + 7z = 50$  with  $x = 2k + 1$ ,  $y = 2l + 1$ ,  $z = 2m + 1$ , and  $k, l, m \in \mathbb{N}$ , determine the value of  $x$ ,  $y$ , and  $z$ .

### ***Analysis of the Truth-Seeking Process in Solving Question Number 1***

Two distinctive groups were evident in the truth-seeking process of the four Mathematics students in solving mathematical problem number 1, described as follows.

1. The truth-seeking process of the three Mathematics students (S1, S2, S3), before solving problem number 1, was commenced by checking the truth of the information and the instruction in the problem. Thus, the third student's truth-seeking process initiated at the beginning before working on the problem. The truth-seeking process was done by writing another representation of the known information stating that  $x$ ,  $y$ , and  $z$  are odd numbers because  $x = 2k + 1$ ,  $y = 2l + 1$ ,  $z = 2m + 1$ , with  $k, l, m \in \mathbb{N}$ . Since  $x$ ,  $y$ , and  $z$  are odd numbers. They concluded that the result of  $x + 3y + 7z$  was also an odd number due to the reason that the sum of three odd numbers equals an odd number. They hence concluded that question number 1 could not be solved because no value could fit in  $x$ ,  $y$ , and  $z$ , which were all odd number. Also, they predicted that the sum was equal to 50, an even number. These findings showed that the three Mathematics students always sought the best understanding before working on the mathematical problem by seeking the correct evidence as well as reasoning and paying attention to the important details related to the terms  $x$ ,  $y$ , and  $z$ , which was all performed to confirm that question number 1 was incorrect. Figure 2 is present the students' answers S2.

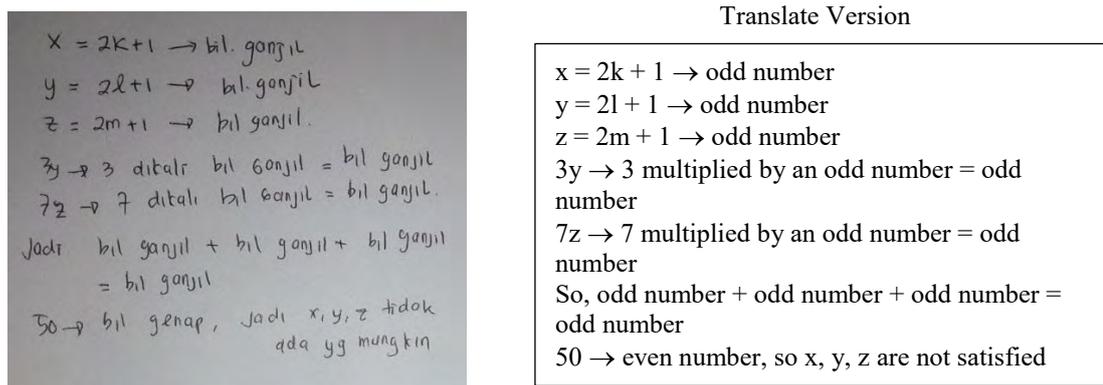


Figure 2. The S2's answers to Question Number 1

2. The truth-seeking process of one other Mathematics student (S4) differed from that of the other three research subjects. The difference occurred when the student was checking the truth after working on question number 1, and he found out that there was an error in the information presented in the question. He changed the form  $x + 3y + 7z = 50$  by substituting  $x$ ,  $y$ , and  $z$  with  $x = 2k + 1$ ,  $y = 2l + 1$ ,  $z = 2m + 1$ , so he came up with  $k + 3l + 7m = \frac{39}{2}$ . Since he could not find the values of  $k$ ,  $l$ , and  $m$ , which could satisfy the equation, he checked the correctness of

the question. That statement indicated that the truth-seeking process of the student emerged during working on the problem. However, the decision taken by the student was similar to that of the other three students, which stated that the given problem was incorrect because it was impossible that the sum of three odd numbers would result in even number. The reason given for confirming the error in question number 1 was corroborated by the concept of even and odd numbers and the operation of the odd number. Figure 3 is present the S4's answer.

Translate Version

$$\begin{aligned}
 &x = 2k + 1, y = 2l + 1, z = 2m + 1 \rightarrow \text{odd number} \\
 &2k + 1 + 3(2l + 1) + 7(2m + 1) = 50 \\
 &2k + 1 + 6l + 3 + 14m + 7 = 50 \\
 &2k + 6l + 14m + 11 = 50 \\
 &2k + 6l + 14m = 50 - 11 \\
 &2(k + 3l + 7m) = 39 \\
 &k + 3l + 7m = \frac{39}{2} \\
 &\text{Nothing the value of } x, y, \text{ and } z \text{ that satisfying } x + 3y + 7z = 50. \text{ Because the value of } x, y, \text{ and } z \text{ must be an odd number and nothing the odd numbers that satisfying } x + 3y + 7z = 50
 \end{aligned}$$

Figure 3. S4's Answer to Question Number 1

### ***Analysis of Student Think-Aloud during the Truth-Seeking in Solving Mathematical Question Number 1 Based on APOS Theory***

During the process of truth-seeking in solving a given mathematical problem performed by four research subjects, there were different ways to construct knowledge based on APOS theory, as evinced by the answer sheets and the think-aloud recordings. Differences in the process of mental mechanism and structure based on the theory of APOS among the four subjects resulted in classifying two distinctive groups by truth-seeking process.

1. The research subjects (S1, S2, and S3) started solving problem number 1 by carefully reading the information in the problem and instruction, that is *known*  $x + 3y + 7z = 50$  with  $x = 2k + 1, y = 2l + 1, z = 2m + 1$ , with  $k, l, m \in \mathbb{N}$ , then determine the value of  $x, y$ , and  $z$ ! It was obvious that they were trying to understand the information and instructions involved. The process of understanding the information denotes interiorization. S1, S2, and S3 were found to go through interiorization in solving question number 1. Furthermore they said that *because*  $x = 2k + 1, y = 2l + 1, z = 2m +$

1, with  $k, l, m \in \mathbb{N}$ , then  $x, y$ , and  $z$  were odd numbers. That statement is indicated that S1, S2, and S3 went through the coordination between mathematical objects in the problem. Also, they stated that *since  $x, y$ , and  $z$  were odd numbers, then  $3y$  and  $5z$  were also odd numbers, so the sum of the three odd numbers would result in odd number as well*. They concluded, from the coordination between objects in the question, that action was pertinent to the premise claiming that *question number 1 was incorrect because they could not determine any numbers to fit in  $x, y$ , and  $z$  and satisfy  $x + 3y + 7z = 50$ . This was owing to the fact that 50 was an even number and  $x + 3y + 7z$  was an odd number*. The research subjects (S1, S2, S3) constructed the existing scheme well and correctly between the concept of the original numbers, odd numbers, and even numbers, and the operation of the original number in the question. In this case, S1, S2, and S3 constructed the mental structures of "action", "process", "object", and "scheme". Furthermore, the process of mental mechanisms conducted by the three subjects constituted interiorization and coordination, implying that not all the mental mechanism processes were performed by S1, S2, and S3. In detail, the students' mental structure and mechanism in solving the first question are presented in Figure 4.

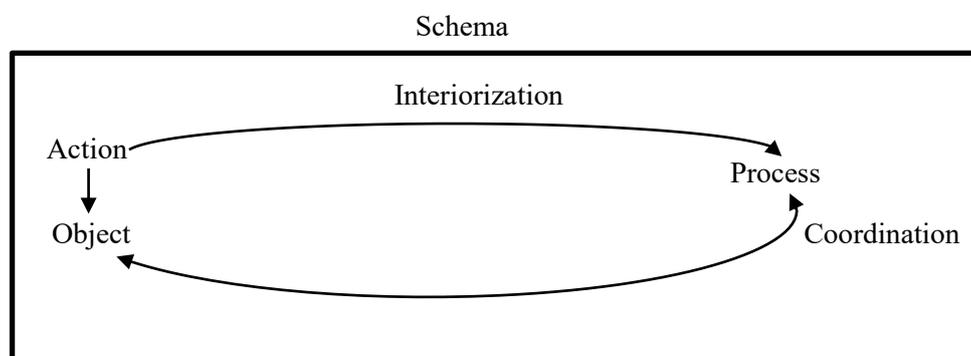


Figure 4. S1, S2, and S3's Mental Mechanism and Structure in Solving Question Number 1

2. Subject S4 started solving the problem by carefully reading the information in the problem and instruction, *that if  $x + 3y + 7z = 50$  with  $x = 2k + 1, y = 2l + 1, z = 2m + 1$ , with  $k, l, m \in \mathbb{N}$ , then determine the values of  $x, y$ , and  $z$ !* That statement indicated that S4 tried to understand the information and instructions in the question. The process of understanding the information manifested interiorization, which was evident in S4's attempt to solve the first question. Also, S4 also did a re-understanding of the information in the question by repeating to himself that *he had to determine  $x, y$ , and  $z$ , so  $x + 3y + 7z = 50$ . In this case, S4's interiorization process was done twice*. Then, S4 stated that he would *change the form  $x + 3y + 7z = 50$  by substituting  $x, y$ , and  $z$  with  $x = 2k + 1, y = 2l + 1, z = 2m + 1$ , so that  $k + 3l + 7m = \frac{39}{2}$* . After starting the process, S4 completed the problem as follows: *knowing  $k + 3l + 7m = \frac{39}{2}$ , then I had to check if there were*

numbers could possibly fit in  $k$ ,  $l$ , and  $m$  and satisfy the equation  $k + 3l + 7m = \frac{39}{2}$ . Since  $k$ ,  $l$ , and  $m \in \mathbb{N}$  then the sum of  $k$ ,  $3l$ , with  $7m$  should be a native number as well. However, after the calculation process was done, resulting in a sum of  $\frac{39}{2}$ , it was conclusive that no figures could meet the requirement of  $k$ ,  $l$ , and  $m$  and satisfy  $k + 3l + 7m = \frac{39}{2}$ . In this case, S4 underwent coordination between objects in the question. However, S4 was confused in determining the values of  $x$ ,  $y$ , and  $z$  which could satisfy equation in the problem. As a result, S4 re-checked what he concluded. The next statement of S4 is as follows.

*Woops! So I had to double check the terms of  $x$ ,  $y$ , and  $z$ . It turned out that the question stated  $x = 2k + 1$ ,  $y = 2l + 1$ ,  $z = 2m + 1$  with  $k, l, m \in \mathbb{N}$ . Since  $x = 2k + 1$ ,  $y = 2l + 1$ ,  $z = 2m + 1$  then the numbers  $x$ ,  $y$ , and  $z$  were odd numbers.*

In this case, S4 underwent a mental mechanism called a reversal because he decided on the process of returning to the object to decide whether the given problem could be solved. Therefore, based on the think-aloud analysis on the mental structure of "action", "process", "object", and "schema" constructed by S4, the mental mechanisms involving interiorization, coordination, and reserves were operative upon solving question number 1. In detail, Figure 5 is present S4's mental structure and mechanism in solving question number 1.

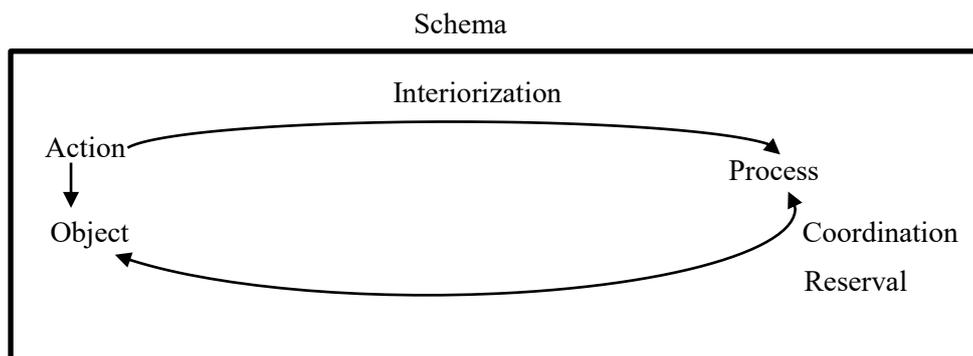


Figure 5. S4's Mental Mechanism and Structure in Solving Question Number 1

### ***The Case of Question Number 2***

The second question given to the four research subjects is described as follows.  
 Given  $x^2 = 4$  and  $x = 2$  and the result is  $x^2 - 4 = x - 2$ . Since  $x^2 - 4 = (x - 2)(x + 2)$  then  $x + 2 = 1$ . With the information stating that  $x = 2$ ,  $= 1$ , are these statements correct or incorrect? If they are incorrect, then where does the error occur in the process of solving the problem? Explain your reasons. If they are correct, then explain why the statements are true!

**Analysis of Truth-Seeking Process in Solving Question Number 2**

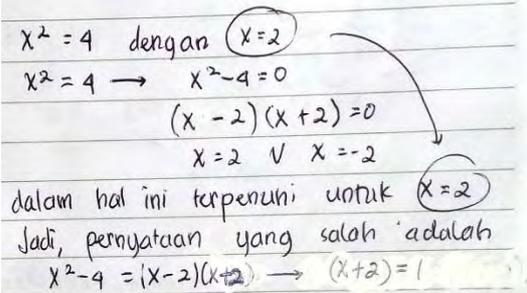
Data indicated that the four research subjects performed a truth-seeking process in solving the question number 2. There were three groups of truth-seeking process evident of the four math students.

1. The S1's truth-seeking when solving question number 2 pertained to analyzing where the error, leading to the statement  $4 = 1$ , occurred. According to the student, the error on problem number 2 lied on the information given, which was  $x = 2$ . The information that should be known in question 2 was  $x^2 = 4$ ,  $x = 2$  and  $x = -2$  since  $x = 2$  and  $x = -2$  were the result of  $x^2 = 4$ . In this case, the student performed correct reasoning based on the roots of the quadratic equation  $x^2 - 4 = 0$  and paid attention to the details of in the question, although the information in the question seemed to be true. Because the information in the problem was incomplete, the cause of the information in the question is  $x^2 - 4 = x - 2$  was also incorrect, so  $x + 2 = 1$  was equivalent to  $x = -1$  contradiction with the known  $x = 2$ . Figure 6 is present the students' answers in number 2.

	<p style="text-align: center;">Translate Version</p> <p>The error is in the statement <math>x^2 = 4</math> and <math>x = 2</math>. Because if <math>x^2 = 4</math> then must be <math>x = \pm 2</math>, because <math>x</math> that satisfying just not <math>x = 2</math> but also <math>x = -2</math>, so <math>x^2 - 4 = (x + 2)(x - 2)</math></p>
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Figure 6. S1's Answer to Question Number 2

2. The truth-seeking process of the first subject differed from that of the other research subjects (S3 and S4). Based on the answer sheets and interview results, the conclusion is in the truth-seeking process, he stated that the error of question 2 was evident when he concluded that  $x^2 - 4 = x - 2$ . The information known in the question was  $x^2 = 4$  and  $x = 2$  was a correct statement (considered true) and the result was  $x^2 - 4 = x - 2$ , which was an incorrect statement because it had to be  $x^2 - 4 = (x - 2)(x + 2)$ . Since the first statement was correct and thus implied that the second statement was incorrect, the known statement in question was incorrect based on mathematical logic. Because the statements were known to be incorrect, the conclusions were also incorrect  $4 = 1$ . In this case, the two students always sought fine-cut understanding of the information in the question by linking the existing material with mathematical logic and quadratic equations. In addition, the two students also emphasized reasoning and argument in order to prove that the statements  $x^2 = 4$  and  $x = 2$  resulting in  $x^2 - 4 = x - 2$  were incorrect. The answer sheet of one of the research subjects, S3, is presented in Figure 7.



Translate Version

$$x^2 = 4 \text{ with } x = 2$$

$$x^2 = 4 \rightarrow x^2 - 4 = 0$$

$$(x - 2)(x + 2) = 0$$

$$x = 2 \vee x = -2$$

So the value  $x$  that satisfying is  $x = 2$

So, the statement that wrong is  $x^2 - 4 = (x + 2)(x - 2) \rightarrow (x + 2) = 1$

Figure 7. S3's Answer to Question Number 2

The results of the interview are as follows.

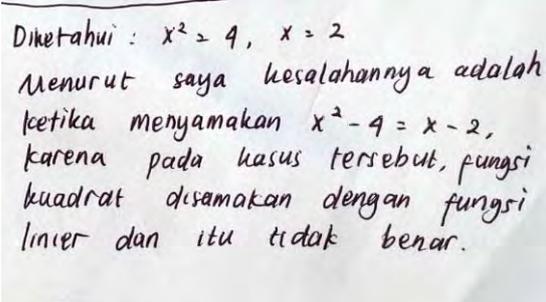
Researcher : *On the answer sheet you write that the statement  $x^2 - 4 = (x + 2)(x - 2) \rightarrow (x + 2) = 1$  is an incorrect statement. Please explain the reasons for your judgment!*

S3 : *Actually, I think it is incorrect when it is concluded that  $x^2 - 4 = x - 2$  resulting in  $x^2 - 4 = (x + 2)(x - 2) \rightarrow (x + 2) = 1$*

Researcher : *Can you explain why  $x^2 - 4 = x - 2$  is incorrect?*

S3 : *Since it should be  $x^2 - 4 = (x + 2)(x - 2)$ , whatever is known in the question. The known statement in question is  $x^2 = 4$  and  $x = 2$  is correct and the result statement is  $x^2 - 4 = x - 2$  is incorrect. As such, if  $P$  is true, it must imply that  $Q$  is incorrect then the conclusion resulting from that place is thus incorrect. However, I did not write it on the answer sheet because I was confused with how to write it.*

3. The truth-seeking process of another student (S2) was different from that of the other three students in that the student stated that the error associated with  $4 = 1$  was in  $x^2 - 4 = x - 2$ . In this case, the student stated that the quadratic equation could not be the same as a linear equation. He compared  $x^2 - 4$  as a quaternary equation to  $x - 2$  as a linear equation. In this case, he used his reasoning and arguments related to the characteristics of quadratic equations and linear equations. Also, he also noticed the details in the question by checking the truth behind the information. Figure 8 present the student's answer.



Translate Version

Let:  $x^2 = 4, x = 2$

In my opinion, the error is when I equalize  $x^2 - 4 = x - 2$ , because in this case, the quadratic function equal with the linear function and it is wrong

Figure 8. S2's Answer to Question Number 2

### ***Analysis of Students' Think-Aloud During the Truth-Seeking Process in Solving Mathematics Problem Number 2 Based on APOS Theory***

During the process of truth-seeking performed by the research subjects, a particular method guided by APOS theory was made operative to accrue insights into the subjects by studying their answer sheets and think-aloud recordings. The mental mechanism and structures of S1, S2, S3, and S4 when solving problem number 2 are as follows.

All research subjects started the process of solving problem two by carefully reading the information in the question and the instruction, that was  $x^2 = 4$  and  $x = 2$ , and the result was  $x^2 - 4 = x - 2$ . Since  $x^2 - 4 = (x - 2)(x + 2)$  then  $x + 2 = 1$ . With the information stating  $x = 2$  and  $4 = 1$ , determine whether the statement is correct or incorrect? If it is incorrect, then where does the error occur in the process of solving the problem? Explain your reasons. If it is correct, then explain why the statement is correct! Findings indicated that they were trying to understand the information and instructions in the question. The process of understanding the information indicated interiorization, clearly implying that all research subjects performed interiorization in solving question number 2. S1 further said that the error on the known information lied in  $x = 2$ . The information known in problem number 2 should be  $x^2 = 4$ ,  $x = 2$  and  $x = -2$  since  $x = 2$  and  $x = -2$  were the result of  $x^2 = 4$ . Since the information in the problem was incomplete, then  $x^2 - 4 = x - 2$  was also incorrect, so  $x + 2 = 1$ , which was equivalent to  $x = -1$ , was contradictory to  $x = 2$ . It denotes that S1 undergoes a coordination process between the mathematical objects present in the problem. Also, S3 and S4 also performed coordination between mathematical objects included in problem number 2. That premise was based on the results of S3 and S4's think-aloud, indicating that the location of the error of the problem number 2 was  $x^2 - 4 = x - 2$ . Information known in the question was  $x^2 = 4$  and  $x = 2$ , and this was assumed to be correct because it was obvious in the problem and the result was  $x^2 - 4 = x - 2$ , which was the incorrect statement because  $x^2 - 4 = (x - 2)(x + 2)$ . Since the first statement was correct and thus concluded that the second statement was incorrect, then, based on mathematical logic, the known statement in question was incorrect. The same case applied to S2, which suggested that the error associated with  $4 = 1$  was  $x^2 - 4 = x - 2$ . This was owing to the fact that equation of the square was equated to the linear equation. Although we knew that the characteristics of quadratic equations and linear equations were very different. The conclusion, known as an action, that they drew from the process of coordination between objects in the problem was that question 2 was an incorrect problem due to the error in the statement  $x^2 - 4 = x - 2$ . Based on the completion process performed by S1, S2, S3, and S4, then they have constructed the scheme of their minds, the principle of quadratic equation, the principle of equality of two integers, and the concept of a solution of quadratic equations well and correctly. In this case, S1, S2, S3, and S4 were capable of constructing the mental structures of "action", "process", "object", and "scheme". Furthermore, the

mental mechanism processes run by the four subjects of the study were evident of interiorization and coordination. As a corollary, it was clear that not all the mental mechanism processes were performed by S1, S2, S3, and S4. In detail, the mental structure and mechanisms performed by S1, S2, S3, and S4 in solving question number 2 are presented in Figure 9.

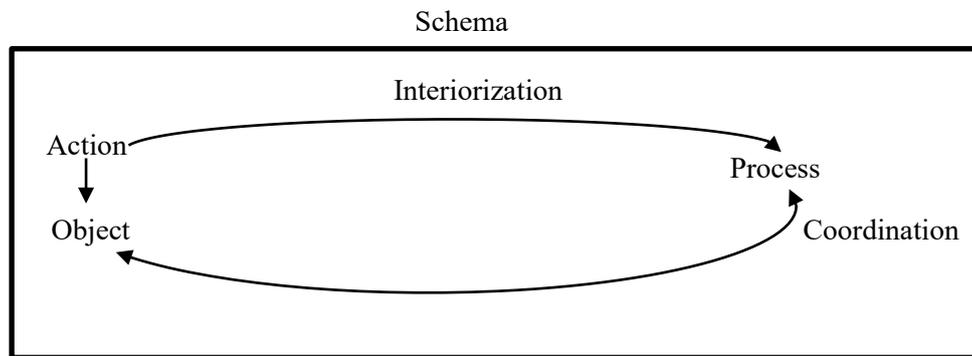


Figure 9. S1, S2, and S3's Mental Mechanism and Structure in Solving Question 2

Based on the results of the analysis of answer sheets, interviews, and think-aloud of the four research subjects, the study has concluded that all research subjects performed a truth-seeking process in solving both mathematical problems in the study. The research subjects' tendency in mental mechanisms and structures in solving the mathematical problems is portrayed as follows.

1. All research subjects have constructed all mental structures, i.e., actions, processes, objects, and schemes
2. There are two kinds of mental mechanism processes evident of the four subjects who go through the truth-seeking process when solving mathematical problems. The process of interiorization and coordination characterizes the first type, and the second one is tailored to the process of interiorization, coordination, and reversal.

The findings of this study have contradicted the results of other studies, which suggest that Mathematics students, upon working on the problem of algebra, only prove object which has been formed while the process, action, and scheme are not constructed. What is more, the tendency of their mental mechanisms is only marked by interiorization and co-ordination (Syamsuri, et al. 2017). But, in the process of constructing their knowledge, the Mathematics students performing the truth-seeking process also perform the process of understanding problems, exploring, formulating, justifying, and proving possible incorrect information in both questions (Astawa, Budayasa, & Juniati, 2018). So and so, the process of Mathematics students performs when constructing knowledge can be used as a basis for making decisions to perform truth-seeking when encountering mathematical problems (Moore, 2010).

The process of the mental mechanism of the Mathematics students conducting the truth-seeking process in this study was incomplete because the encapsulation and de-encapsulation process was not

evident in the problem-resolution process. On the other hand, the process of a person's mental mechanism when solving mathematical problems has to be performed regardless of the sequence (Dubinsky, 2002; Stoilescu, 2016) This is because when all processes of mental mechanism are done then the process of proving a statement or proposition in a question can be precisely determined to reveal its true value. Therefore, it is necessary to apply a model of problem-based learning as an effort to develop high-level thinking skills, especially in critical thinking disposition research associated with APOS theory. By so doing, all processes of students' mental structure and mechanisms can be developed to their utmost (Mudrikah, 2016; Widyatiningtyas, Kusumah, Sumarmo, & Sabandar, 2015).

Furthermore, the truth-seeking process performed by Mathematics students when solving mathematical problems was based on the process of mental structure and mechanism that tend to perform the process of interiorization. That statement was evident of checking the truth behind the information associated with the question and coordination to make decisions on problem-solving mediated by all pertinent objects. These findings, in fact, comply with the characteristics of people capable of performing truth-seeking, as elaborated by Ennis (1985). This capability is characterized by (1) seeking clear statements about theories, especially those regarding the odd number and quadratic equations, (2) explaining each argument to support the decisions taken, (3) trying to find the best information, and (4) keeping in mind the original problem and/or the basis of the problem.

## **CONCLUSION**

The Mathematics students performing a truth-seeking process in solving mathematical problems always construct knowledge using their mental structure, as elaborated in APOS theory, comprising of action, process, object, and scheme. All mental structures are used as the basis for decision-making in truth-seeking when students solving the mathematical problems. Furthermore, the tendency of mental mechanisms performed by the students during the truth-seeking process in solving math problems only includes interiorization and coordination, although one student activates the mental mechanisms of interiorization, coordination, and reversal.

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